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PAUL ROBERTS

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TWO APPLICATIONS OF DUALIZING COMPLEXES OVER LOCAL RINGS

BY PAUL ROBERTS

We prove two results on Noetherian local rings, related by the fact that both use properties of dualizing complexes. The first is a result on the annihilators of the homology modules of a perfect complex, which has as a consequence the Intersection Theorem (*cf.* Peskine-Szpiro [5]) for rings of positive characteristic. The second result determines the integers i for which the μ_i of Bass vanish, answering a question of Foxby [2].

Let A be a commutative Noetherian local ring, $\mathfrak{m}$ its maximal ideal, and k its residue field.

DEFINITION. — A *dualizing complex* over A is a complex $(D_.)$ of A-modules such that for each integer i :

(1) $$D_i \cong \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec}(A) \\ \dim(A/\mathfrak{p}) = i}} E(A/\mathfrak{p}),$$

where $E(A/\mathfrak{p})$ is the injective hull of $A/\mathfrak{p}$.

(2) The homology $H_i(D_.)$ is a finitely generated A-module.

We note that a dualizing complex consists of injective modules, and is bounded, with $D_i = 0$ for $i < 0$ and $i > \dim A$. If A is complete, there exists a dualizing complex over A ([3], p. 299). Since we can assume A complete in our applications, we will henceforth assume that a dualizing complex exists, and denote it $(D_.)$.

If $F^{\cdot}$ and $G^{\cdot}$ are complexes, we will let $\operatorname{Hom}(F^{\cdot}, G^{\cdot})$ denote the double complex $\{\operatorname{Hom}(F^j, G^k)\}$, and $\underline{\operatorname{Hom}}(F^{\cdot}, G^{\cdot})$ will denote the associated simple complex, with

$$(\underline{\operatorname{Hom}}(F^{\cdot}, G^{\cdot}))^n = \prod_{i \in \mathbf{Z}} \operatorname{Hom}(F^i, G^{i+n}) \quad \text{for all } n.$$

I. Let $n = \dim(A)$. For $i = 0, 1, \ldots, n$, let $\mathfrak{a}_i = \operatorname{Annih}(H_i(D_.))$, and let $\mathfrak{b}_i$ be the product $\mathfrak{a}_i \mathfrak{a}_{i-1} \ldots \mathfrak{a}_0$.

PROPOSITION 1. — *For* $i = 0, 1, \ldots, n$, *we have* $\dim(A/\mathfrak{b}_i) \leqq i$.

Proof. — Let $\mathfrak{q}$ be a prime ideal such that $\dim(A/\mathfrak{q}) > i$. Since $D_i \cong \bigoplus_{\dim A/\mathfrak{p}=i} E(A/\mathfrak{p})$, the localization $(D_i)_{\mathfrak{q}}$ is zero, and hence $(H_i(D_{\cdot}))_{\mathfrak{q}} = 0$. Since $H_i(D_{\cdot})$ is a finitely generated A-module, this implies that $\mathfrak{a}_i = \text{Annih}(H_i(D_{\cdot})) \nsubseteq \mathfrak{q}$. Hence $\mathfrak{a}_j \nsubseteq \mathfrak{q}$ for $j \leqq i$, so $\mathfrak{b}_i = \mathfrak{a}_i \mathfrak{a}_{i-1} \ldots \mathfrak{a}_0 \nsubseteq \mathfrak{q}$, and $\dim(A/\mathfrak{b}_i) \leqq i$.

THEOREM 1. — *Let* $F^{\cdot} = 0 \to F^0 \to \ldots \to F^r \to 0$ *be a complex of finitely generated free modules such that* $H^i(F^{\cdot})$ *is a module of finite length for all i. Then* $\mathfrak{b}_i$ *annihilates* $H^i(F^{\cdot})$ *for* $i = 0, 1, \ldots, n$.

Proof. — We consider the two spectral sequences of the double complex $\text{Hom}(F^{\cdot}, D_{\cdot})$, converging to the homology of the associated simple complex.

Fixing k, we have $H_j(\text{Hom}(F^{\cdot}, D_k)) \cong \text{Hom}(H^j(F^{\cdot}), D_k)$ for all j, since D_k is injective. Since $H^j(F^{\cdot})$ has finite length, $\text{Hom}(H^j(F^{\cdot}), E(A/\mathfrak{p})) = 0$ unless $\mathfrak{p} = \mathfrak{m}$, and hence $\text{Hom}(H^j(F^{\cdot}), D_k) = 0$ unless $k = 0$. Thus the spectral sequence degenerates, and we have

$$H_i(\underline{\text{Hom}}(F^{\cdot}, D_{\cdot})) \cong \text{Hom}(H^i(F^{\cdot}), D_0) \cong \text{Hom}(H^i(F^{\cdot}), E(k)).$$

Fixing j, we have $H_k(\text{Hom}(F^j, D_{\cdot})) \cong \text{Hom}(F^j, H_k(D_{\cdot}))$; thus $H_k(\text{Hom}(F^j, D_{\cdot}))$ is annihilated by $\mathfrak{a}_k$. Hence $\mathfrak{a}_k$ annihilates the (j, k)-term in every subsequent stage of the spectral sequence. Taking those terms in the limit for which $j+k = i$, we thus arrive at a filtration of $H_i(\underline{\text{Hom}}(F^{\cdot}, D_{\cdot}))$ of the form

$$H_i(\underline{\text{Hom}}(F^{\cdot}, D_{\cdot})) = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_i \supseteq M_{i+1} = 0,$$

where $\mathfrak{a}_k$ annihilates M_k/M_{k+1} for each k. Hence $\mathfrak{b}_i = \mathfrak{a}_i \mathfrak{a}_{i-1} \ldots \mathfrak{a}_0$ annihilates $H_i(\underline{\text{Hom}}(F^{\cdot}, D_{\cdot}))$. Since

$$H_i(\underline{\text{Hom}}(F^{\cdot}, D_{\cdot})) \cong \text{Hom}(H^i(F^{\cdot}), E(k))$$

and

$$\text{Annih}(\text{Hom}(H^i(F^{\cdot}), E(k))) = \text{Annih}(H^i(F^{\cdot})),$$

we thus have that $\mathfrak{b}_i$ annihilates $H^i(F^{\cdot})$.

We now show how this implies the Intersection Theorem for rings of characteristic $p > 0$. (This result appears in Peskine and Szpiro [5], and also in [6].)

INTERSECTION THEOREM. — *Assume* A *has characteristic* $p > 0$. *If*

$$F^{\cdot} = 0 \to F^0 \to \ldots \to F^r \to 0$$

is as in Theorem 1, *and if* $r < \dim(A)$, *then* $F^{\cdot}$ *is exact.*

Proof. — If $F^{\cdot}$ is not exact, we can assume by splitting off irrelevant components that $F^r \neq 0$ and $F^{r-1} \to F^r$ is defined by a matrix (a_{ij}) with entries in $\mathfrak{m}$. Tensoring with the nth power of the Frobenius morphism (*cf.* [4]) gives a new complex $F(n)^{\cdot}$ with

$F(n)^i = F^i$ for all i and still satisfying the hypotheses of Theorem 1, but with $F(n)^{r-1} \to F(n)^r$ defined by the matrix $((a_{ij})^{p^n})$. By Theorem 1,

$$H^r(F(n)^{\cdot}) = \operatorname{Coker}(F(n)^{r-1} \to F(n)^r)$$

is annihilated by $\mathfrak{b}_r$, and since $(a_{ij})^{p^n}$ is in $\mathfrak{m}^{p^n}$ for all i, j, this implies that $\mathfrak{b}_r \subseteq \mathfrak{m}^{p^n}$. Hence $\mathfrak{b}_r \subseteq \bigcap_{n \in Z} \mathfrak{m}^{p^n} = 0$, so $\mathfrak{b}_r = 0$. However, by Proposition 1,

$$\dim(A/\mathfrak{b}_r) \leqq r < \dim(A),$$

a contradiction.

II. THEOREM 2. — *Let* M *be a non-zero* A-*module of finite type, and, for each integer* $i \geqq 0$, *let* $\mu_i(M) = \dim_k \operatorname{Ext}_A^i(k, M)$ (*as in Bass* [1]). *Then* $\mu_i(M) \neq 0$ *if and only if* depth (M) $\leqq i \leqq$ inj dim (M) (*note: the injective dimension of* M *may be infinite*).

Proof. — For the proof of this theorem we will need the following property of dualizing complexes: if M is a module of finite type, then M, considered as a complex in degree zero, is quasi-isomorphic to the complex $\underline{\operatorname{Hom}}(\underline{\operatorname{Hom}}(M, D_{\cdot}), D_{\cdot})$ (*see* [3], p. 258).

Since the theorem can easily be reduced to the case of a complete local ring, we will assume that A is complete.

We note first that it is known that

$$\inf\{\, i \mid \mu_i(M) \neq 0 \,\} = \operatorname{depth}(M) \qquad \text{and} \qquad \sup\{\, i \mid \mu_i(M) \neq 0 \,\} = \operatorname{inj\,dim}(M),$$

and, in addition, that depth (N) $\leqq$ inj dim (M) for all finitely generated modules N ([1], § 3).

We assume the theorem is false, then there is a j strictly between depth (M) and inj dim (M) with $\mu_j(M) = 0$. Let $I^{\cdot}$ be a minimal injective resolution of M, and let $J^{\cdot}$ be the subcomplex of $I^{\cdot}$ consisting of all elements with support in $\{\mathfrak{m}\}$. Then $J^i \cong E(k)^{\mu_i(M)}$ for all i. Let $F_{\cdot} = \operatorname{Hom}(J^{\cdot}, E(k))$; since we are assuming A complete, $(F_{\cdot})$ is a complex of free modules, and $F_i \cong A^{\mu_i(M)}$ for all i.

LEMMA. — $(F_{\cdot})$ *is a quasi-isomorphic to* $\underline{\operatorname{Hom}}(M, D_{\cdot})$.

Proof. — By the Local Duality Theorem ([3], Theorem V.6.2), $J^{\cdot}$ is quasi-isomorphic to $\underline{\operatorname{Hom}}(\underline{\operatorname{Hom}}(M, D_{\cdot}), E(k))$. Hence, applying $\underline{\operatorname{Hom}}(-, E(k))$ to both complexes and using the completeness of A, together with the fact that $\underline{\operatorname{Hom}}(M, D_{\cdot})$ has finitely generated homology, we deduce that $(F_{\cdot})$ is quasi-isomorphic to $\underline{\operatorname{Hom}}(M, D_{\cdot})$.

Since $(F_{\cdot})$ is quasi-isomorphic to $\underline{\operatorname{Hom}}(M, D_{\cdot})$, $\underline{\operatorname{Hom}}(F_{\cdot}, D_{\cdot})$ is quasi-isomorphic to $\underline{\operatorname{Hom}}(\underline{\operatorname{Hom}}(M, D_{\cdot})\, D_{\cdot})$, and hence also to M. Let $r = \operatorname{depth}(M)$; then, since $F_i \cong A^{\mu_i(M)}$ for all i and $\mu_j(M) = 0$, $(F_{\cdot})$ looks like:

$$\ldots \to F_i \to F_{i-1} \to \ldots \to F_{j+1} \to 0 \to F_{j-1} \to \ldots \to F_r \to 0 \to \ldots,$$

where $F_r \neq 0$ and some $F_i \neq 0$ for $i > j$.

Let $(F'_\bullet)$ be the part of $(F_\bullet)$ below j and $(F''_\bullet)$ the part above j; that is, $F'_i = F_i$ if $i < j$ and 0 if $i \geqq j$, and $F''_i = 0$ if $i < j$ and F_i if $i \geqq j$, with boundary maps inherited from $(F_\bullet)$. Since $F_j = 0$, we have $F_\bullet = (F'_\bullet \oplus F''_\bullet)$. Hence

$$\underline{\mathrm{Hom}}(F_\bullet, D_\bullet) \cong \underline{\mathrm{Hom}}(F'_\bullet, D_\bullet) \oplus \underline{\mathrm{Hom}}(F''_\bullet, D_\bullet).$$

Taking homology, and using the fact that $\underline{\mathrm{Hom}}(F_\bullet, D_\bullet)$ is quasi-isomorphic to M, we get that $M \cong M' \oplus M''$, where $M' = H^0(\underline{\mathrm{Hom}}(F'_\bullet, D_\bullet))$ and $M'' = H^0(\underline{\mathrm{Hom}}(F''_\bullet, D_\bullet))$; in addition, $\underline{\mathrm{Hom}}(F'_\bullet, D_\bullet)$ and $\underline{\mathrm{Hom}}(F''_\bullet, D_\bullet)$ are exact in degrees other than zero. Thus, since they are complexes of injective modules, they are injective resolutions of M′ and M″ respectively. Calculation of $\mathrm{Ext}^i(k, M')$ and $\mathrm{Ext}^i(k, M'')$ using these complexes shows that $\mu_i(M') = \mu_i(M)$ for $i < j$ and 0 for $i \geqq j$, and $\mu_i(M'') = 0$ for $i \leqq j$ and $\mu_i(M)$ for $i > j$. Hence M′ and M″ are non-zero modules of finite type with

$$\text{inj dim}(M') < j < \text{depth}(M''),$$

and, as mentioned above, this situation is known to be impossible. Hence we must have had $\mu_j(M) \neq 0$, and this proves the theorem.

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Paul Roberts,
McGill University,
Department of Mathematics,
P. O. Box 6070, Station A,
Montréal, Québec, Canada.