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TWO APPLICATIONS
OF DUALIZING COMPLEXES OVER LOCAL RINGS

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We prove two results on Noetherian local rings, related by the fact that both use properties of dualizing complexes. The first is a result on the annihilators of the homology modules of a perfect complex, which has as a consequence the Intersection Theorem (cf. Peskine-Szpiro [5]) for rings of positive characteristic. The second result determines the integers $i$ for which the $\mu_i$ of Bass vanish, answering a question of Foxby [2].

Let $A$ be a commutative Noetherian local ring, $m$ its maximal ideal, and $k$ its residue field.

DEFINITION. - A dualizing complex over $A$ is a complex $(D)$ of $A$-modules such that for each integer $i$:

1. $D_i \cong \bigoplus_{\substack{p \in \text{Spec}(A) \mid \dim(A/p) = i}} E(A/p)$,

where $E(A/p)$ is the injective hull of $A/p$.

2. The homology $H_i(D)$ is a finitely generated $A$-module.

We note that a dualizing complex consists of injective modules, and is bounded, with $D_i = 0$ for $i < 0$ and $i > \dim A$. If $A$ is complete, there exists a dualizing complex over $A$ ([3], p. 299). Since we can assume $A$ complete in our applications, we will henceforth assume that a dualizing complex exists, and denote it $(D_i)$.

If $F'$ and $G'$ are complexes, we will let $\text{Hom}(F', G')$ denote the double complex $\{ \text{Hom}(F^i, G^j) \}$, and $\overline{\text{Hom}}(F', G')$ will denote the associated simple complex, with

$$(\overline{\text{Hom}}(F', G'))^n = \prod_{i+z} \text{Hom}(F^i, G^{i+n}) \text{ for all } n.$$  

I. Let $n = \dim(A)$. For $i = 0, 1, \ldots, n$, let $a_i = \text{Annih}(H_i(D))$, and let $b_i$ be the product $a_ia_{i-1} \cdots a_0$.  

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PROPOSITION 1. — For $i = 0, 1, \ldots, n$, we have $\dim (A/b_i) \leq i$.

Proof. — Let $q$ be a prime ideal such that $\dim (A/q) > i$. Since $D_i \cong \bigoplus_{\dim A/p = i} E(A/p)$, the localization $(D_i)_q$ is zero, and hence $(H_i(D))_q = 0$. Since $H_i(D)$ is a finitely generated $A$-module, this implies that $a_i = \operatorname{Annih}(H_i(D)) \cong q$. Hence $a_j \cong q$ for $j \leq i$, so $b_i = a_1 a_{i-1} \ldots a_0 \cong q$, and $\dim (A/b_i) \leq i$.

THEOREM 1. — Let $F^r = 0 \to F^0 \to \ldots \to F^r \to 0$ be a complex of finitely generated free modules such that $H^i(F')$ is a module of finite length for all $i$. Then $b_i$ annihilates $H^i(F')$ for $i = 0, 1, \ldots, n$.

Proof. — We consider the two spectral sequences of the double complex $\operatorname{Hom}(F', D)$, converging to the homology of the associated simple complex.

Fixing $k$, we have $H_j(\operatorname{Hom}(F', D)) \cong \operatorname{Hom}(H^j(F'), D_k)$ for all $j$, since $D_k$ is injective. Since $H^j(F')$ has finite length, $\operatorname{Hom}(H^j(F'), E(A/p)) = 0$ unless $p = m$, and hence $\operatorname{Hom}(H^j(F'), D_k) = 0$ unless $k = 0$. Thus the spectral sequence degenerates, and we have

$H_i(\operatorname{Hom}(F', D)) = \operatorname{Hom}(H^i(F'), E(k))$.

Fixing $j$, we have $H_k(\operatorname{Hom}(F', D)) \cong \operatorname{Hom}(H^j, H_k(D))$; thus $H_k(\operatorname{Hom}(F', D))$ is annihilated by $a_k$. Hence $a_k$ annihilates the $(j, k)$-term in every subsequent stage of the spectral sequence. Taking those terms in the limit for which $j+k = i$, we thus arrive at a filtration of $H_i(\operatorname{Hom}(F', D))$ of the form

$H_i(\operatorname{Hom}(F', D)) = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_t \supseteq M_{t+1} = 0,$

where $a_k$ annihilates $M_k/M_{k+1}$ for each $k$. Hence $b_i = a_1 a_{i-1} \ldots a_0$ annihilates $H_i(\operatorname{Hom}(F', D))$. Since

$H_i(\operatorname{Hom}(F', D)) \cong \operatorname{Hom}(H^i(F'), E(k))$

and

$\operatorname{Annih}(H^i(F'), E(k)) = \operatorname{Annih}(H^i(F'))$,

we thus have that $b_i$ annihilates $H^i(F')$.

We now show how this implies the Intersection Theorem for rings of characteristic $p > 0$. (This result appears in Peskine and Szpiro [5], and also in [6].)

INTERSECTION THEOREM. — Assume $A$ has characteristic $p > 0$. If

$F^r = 0 \to F^0 \to \ldots \to F^r \to 0$

is as in Theorem 1, and if $r < \dim (A)$, then $F^r$ is exact.

Proof. — If $F^r$ is not exact, we can assume by splitting off irrelevant components that $F' \neq 0$ and $F'^{r-1} \to F^r$ is defined by a matrix $(a_{ij})$ with entries in $m$. Tensoring with the $n$th power of the Frobenius morphism (cf. [4]) gives a new complex $F(n)'$ with
F(n)^i = F^i for all i and still satisfying the hypotheses of Theorem 1, but with F(n)^{r-1} \to F(n)^r defined by the matrix \((a_{ij})^m\). By Theorem 1,

\[ H^r(F(n)^r) = \text{Coker}(F(n)^{r-1} \to F(n)^r) \]

is annihilated by \(b_r\), and since \((a_{ij})^m\) is in \(m^n\) for all \(i, j\), this implies that \(b_r \subseteq m^n\). Hence \(b_r \subseteq \bigcap_{n \in \mathbb{Z}} m^n = 0\), so \(b_r = 0\). However, by Proposition 1,

\[ \dim(A/b_r) \leq r < \dim(A), \]

a contradiction.

II. THEOREM 2. — Let \(M\) be a non-zero \(A\)-module of finite type, and, for each integer \(i \geq 0\), let \(\mu_i(M) = \dim_k \text{Ext}^i_q(k, M)\) (as in Bass [1]). Then \(\mu_i(M) \neq 0\) if and only if \(\text{depth}(M) \leq i \leq \text{inj dim}(M)\) (note: the injective dimension of \(M\) may be infinite).

Proof. — For the proof of this theorem we will need the following property of dualizing complexes: if \(M\) is a module of finite type, then \(M\), considered as a complex in degree zero, is quasi-isomorphic to the complex \(\text{Hom}(\text{Hom}(M, D), D)\) (see [3], p. 258).

Since the theorem can easily be reduced to the case of a complete local ring, we will assume that \(A\) is complete.

We note first that it is known that

\[ \inf \{ i \mid \mu_i(M) \neq 0 \} = \text{depth}(M) \quad \text{and} \quad \sup \{ i \mid \mu_i(M) \neq 0 \} = \text{inj dim}(M), \]

and, in addition, that \(\text{depth}(N) \leq \text{inj dim}(M)\) for all finitely generated modules \(N\) ([1], § 3).

We assume the theorem is false, then there is a \(j\) strictly between \(\text{depth}(M)\) and \(\text{inj dim}(M)\) with \(\mu_j(M) = 0\). Let \(I^j\) be a minimal injective resolution of \(M\), and let \(J^j\) be the subcomplex of \(I^j\) consisting of all elements with support in \(\{ m \}\). Then \(J^j \cong E(k)^{\mu_j(M)}\) for all \(i\). Let \(F^j = \text{Hom}(J^j, E(k))\); since we are assuming \(A\) complete, \((F^j)\) is a complex of free modules, and \(F^j \cong A^{\mu_j(M)}\) for all \(i\).

LEMMA. — \((F^j)\) is a quasi-isomorphic to \(\text{Hom}(M, D)\).

Proof. — By the Local Duality Theorem ([3], Theorem V.6.2), \(J^j\) is quasi-isomorphic to \(\text{Hom}(\text{Hom}(M, D), E(k))\). Hence, applying \(\text{Hom}(-, E(k))\) to both complexes and using the completeness of \(A\), together with the fact that \(\text{Hom}(M, D)\) has finitely generated homology, we deduce that \((F^j)\) is quasi-isomorphic to \(\text{Hom}(M, D)\).

Since \((F^j)\) is quasi-isomorphic to \(\text{Hom}(M, D)\), \(\text{Hom}(F^j, D)\) is quasi-isomorphic to \(\text{Hom}(\text{Hom}(M, D), D)\), and hence also to \(M\). Let \(r = \text{depth}(M)\); then, since \(F^j \cong A^{\mu_j(M)}\) for all \(i\) and \(\mu_j(M) = 0\), \((F^j)\) looks like:

\[ \ldots \to F^j \to F^j_{i-1} \to \ldots \to F^j_{j+1} \to 0 \to F^j_{j-1} \to \ldots \to F^j_r \to 0 \to \ldots, \]

where \(F_r \neq 0\) and some \(F_{i} \neq 0\) for \(i > j\).
Let \((F')\) be the part of \((F)\) below \(j\) and \((F'')\) the part above \(j\); that is, \(F'_i = F_i\) if \(i < j\) and 0 if \(i \geq j\), and \(F''_i = 0\) if \(i < j\) and \(F_i\) if \(i \geq j\), with boundary maps inherited from \((F)\). Since \(F_j = 0\), we have \(F_j = (F'_j \oplus F''_j)\). Hence
\[
\text{Hom}(F_j, D) \cong \text{Hom}(F'_j, D) \oplus \text{Hom}(F''_j, D).
\]

Taking homology, and using the fact that \(\text{Hom}(F_j, D)\) is quasi-isomorphic to \(M\), we get that \(M \cong M' \oplus M''\), where \(M' = H^0(\text{Hom}(F'_j, D))\) and \(M'' = H^0(\text{Hom}(F''_j, D))\); in addition, \(\text{Hom}(F'_j, D)\) and \(\text{Hom}(F''_j, D)\) are exact in degrees other than zero. Thus, since they are complexes of injective modules, they are injective resolutions of \(M'\) and \(M''\) respectively. Calculation of \(\text{Ext}^i(k, M')\) and \(\text{Ext}^i(k, M'')\) using these complexes shows that \(\mu_i(M') = \mu_i(M)\) for \(i < j\) and 0 for \(i \geq j\), and \(\mu_i(M'') = 0\) for \(i \leq j\) and \(\mu_i(M)\) for \(i > j\). Hence \(M'\) and \(M''\) are non-zero modules of finite type with
\[
\text{inj dim}(M') < j < \text{depth}(M''),
\]
and, as mentioned above, this situation is known to be impossible. Hence we must have had \(\mu_j(M) \neq 0\), and this proves the theorem.

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