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Local Chern classes

Annales scientifiques de l’É.N.S. 4e série, tome 9, n° 1 (1976), p. 155-169

<http://www.numdam.org/item?id=ASENS_1976_4_9_1_155_0>
The purpose of this paper is to give a construction of local Chern classes as conjectured by Grothendieck [6] (XIV 7.2).

The construction is given in the framework of complex vector bundles on topological spaces where it appears as a generalization of the relative Chern classes obtained from the “difference construction” in K-theory notably used by Atiyah ([1]-[4]).

It will be clear that the constructions performed work equally well in other theories, especially the etale cohomology of algebraic geometry.

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1. Introduction

By a complex $K^*$ of vector bundles on a topological space $X$ we understand a finite complex of $C$-vector bundles each having constant rank. By the support of $K^*$ we understand the complement to the set of points $x \in X$ for which $K^*_x$ is an exact complex of vector spaces.

For a space $X$, $H^i(X; Z)$ denotes integral cohomology in the sense of sheaf theory, $\hat{H}^i(X; Z) = \prod H^i(X; Z)$. For a closed subset we use interchangeably

$$H^i_Z(X; Z) = H^i(X, X-Z; Z)$$

for cohomology with support in $Z$. 

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A theory of local Chern classes consists in assigning to a complex $K^\ast$ on $X$ with support in $Z$ a cohomology class

$$c^Z(K^\ast) \in \hat{H}_Z(X; \mathbb{Z})$$

with the following two properties

(1.1) For a continuous map $f: X \rightarrow Y$, closed subsets $Z \subseteq X$, $V \subseteq Y$ with $f(X-Z) \subseteq Y-V$ and a complex $L^\ast$ on $Y$ with support in $V$:

$$c^Z(f^\ast L^\ast) = f^\ast c^V(L^\ast).$$

(1.2) Let $r: \hat{H}_Z(X; Z) \rightarrow \hat{H}_Z(X; Z)$ denote the canonical map. Then

$$r(c^Z(K^\ast)) + 1 = \prod_i c_i(K^{2i}) c_i(K^{2i-1})^{-1}.$$

The main result of this paper is

THEOREM 1.3. — A theory of local Chern classes exists and is unique.

As usual we introduce a local Chern character

$$\text{ch}^Z(K^\ast) \in \hat{H}_Z(X; \mathbb{Q})$$

with the following properties:

(1.4) Functoriality. — $f^\ast \text{ch}^V(L^\ast) = \text{ch}^Z(f^\ast L^\ast)$.

(1.5) $r(\text{ch}^Z(K^\ast)) = \sum_i (-1)^i \text{ch}(K^i)$.

(1.6) Decalage. — $\text{ch}^Z(K^\ast[1]) = - \text{ch}^Z(K^\ast)$.

(1.7) Additivity. — For complexes $K^\ast$ and $L^\ast$ on $X$ with support in $Z$:

$$\text{ch}^Z(K^\ast \oplus L^\ast) = \text{ch}^Z(K^\ast) + \text{ch}^Z(L^\ast).$$

(1.8) Multiplicativity. — Let $K^\ast$ and $L^\ast$ be complexes on $X$ with support in $Z$ and $V$, respectively. Then

$$\text{ch}^Z \cap V(K^\ast \otimes L^\ast) = \text{ch}^Z(K^\ast) \text{ch}^V(L^\ast).$$

The proof of 1.3 is given in paragraphs 2 and 3 while paragraphs 4 and 5 derives multiplicative and additive properties of $c^Z$ and $\text{ch}^Z$.

In paragraph 6 we derive Riemann-Roch formulas for the Thom class and paragraph 7 initiates applications to algebraic geometry.
In cases where $X$ is an oriented topological manifold of dimension $n$, Poincaré duality
\[ H^*_Z(X; \mathbb{Z}) \iso H_{n-*}(Z; \mathbb{Z}) \]
transforms our local cohomology classes into homology classes. In cases where $X$ is a smooth algebraic variety/$\mathbb{C}$, this should be compared with the homology classes constructed by means of MacPherson’s graph construction [5] compare [10], [14], [16].

It should also be mentioned that Illusie ([13] V.6) has constructed local Chern classes “à la Atiyah” in Hodge cohomology.

I should like to thank K. Suominen for stimulating my interest in these matters.

2. The canonical complex

Throughout this paragraph we shall work with the following data.

A topological space $X$, a sequence of vector bundles $(K^i)_{i \in \mathbb{Z}}$ on $X$ with $K^i = 0$ except for finitely many $i \in \mathbb{Z}$.

\[ v_i = \text{rank } K^i. \]

We shall assume that there exists a sequence $(\lambda_i)_{i \in \mathbb{Z}}$ of integers with
\[ \lambda_i + \lambda_{i+1} = v_i, \quad i \in \mathbb{Z}, \]
\[ \lambda_i \geq 0, \quad i \in \mathbb{Z}. \]

Put $K = \bigoplus_{i \in \mathbb{Z}} K^i$. The flag manifold whose sections are flags in $K$ of nationality $v$ will be denoted $\text{Fl}_v$. The fixed flag defined by
\[ F_i = \bigoplus_{t \leq i} K^t \]
is denoted $F_\circ$.

**Definition 2.1.** — $T \subseteq \text{Fl}_v$ denote the closed subspace whose sections are flags $D$, with the property that
\[ F_{i-1} \subseteq D_i \subseteq F_{i+1}, \quad i \in \mathbb{Z}. \]

The canonical projection is denoted $p: T \to X$. The restriction to $T$ of the canonical flag on $\text{Fl}_v$ will be denoted $E_*$. On $T$ we have a canonical complex $C^*$ given by
\[ C^i = E_i/p^*F_{i-1}, \]
\[ \partial^i : E_i/p^*F_{i-1} \to E_{i+1}/p^*F_i. \]
is induced by the inclusion $E_i \subseteq E_{i+1}$. \( \partial^{i+1} \partial^i = 0 \) since
\[ p^*F_i \subseteq E_i \subseteq p^*F_{i+1}, \quad i \in \mathbb{Z}. \]

Finally $T_\varphi$ is the complement in $T$ of the support of $C$, and $p_\varphi : T_\varphi \to X$ denotes the restriction of $p$ to $T_\varphi$.

**Lemma 2.2.** — A section of $T$ over $X$ represented by a flag $D_i$ is a section of $T_\varphi$ if and only if for all $x \in X$:
\[ \text{rank} (D_{i,x} \cap F_{i,x}/F_{i-1,x}) = \lambda_i. \]

**Proof.** — By definition $D_i$ represents a section of $T_\varphi$ if and only if the complex
\[ \to D_{i-1}/F_{i-2} \to D_i/F_{i-1} \to D_{i+1}/F_i \to \]
has exact fibres for all $x \in X$. Note that $D_i/F_{i-1}$ has rank $\nu_i$, and the lemma follows from the definition of $(\lambda_i)_{i \in \mathbb{Z}}$.

**Theorem 2.3.** — Let $i_\varphi : T_\varphi \to T$ denote the inclusion. Then
\[ i_\varphi^* : H^*(T; \mathbb{Z}) \to H^*(T_\varphi; \mathbb{Z}) \]
is surjective.

**Proof.** — Define
\[ G_\lambda = \prod \text{Grass}_{\lambda_i} (K^i) \to X, \]
where $p_i : \text{Grass}_{\lambda_i} (K^i) \to X$ is the fibre space whose sections are rank $\lambda_i$-subbundles of $K^i$. \[ f_\lambda : T_\varphi \to G_\lambda \]
denotes the map which on the level of sections (compare 2.2) transforms
\[ D_i \mapsto (D_i \cap F_i/F_{i-1})_{i \in \mathbb{Z}}. \]

We shall first prove
\[ f_\lambda^* : H^*(G_\lambda; \mathbb{Z}) \to H^*(T_\varphi; \mathbb{Z}) \]
is an isomorphism.

We shall prove that $f_\lambda$ is a fibration with fibres of type $A^d(A^d : \text{affine space of dimension } d = \sum \lambda_i^2)$. For this assume $X = P^l$. The fibre of $f_\lambda$ above $B \in G_\lambda$ consists of sequences $(G_i)_{i \in \mathbb{Z}}$, where $G_i$ is a $\lambda_{i+1}$-plane in $2 \lambda_{i+1}$-space $B^{i+1}/B^i$ intersection the $\lambda_{i+1}$-plane $F_i/B^i$ in zero.
Next define

\[ G_v = \prod_i \text{Grass}_i(K^i \oplus K^{i+1}) \]

and maps

\[
\begin{align*}
    f_v : & T \to G_v, & D_i \mapsto (D_i/F_{i-1})_{i \in \mathbb{Z}}; \\
    g : & G_h \to G_v, & B_i \mapsto (B^i \oplus B^{i+1})_{i \in \mathbb{Z}}; \\
    s_h : & G_h \to T_{v}; & B_i \mapsto (\oplus K^i \oplus B^i \oplus B^{i+1})_{i < 1},
\end{align*}
\]

where in each case the transformation on the level of sections is given.

We have the following diagram

\[
\begin{array}{ccc}
T & \xleftarrow{f_v} & T_v \\
\downarrow{f_h} & & \downarrow{g} \\
G_v & \xleftarrow{s_h} & G_h
\end{array}
\]

with

\[ f_v s_h = g, \quad f_h s_h = 1 \]

\((f_v s_h \neq g f_h)\).

Let us grant (2.5 below) that \(g^*\) is surjective.

\[ s_h f_h^* = 1 \quad \text{and whence by 2.4;} \]
\[ f_h s_h^* = 1, \quad \text{on the other hand;} \]
\[ s_h^* f_h^* = g^* \quad \text{and whence;} \]
\[ i^*_h f_v^* = f_h^* g^*. \quad \text{Thus } i^*_h \text{ surjective.} \]

Q. E. D.

**Lemma 2.5.** — *The map*

\[
g : \prod_i \text{Grass}_i K^i \to \prod_i \text{Grass}_i K^i \oplus K^{i+1},
\]

\[ B^i \mapsto (B^i \oplus B^{i+1})_{i \in \mathbb{Z}} \]

*induces a surjective map* \(g^*\) *on integral cohomology.*

**Proof.** — Let \(P_i\) denote the canonical \(\lambda_i\)-bundle on \(\text{Grass}_i(K^i)\). Consider

\[ H^*(\prod_i \text{Grass}_i K^i; \mathbb{Z}) \]

as a \(H^* (X; \mathbb{Z})\)-algebra. As is well known this algebra is generated by the homogeneous components of

\[ \text{pr}_{i*} c.(P_i), \quad i \in \mathbb{Z}. \]
Consider the composite of $g$ and the $i$'th projection
$$\prod_i \text{Grass}_i K^i \rightarrow \text{Grass}_i K^i \oplus K^{i+1}$$
to see that
$$\text{pr}_i^* c.(P_i) \text{pr}_{i+1}^* c.(P^{i+1})$$
and the inverse to that element belongs to the image of $g^*$. It is now clear by decreasing induction that $\text{pr}_i^* c.(P_i)$ and $\text{pr}_{i+1}^* c.(P_{i+1})^{-1}$ belong to the image of $g^*$.

Q. E. D.

**Proposition 2.6.** — The $H^*(X; \mathbb{Z})$-module $H^*(T_X; \mathbb{Z})$ is finitely generated free and for any map $X' \rightarrow X$.

$$H^*(T_X; \mathbb{Z}) \otimes_{H^*(X; \mathbb{Z})} H^*(X'; \mathbb{Z}) \rightarrow H^*(T_X \times X'; \mathbb{Z})$$
is an isomorphism.

**Proof.** — By 2.4 we may replace $T_X$ by a product of Grassmannian bundles for which this is well known.

Q. E. D.

3. Construction of the local Chern class

With the notation of paragraph 2 let $(\partial^i)_{i \in \mathbb{Z}}$ be a family of linear maps $\partial^i: K^i \rightarrow K^{i+1}$ with $\partial^{i+1} \partial^i = 0$, $i \in \mathbb{Z}$. Define a flag $s^*(\partial^i)$ in $K = \bigoplus_i K^i$ as follows: $s^*(\partial^i)$ is the graph of the map

$$\bigoplus_{i \leq i} K^i \rightarrow \bigoplus_{i \geq i} K^i,$$

$$(\ldots, k_{i-1}, k_i) \mapsto (\partial^i k_i, 0, \ldots).$$

Clearly,

$$F_{i-1} \subseteq s^*(\partial^i) \subseteq F_{i+1}, \quad i \in \mathbb{Z}.$$

Thus we may interpret $s^*(\partial^i)$ as a section of $p: T \rightarrow X$

$$s^*(\partial^i): X \rightarrow T.$$

Clearly

$$(3.1) \quad s^*(\partial^i)^* C' = (K^i, \partial^i).$$
Let now \( Z \subseteq X \) denote a closed subset such that \( \text{Supp} (K', \delta') \subseteq Z \) then
\[
\sigma_\gamma (X-Z) \subseteq T_\gamma.
\]

Consider the exact sequence, (2.3):
\[
0 \to \hat{H}^1 (T, T_\gamma; Z) \xrightarrow{i_\gamma} \hat{H}^1 (T; Z) \xrightarrow{\delta'} \hat{H}^1 (T_\gamma; Z) \to 0.
\]

The image by \( i_\gamma^* \) of the cohomology class
\[
c_\gamma (C) - 1 = \prod_i c_2 (C^{2i}) c_2 (C^{2i-1}) - 1
\]
is zero since \( C \) is exact on \( T_\gamma \). Let
\[
\gamma_T \in \hat{H}^1 (T, T_\gamma; Z)
\]
denote the cohomology class characterized by
\[
(3.2) \quad r_\gamma (\gamma_T) + 1 = c_\gamma (C).
\]

**Definition 3.3.** – Consider the map induced by \( s_\gamma (\delta') \)
\[
s_\gamma (\delta')^* : H^1 (T, T_\gamma; Z) \to H^1 (T; Z)
\]
and define the local Chern class of \((K', \delta')\) supported in \( Z \) by
\[
c^z_\gamma (K', \delta') = s_\gamma (\delta')^* \gamma_T.
\]

**Proof of 1.3.** – Follows from 3.1 and 3.2.

As above we consider the cohomology class
\[
\gamma_{\chi_T} \in \hat{H}^1 (T, T_\gamma; Q)
\]
characterized by
\[
(3.4) \quad r_\gamma (\gamma_{\chi_T}) = \sum_i (-1)^i \text{ch} (C^i).
\]

**Definition 3.5:**
\[
\text{ch}^z (K', \delta') = s_\gamma (\delta')^* \gamma_{\chi_T}.
\]

The local Chern character thus defined satisfies clearly 1.4-6. Let us remark that \( \text{ch}^z \) can be derived directly from \( c^z_\gamma \) by means of the theory of \( \lambda \)-rings, compare paragraph 5.
4. Properties of the local Chern character

In this paragraph we shall prove the multiplication property 1.8 of \( \text{ch}^2 \). The proof of the additive property 1.7 is similar but simpler and will not be given. Finally, we give some variants of the additive property.

**Proof of 1.8.** — Let us first note that 1.8 is true if the canonical map

\[ H^*_Z(n) \rightarrow H^*_Z(n) \]

is injective. We are going to reduce the problem to this case. Let \( T = T(K') \) and \( S = T(L') \) with a slight abuse of notation. It will now suffice to prove that

\[ H^*(T \times S; Z) \rightarrow H^*(S \times T_{\varphi} \cup T \times S_{\varphi}; Z) \]

is surjective. Here and in the following all products are formed in the category of spaces/\( X \). \( H^*(\_\_\_) \) denotes integral cohomology. Let us first recall that if \( Z \subseteq Y \) is a closed subset of the space \( Y \) and if \( U \subseteq Y \) is an open subset, then there is a canonical exact sequence

\[ H^u(U) \rightarrow H^u(U) \rightarrow H^u(X) \rightarrow H^u(Y) \rightarrow H^u(Y) \rightarrow \cdots \]

Put \( X = S - S_{\varphi} \) and \( Y = T - T_{\varphi} \). It follows from 2.6 that the following commutative diagram is exact \([\otimes \text{ is formed in the category of } H^*(X)\text{-modules}]\):

\[
\begin{array}{cccc}
0 & \rightarrow & H^*_X(S \times T) & \rightarrow H^*(S \times T) \rightarrow H^*(S \times T) \\
& & \downarrow & \downarrow \\
& & H^*(S_{\varphi}) \otimes H^*_Y(T) & \rightarrow H^*_Y(T) \\
0 & \rightarrow & H^*_X T \rightarrow H^*(S \times T) \rightarrow & \rightarrow H^*(S \times T_{\varphi}) \\
& & \downarrow & \downarrow \\
& & 0 & \rightarrow 0
\end{array}
\]

From this follows that

\[ H^*_X(S \times T) \rightarrow H^*_X T_{\varphi}(S \times T_{\varphi}) \]

is surjective by remarking that \( H^*(S) \otimes H^*_Y(T) \rightarrow H^*(S_{\varphi}) \otimes H^*_Y(T) \) is surjective, taking into account the map from \( H^*(S) \otimes H^*_Y(T) \) into the kernel of \( H^*(S \times T) \rightarrow H^*(S \times T_{\varphi}) \). Next, apply the above long exact sequence to \((S \times T, S \times T_{\varphi}, X \times T)\) to get the exact sequence

\[ \rightarrow H^*_X Y(S \times T) \rightarrow H^*_X T(S \times T) \rightarrow H^*_X T_{\varphi}(S \times T_{\varphi}) \rightarrow \]

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from which we conclude that

\[ H_{X \times Y}(S \times T) \to H_{X \times T}(S \times T) \]

is injective. From the following exact sequence and 2.6

\[ \to H_{X \times T}^*(S \times T) \to H^i(S \times T) \to H^i(S_x \times T) \to \]

follows that

\[ H_{X \times T}^*(S \times T) \to H^i(S \times T) \]

is injective. Compose the last two results and write still another long exact sequence to derive the result.

Q. E. D.

**Proposition 4.1.** — Let \( K'' \) denote a finite double complex on the topological space \( X \). Suppose \( Z \) is a closed subset of \( X \) such that \( K'^p \), has support in \( Z \) for all \( p \in \mathbb{Z} \). Then

\[
\text{ch}^2(\text{tot } K'') = \sum (-1)^i \text{ch}^2(K'^i),
\]

where \( \text{tot } K'' \) denotes the total single complex associated to \( K'' \).

**Proof.** — We shall first change notation and let \( K'' \) denote the double indexed family of vector bundles on \( X \) underlying the above double complex. Let \( C(K'') \) denote the fibre space over \( X \) whose sections are pairs \( (\partial', \partial'') \) of endomorphisms of \( K'' \) such that \( (K'', \partial', \partial'') \) form a double complex. Let \( E'' \) denote the canonical double complex on \( C(K'') \) and \( C_Y \) the complement of the support of \( \text{tot } E'' \).

Consider now a fixed pair \( (\partial', \partial'') \) as above and assume that \( (K'', 0, \partial') \) has support in \( Z \). Consider the map of spaces/\( X \):

\[ \theta : X \times \mathbb{A}^1 \to C(K'') \]

which on the section level is given by

\[ t \mapsto (K'', t \partial', \partial''). \]

Clearly

\[ \theta(X - Z) \subseteq C_Y \]

and

\[ \theta_*^*(\text{tot } E'') = \text{tot } (K'', t \partial', \partial''). \]

Conclusion by (1.6), (1.7) and a simple homotopy argument.

Q. E. D.
**Corollary 4.2.** — Consider an exact sequence of complexes of vector bundles on $X$:

$$0 \to K' \to L' \to M' \to 0$$

and suppose all three complexes have support in the closed subset $Z$ of $X$. Then

$$\text{ch}^Z(L') = \text{ch}^Z(K') + \text{ch}^Z(M').$$

**Proof.** — Consider an appropriate double complex and apply 4.1 twice.

Q. E. D.

**Corollary 4.3.** — Let $f: K' \to L'$ be a linear map of complexes on $X$ and let $K'$ and $L'$ have support in $Z$. If for all $x \in X$:

$$H'(f_x): H'(K'_x) \to H'(L'_x)$$

is an isomorphism, then

$$\text{ch}^Z(K') = \text{ch}^Z(L').$$

**Proof.** — Construct the mapping cone and apply 4.2.

Q. E. D.

5. **Formulas without denominators**

Let $Z$ be a closed subspace of the space $X$ and consider the commutative graded ring with 1:

$$Z \oplus H^e_Z(X; Z^+).$$

To this we associate

$$1 + \hat{H}^e_Z(X; Z)^+ = 1 + \prod_{i \geq 1} H^e_{2i}Z(X; Z)$$

which is an abelian group under cup product. Recall that $1 + \hat{H}^e_Z(X; Z)^+$ comes equipped with a product $\star$ with the property

$$\text{(1 + x}_m + \text{higher terms)} \star (1 + y_n + \text{higher terms}) =$$

$$1 - \frac{(n+m-1)!}{(m-1)!(n-1)!} x_m y_n + \text{higher terms}$$

(5.1)

[6] (0, App. § 3).
If $K'$ is a complex on $X$ with support in $Z$, we put

$$
\tilde{c}^Z(K') = 1 + c^Z(K'),
$$

$$
\tilde{c}^Z(K') \in 1 + \hat{H}^*(X; Z)^+.
$$

With the notation of the corresponding formulas for $c^Z$, 1.4-8, we have

(5.2) \hspace{1cm} \tilde{c}^Z(f^*L') = f^*\tilde{c}^Z(L'),

(5.3) \hspace{1cm} r(\tilde{c}^Z(K')) = \prod_i c_i(K^{2i}) c_i(K^{2i-1})^{-1},

(5.4) \hspace{1cm} \tilde{c}^Z(K'[1]) = \tilde{c}^Z(K')^{-1},

(5.5) \hspace{1cm} \tilde{c}^Z(K' \oplus L') = \tilde{c}^Z(K') \tilde{c}^Z(L'),

(5.6) \hspace{1cm} \tilde{c}^{Z \cap V}(K' \otimes L') = \tilde{c}^Z(K') \ast \tilde{c}^Z(L').

These formulas are easily derived by the method developed in paragraph 4. From loc. cit. follows

(5.7) Suppose $c^Z(K') = a_n + \text{higher terms}$, then $\text{ch}^Z(K') = 1/(-1)^{n-1} (n-1)! a_n + \text{higher terms}$

6. Riemann-Roch formula for the Thom class

Let $\pi: E \to X$ denote a rank $n$ vector bundle, and let $\lambda_E$ denote the canonical complex on $E$. Recall that $(\lambda_E)^i = \Lambda^i \pi^*E$. The Koszul complex, i.e. the complex dual to $\lambda_E$ will be denoted $\lambda_{\tilde{E}}$.

**Proposition 6.1.** With the above notation

$$
(-1)^n \text{Todd}(E^\vee) \text{ch}^Z(\lambda_E) = \text{Todd}(E) \text{ch}^Z(\lambda_{\tilde{E}}) = \text{Thom class of } E.
$$

**Proof.** Let $\tilde{E} = \text{Proj}(E \oplus 1)$ and let $H$ denote the canonical line bundle on $\tilde{E}$. From the canonical imbedding ([1], p. 100):

$$
H^\vee \subseteq E \oplus 1
$$

we derive the canonical section

$$
s \in \Gamma(\tilde{E}, E \otimes H \oplus H).
$$

The projection of $s$ onto $E \otimes H$ will be denoted

$$
t \in \Gamma(\tilde{E}, E \otimes H).
$$
The zero's of \( t \) all lie on the canonical section \( X \to \tilde{E} \). Consider the commutative diagram

\[
\begin{array}{ccc}
\hat{\tau} \in \hat{H}_X(E; \mathbb{Q}) & \rightarrow & \hat{H}^*(E; \mathbb{Q}) \\
\downarrow & & \downarrow \\
\tau \in H_X(E; \mathbb{Q}) & \rightarrow & H^*(E; \mathbb{Q})
\end{array}
\]

where \( \tau \) denotes the Thom class. Let us first prove that

\[
\tilde{\tau}(\hat{\tau}) = c_n(E \otimes H).
\]

For this let us note that \( \tilde{\tau} \) is injective. Namely, \( H^*(\tilde{E}; \mathbb{Q}) \to H^*(\tilde{E}-X; \mathbb{Q}) \) is surjective since the restriction to \( \tilde{E}-X \) of

\[
1, c_1(H), \ldots, c_1(H)^{n-1}
\]

form a basis for the \( H^*(X; \mathbb{Q}) \)-module \( H^*(\tilde{E}-X; \mathbb{Q}) \). Note that the restriction of \( c_n(E \otimes H) \) to \( \tilde{E}-X \) is zero because of the section \( t \). Let \( \sigma \in H_X(\tilde{E}) \) be such that \( \tilde{\tau}(\sigma) = c_n(E \otimes H) \).

We shall show that \( \sigma \) is the Thom class. For this it suffices to treat the case \( X = P^r \). In this case \( c_n(E \otimes H) = c_1(H)^n \) and the statement is clear.

We shall now prove the first formula. Let \( \lambda^\sim \) denote the Koszul complex associated with the section \( t \) of \( E \otimes H \). The restriction of \( \lambda^\sim \) to \( E \) is \( \lambda_{\tilde{E}} \). Let us recall [8], Lemma 18 that for a rank \( n \) bundle \( N \) we have

\[
(6.2) \quad ch(\lambda_{-1} N^\sim) = c_n(N) Todd(N)^{-1}.
\]

The formula will now follow by applying (1.5) to \( \lambda^\sim \)

\[
\tilde{\tau}(ch^X \lambda^\sim) = ch(\lambda_{-1} \tilde{E} \otimes H) = c_n(E \otimes H) Todd(E \otimes H)^{-1},
\]

\[
Todd(E \otimes H)^{-1} \equiv Todd(E)^{-1} \mod c_1(H),
\]

\[
c_n(E \otimes H) c_1(H) = 0
\]

as it follows from the fact that \( t \in \Gamma(\tilde{E}, E \otimes H \otimes H) \) has no zeros. Whence

\[
\tilde{\tau}(ch^X \lambda^\sim) = c_n(E \otimes H) Todd(E)^{-1}.
\]

Q. E. D.

Remark. — The above formula should be considered as generalizations of formulas used in [2], [3], [4].
7. Multiplicity in algebraic geometry

In this paragraph we shall work in the framework of [7] and prove a fundamental relation 7.1 between local Chern classes and the multiplicity of local algebra [15], compare [2], 6.2.

Let $V$ denote a smooth (connected) algebraic variety $\mathbb{C}$ and $X \subseteq V$ a closed subvariety of codimension $d$. The local fundamental class will be denoted

$$c^X \in H^{2d}_X(V; \mathbb{Z}).$$

The fundamental class of $X$, i.e. the image of $c^X$ in $H^{2d}(V; \mathbb{Z})$ will be denoted

$$\text{cl}(X) \in H^{2d}(V; \mathbb{Z}).$$

For a coherent (algebraic) sheaf $M$ on $V$ with support in $X$, $l(M)$ denotes the length of the stalk of $M$ at the generic point of $X$.

**Theorem 7.1.** — Let $E'$ denote a complex of locally free coherent (algebraic) sheaves on $V$ with $\text{Supp}(E') \subseteq X$. Then

$$\text{ch}^X(E') = \sum (-1)^i l(H^i E') c^X + \text{higher terms}.$$ 

**Proof.** — Let $O$ denote the local ring of $V$ at the generic point of $X$, $m$ denotes the maximal ideal of $O$. Let $K_m(O)$ denote the Grothendieck group of the category of finite complexes of finitely generated free $O$-modules with homology of finite length (modulo exact complexes). We are going to define a topological character

$$l : K_m(O) \to \mathbb{Z}.$$ 

Recall first that if $U$ is a Zariski open subset of $V$ with $X \cap U \neq \emptyset$, then the restriction map

$$H^{2d}_X(V; \mathbb{Z}) \to H^{2d}_{X \cap U}(U; \mathbb{Z})$$

is an isomorphism which carries $c^X$ to $c^{X \cap U}$. From this follows that there is a character $l$ as above such that for any complex $E'$ as in the theorem

$$\text{ch}^X(E') = l(E') c^X + \text{higher terms}.$$ 

As is well known $K_m(O) \cong \mathbb{Z}$ since $O$ is a regular local ring [6]. Thus it will suffice to find a resolution $E'$ of $O/m$ by finitely generated free sheaves with $l(E') = 1$. Let us first consider the case $V = \mathbb{A}^d$, $X = \{0\}$. In this case we can take for $E'$ the standard Koszul complex. That $l(E') = 1$ follows from 6.1.
In the general case choose a Zariski open set $U$ of $V$ and $f_1, \ldots, f_d \in \Gamma(U, \mathcal{O}_V)$ which defines $X \cap U$. This defines a map

$$f : U \rightarrow \mathbb{A}^d$$

with $f^{-1}(\{0\}) = U \cap X$. It follows that

$$f^* : H^{2d}_{\{0\}}(\mathbb{A}^d; \mathbb{Z}) \rightarrow H^{2d}_{X \cap U}(U; \mathbb{Z})$$

is an isomorphism. The pull-back of the complex considered before will now do the job.

Q.E.D.

Remark 7.2. — Taking in particular a resolution of the structure sheaf $\mathcal{O}_X$ of $X$ we obtain by means of (5.7):

$$c_d(\mathcal{O}_X) = (-1)^{d-1}(d-1)!\chi(X)$$


Remark 7.3. Combining 7.1 and 1.8 we obtain Serre’s “alternating Tor-formula” [15] for the topological intersection number.

REFERENCES


(Manuscrit reçu le 27 octobre 1975.)

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