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LOCAL CHERN CLASSES

BY BIRGER IVERSEN

The purpose of this paper is to give a construction of local Chern classes as conjectured by Grothendieck [6] (XIV 7.2).

The construction is given in the framework of complex vector bundles on topological spaces where it appears as a generalization of the relative Chern classes obtained from the “difference construction” in K-theory notably used by Atiyah ([1]-[4]).

It will be clear that the constructions performed work equally well in other theories, especially the étale cohomology of algebraic geometry.

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1. Introduction

By a complex K' of vector bundles on a topological space X we understand a finite complex of C -vector bundles each having constant rank. By the support of K' we understand the complement to the set of points $x \in X$ for which K'_x is an exact complex of vector spaces.

For a space X , $H^i(X; Z)$ denotes integral cohomology in the sense of sheaf theory, $\hat{H}^i(X; Z) = \prod_i H^i(X; Z)$. For a closed subset we use interchangeably

$$H_Z^i(X; Z) = H^i(X, X - Z; Z)$$

for cohomology with support in Z .

A theory of local Chern classes consists in assigning to a complex K^\bullet on X with support in Z a cohomology class

$$c^Z(K^\bullet) \in \hat{H}_Z^*(X; \mathbf{Z})$$

with the following two properties

(1.1) For a continuous map $f: X \rightarrow Y$, closed subsets $Z \subseteq X$, $V \subseteq Y$ with $f(X-Z) \subseteq Y-V$ and a complex L^\bullet on Y with support in V :

$$c^Z(f^*L^\bullet) = f^*c^V(L^\bullet).$$

(1.2) Let $r: \hat{H}_Z^*(X; \mathbf{Z}) \rightarrow \hat{H}^*(X; \mathbf{Z})$ denote the canonical map.

Then

$$r(c^Z(K^\bullet)) + 1 = \prod_i c_i(K^{2i}) c_i(K^{2i-1})^{-1}.$$

The main result of this paper is

THEOREM 1.3. — *A theory of local Chern classes exists and is unique.*

As usual we introduce a local Chern character

$$\text{ch}^Z(K^\bullet) \in \hat{H}_Z^*(X; \mathbf{Q})$$

with the following properties:

(1.4) **FUNCTORIALITY.** — $f^* \text{ch}^V(L^\bullet) = \text{ch}^Z(f^*L^\bullet)$.

(1.5) $r(\text{ch}^Z(K^\bullet)) = \sum_i (-1)^i \text{ch}(K^i)$.

(1.6) **DECALAGE.** — $\text{ch}^Z(K^\bullet[1]) = -\text{ch}^Z(K^\bullet)$.

(1.7) **ADDITIVITY.** — For complexes K^\bullet and L^\bullet on X with support in Z :

$$\text{ch}^Z(K^\bullet \oplus L^\bullet) = \text{ch}^Z(K^\bullet) + \text{ch}^Z(L^\bullet).$$

(1.8) **MULTIPLICATIVITY.** — Let K^\bullet and L^\bullet be complexes on X with support in Z and V , respectively. Then

$$\text{ch}^{Z \cap V}(K^\bullet \otimes L^\bullet) = \text{ch}^Z(K^\bullet) \text{ch}^V(L^\bullet).$$

The proof of 1.3 is given in paragraphs 2 and 3 while paragraphs 4 and 5 derives multiplicative and additive properties of c^Z and ch^Z .

In paragraph 6 we derive Riemann-Roch formulas for the Thom class and paragraph 7 initiates applications to algebraic geometry.

In cases where X is an oriented topological manifold of dimension n , Poincaré duality

$$H_{\mathbf{Z}}^i(X; \mathbf{Z}) \xrightarrow{\sim} H_{n-i}(X; \mathbf{Z})$$

transforms our local cohomology classes into homology classes. In cases where X is a smooth algebraic variety/ \mathbf{C} , this should be compared with the homology classes constructed by means of MacPherson's graph construction [5] compare [10], [14], [16].

It should also be mentioned that Illusie ([13] V.6) has constructed local Chern classes "à la Atiyah" in Hodge cohomology.

I should like to thank K. Suominen for stimulating my interest in these matters.

2. The canonical complex

Throughout this paragraph we shall work with the following data.

A topological space X , a sequence of vector bundles $(K^i)_{i \in \mathbf{Z}}$ on X with $K^i = 0$ except for finitely many $i \in \mathbf{Z}$.

$$v_i = \text{rank } K^i.$$

We shall assume that there exists a sequence $(\lambda_i)_{i \in \mathbf{Z}}$ of integers with

$$\begin{aligned} \lambda_i + \lambda_{i+1} &= v_i, & i \in \mathbf{Z}, \\ \lambda_i &\geq 0, & i \in \mathbf{Z}. \end{aligned}$$

Put $K = \bigoplus_{i \in \mathbf{Z}} K^i$. The flag manifold whose sections are flags in K of nationality v . will be denoted Fl_v . The fixed flag defined by

$$F_i = \bigoplus_{t \leq i} K^t$$

is denoted F .

DEFINITION 2.1. — $T \subseteq \text{Fl}_v$ denote the closed subspace whose sections are flags D , with the property that

$$F_{i-1} \subseteq D_i \subseteq F_{i+1}, \quad i \in \mathbf{Z}.$$

The canonical projection is denoted $p: T \rightarrow X$. The restriction to T of the canonical flag on Fl_v will be denoted E . On T we have a canonical complex C given by

$$\begin{aligned} C^i &= E_i/p^* F_{i-1}, \\ \partial^i : E_i/p^* F_{i-1} &\rightarrow E_{i+1}/p^* F_i \end{aligned}$$

is induced by the inclusion $E_i \subseteq E_{i+1}$. $\partial^{i+1} \partial^i = 0$ since

$$p^* F_{i-1} \subseteq E_i \subseteq p^* F_{i+1}, \quad i \in \mathbb{Z}.$$

Finally T_Ψ is the complement in T of the support of C' , and $p_\Psi: T_\Psi \rightarrow X$ denotes the restriction of p to T_Ψ .

LEMMA 2.2. — *A section of T over X represented by a flag D , is a section of T_Ψ if and only if for all $x \in X$:*

$$\text{rank}(D_{i,x} \cap F_{i,x}/F_{i-1,x}) = \lambda_i.$$

Proof. — By definition D represents a section of T_Ψ if and only if the complex

$$\rightarrow D_{i-1}/F_{i-2} \rightarrow D_i/F_{i-1} \rightarrow D_{i+1}/F_i \rightarrow$$

has exact fibres for all $x \in X$. Note that D_i/F_{i-1} has rank ν_i , and the lemma follows from the definition of $(\lambda_i)_{i \in \mathbb{Z}}$.

THEOREM 2.3. — *Let $i_\Psi: T_\Psi \rightarrow T$ denote the inclusion. Then*

$$i_\Psi^*: H^*(T; \mathbb{Z}) \rightarrow H^*(T_\Psi; \mathbb{Z})$$

is surjective.

Proof. — Define

$$G_\lambda = \prod_i \text{Grass}_{\lambda_i}(K^i) \rightarrow X,$$

where $p_i: \text{Grass}_{\lambda_i}(K^i) \rightarrow X$ is the fibre space whose sections are rank λ_i -subbundles of K^i .

$$f_\lambda: T_\Psi \rightarrow G_\lambda$$

denotes the map which on the level of sections (compare 2.2) transforms

$$D \mapsto (D_i \cap F_i/F_{i-1})_{i \in \mathbb{Z}}.$$

We shall first prove

$$(2.4) \quad f_\lambda^*: H^*(G_\lambda; \mathbb{Z}) \rightarrow H^*(T_\Psi; \mathbb{Z})$$

is an isomorphism.

We shall prove that f_λ is a fibration with fibres of type \mathbb{A}^d (\mathbb{A}^d : affine space of dimension $d = \sum \lambda_i^2$). For this assume $X = \mathbb{P}^t$. The fibre of f_λ above $B' \in G_\lambda$ consists of sequences $(G^i)_{i \in \mathbb{Z}}$, where G^i is a λ_{i+1} -plane in $2\lambda_{i+1}$ -space B^{i+1}/B^i intersection the λ_{i+1} -plane F_i/B^i in zero.

Next define

$$G_v = \prod_i \text{Grass}_{v_i}(K^i \oplus K^{i+1})$$

and maps

$$\begin{aligned} f_v : T &\rightarrow G_v, & D_i &\mapsto (D_i/F_{i-1})_{i \in \mathbf{Z}}; \\ g : G_\lambda &\rightarrow G_v, & B_i &\mapsto (B^i \oplus B^{i+1})_{i \in \mathbf{Z}}; \\ s_\lambda : G_\lambda &\rightarrow T_\Psi; \\ B' &\mapsto (\bigoplus_{t < i} K^t \oplus B^i \oplus B^{i+1})_{i \in \mathbf{Z}}, \end{aligned}$$

where in each case the transformation on the level of sections is given.

We have the following diagram

$$\begin{array}{ccc} T & \xleftarrow{i_\Psi} & T_\Psi \\ f_v \downarrow & & \downarrow f_\lambda \\ G_v & \xleftarrow{g} & G_\lambda \end{array} \quad \begin{array}{c} \uparrow s_\lambda \\ \downarrow \end{array}$$

with

$$f_v i_\Psi s_\lambda = g, \quad f_\lambda s_\lambda = 1$$

($f_v i_\Psi \neq g f_\lambda$).

Let us grant (2.5 below) that g^* is surjective.

$s_\lambda^* f_\lambda^* = 1$ and whence by 2.4;

$f_\lambda^* s_\lambda^* = 1$, on the other hand;

$s_\lambda^* i_\Psi^* f_v^* = g^*$ and whence;

$i_\Psi^* f_v^* = f_\lambda^* g^*$. Thus i_Ψ^* surjective.

Q. E. D.

LEMMA 2.5. — *The map*

$$\begin{aligned} g : \prod_i \text{Grass}_{\lambda_i} K^i &\rightarrow \prod_i \text{Grass}_{v_i} K^i \oplus K^{i+1}, \\ B' &\mapsto (B^i \oplus B^{i+1})_{i \in \mathbf{Z}} \end{aligned}$$

induces a surjective map g^* on integral cohomology.

Proof. — Let P^i denote the canonical λ_i -bundle on $\text{Grass}_{\lambda_i}(K^i)$. Consider

$$H^*(\prod_i \text{Grass}_{\lambda_i} K^i; \mathbf{Z})$$

as a $H^*(X; \mathbf{Z})$ -algebra. As is well known this algebra is generated by the homogeneous components of

$$\text{pr}_i^* c_i(P^i), \quad i \in \mathbf{Z}.$$

Consider the composite of g and the i 'th projection

$$\prod_i \text{Grass}_{\lambda_i} K^i \rightarrow \text{Grass}_{\nu_i} K^i \oplus K^{i+1}$$

to see that

$$\text{pr}_i^* c.(P^i) \text{pr}_{i+1}^* c.(P^{i+1})$$

and the inverse to that element belongs to the image of g^* . It is now clear by decreasing induction that $\text{pr}_i^* c.(P_i)$ and $\text{pr}_i^* c.(P_i)^{-1}$ belong to the image of g^* .

Q. E. D.

PROPOSITION 2.6. — *The $H^*(X; \mathbf{Z})$ -module $H^*(T_\Psi; \mathbf{Z})$ is finitely generated free and for any map $X' \rightarrow X$.*

$$H^*(T_\Psi; \mathbf{Z}) \otimes_{H^*(X; \mathbf{Z})} H^*(X'; \mathbf{Z}) \rightarrow H^*(T_\Psi \times_X X'; \mathbf{Z})$$

is an isomorphism.

Proof. — By 2.4 we may replace T_Ψ by a product of Grassmannian bundles for which this is well known.

Q. E. D.

3. Construction of the local Chern class

With the notation of paragraph 2 let $(\partial^i)_{i \in \mathbf{Z}}$ be a family of linear maps $\partial^i: K^i \rightarrow K^{i+1}$ with $\partial^{i+1} \partial^i = 0$, $i \in \mathbf{Z}$. Define a flag $s.(\partial^i)$ in $K = \bigoplus_i K^i$ as follows: $s.(\partial^i)$ is the graph of the map

$$\begin{aligned} \bigoplus_{t \leq i} K^t &\rightarrow \bigoplus_{t > i} K^t, \\ (\dots, k_{i-1}, k_i) &\mapsto (\partial^i k_i, 0, \dots). \end{aligned}$$

Clearly,

$$F_{i-1} \subseteq s.(\partial^i) \subseteq F_{i+1}, \quad i \in \mathbf{Z}.$$

Thus we may interpret $s.(\partial^i)$ as a section of $p: T \rightarrow X$

$$s.(\partial^i) : X \rightarrow T.$$

Clearly

$$(3.1) \quad s.(\partial^i)^* C^* = (K^i, \partial^i).$$

Let now $Z \subseteq X$ denote a closed subset such that $\text{Supp}(K^*, \partial^*) \subseteq Z$ then

$$s.(\partial^*)(X-Z) \subseteq T_{\Psi^*}.$$

Consider the exact sequence, (2.3):

$$0 \rightarrow \hat{H}^*(T, T_{\Psi}; Z) \xrightarrow{r_{\Psi}} \hat{H}^*(T; Z) \xrightarrow{i_{\Psi}^*} \hat{H}^*(T_{\Psi}; Z) \rightarrow 0.$$

The image by i_{Ψ}^* of the cohomology class

$$c.(C^*) - 1 = \prod_i c.(C^{2i}) c.(C^{2i-1})^{-1} - 1$$

is zero since C^* is exact on T_{Ψ} . Let

$$\gamma_T \in \hat{H}^*(T, T_{\Psi}; Z)$$

denote the cohomology class characterized by

$$(3.2) \quad r_{\Psi}(\gamma_T) + 1 = c.(C^*).$$

DEFINITION 3.3. — Consider the map induced by $s.(\partial^*)$

$$s.(\partial^*)^* : H^*(T, T_{\Psi}; Z) \rightarrow H_Z^*(X; Z)$$

and define the local Chern class of (K^*, ∂^*) supported in Z by

$$c^Z(K^*, \partial^*) = s.(\partial^*)^* \gamma_T.$$

Proof of 1.3. — Follows from 3.1 and 3.2.

Q. E. D.

As above we consider the cohomology class

$$\gamma_{\mathcal{K}_T} \in \hat{H}^*(T, T_{\Psi}; \mathbf{Q})$$

characterized by

$$(3.4) \quad r_{\Psi}(\gamma_{\mathcal{K}_T}) = \sum_i (-1)^i \text{ch}(C^i).$$

DEFINITION 3.5:

$$\text{ch}^Z(K^*, \partial^*) = s.(\partial^*)^* \gamma_{\mathcal{K}_T}.$$

The local Chern character thus defined satisfies clearly 1.4-6. Let us remark that ch^Z can be derived directly from c^Z by means of the theory of λ -rings, compare paragraph 5.

4. Properties of the local Chern character

In this paragraph we shall prove the multiplication property 1.8 of ch^Z . The proof of the additive property 1.7 is similar but simpler and will not be given. Finally, we give some variants of the additive property.

Proof of 1.8. — Let us first note that 1.8 is true if the canonical map

$$H^*(Z \cap V; \mathbf{Z}) \rightarrow H^*(X; \mathbf{Z})$$

is injective. We are going to reduce the problem to this case. Let $T = T(K')$ and $S = T(L')$ with a slight abuse of notation. It will now suffice to prove that

$$H^*(T \times S; \mathbf{Z}) \rightarrow H^*(S \times T_\Psi \cup T \times S_\Psi; \mathbf{Z})$$

is surjective. Here and in the following all products are formed in the category of spaces/ X . $H^*(-)$ denotes integral cohomology. Let us first recall that if $Z \subseteq Y$ is a closed subset of the space Y and if $U \subseteq Y$ is an open subset, then there is a canonical exact sequence

$$\rightarrow H_{Z-U}^i(X) \rightarrow H_Z^i(X) \rightarrow H_{Z \cap U}^i(U) \rightarrow H_{Z-U}^{i+1}(X) \rightarrow.$$

Put $X = S - S_\Psi$ and $Y = T - T_\Psi$. It follows from 2.6 that the following commutative diagram is exact [\otimes is formed in the category of $H(X)$ -modules]:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & H^*(S_\Psi) \otimes H_Y^*(T) & & \\ & & & & \downarrow & & \\ 0 \rightarrow & H_{X \times T}^*(S \times T) & \rightarrow & H^*(S \times T) & \rightarrow & H^*(S_\Psi \times T) & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & H_{X \times T_\Psi}^*(S \times T_\Psi) & \rightarrow & H^*(S \times T_\Psi) & \rightarrow & H^*(S_\Psi \times T_\Psi) & \\ & & & \downarrow & & \downarrow & \\ & & & 0 & & 0 & \end{array}$$

From this follows that

$$H_{X \times T}^*(S \times T) \rightarrow H_{X \times T_\Psi}^*(S \times T_\Psi)$$

is surjective by remarking that $H^*(S) \otimes H_Y^*(T) \rightarrow H^*(S_\Psi) \otimes H_Y^*(T)$ is surjective, taking into account the map from $H^*(S) \otimes H_Y^*(T)$ into the kernel of $H^*(S \times T) \rightarrow H^*(S \times T_\Psi)$. Next, apply the above long exact sequence to $(S \times T, S \times T_\Psi, X \times T)$ to get the exact sequence

$$\rightarrow H_{X \times Y}^*(S \times T) \rightarrow H_{X \times T}^*(S \times T) \rightarrow H_{X \times T_\Psi}^*(S \times T_\Psi) \rightarrow$$

from which we conclude that

$$H_{X \times Y}^*(S \times T) \rightarrow H_{X \times T}^*(S \times T)$$

is injective. From the following exact sequence and 2.6

$$\rightarrow H_{X \times T}^i(S \times T) \rightarrow H^i(S \times T) \rightarrow H^i(S_\Psi \times T) \rightarrow$$

follows that

$$H_{X \times T}^*(S \times T) \rightarrow H^*(S \times T)$$

is injective. Compose the last two results and write still another long exact sequence to derive the result.

Q. E. D.

PROPOSITION 4.1. — *Let K'' denote a finite double complex on the topological space X . Suppose Z is a closed subset of X such that K^p , has support in Z for all $p \in \mathbb{Z}$. Then*

$$\text{ch}^Z(\text{tot } K'') = \sum (-1)^i \text{ch}^Z(K^p),$$

where $\text{tot } K''$ denotes the total single complex associated to K'' .

Proof. — We shall first change notation and let K'' denote the double indexed family of vector bundles on X underlying the above double complex. Let $C(K'')$ denote the fibre space over X whose sections are pairs (∂', ∂'') of endomorphisms of K'' such that $(K'', \partial', \partial'')$ form a double complex. Let E'' denote the canonical double complex on $C(K'')$ and C_Ψ the complement of the support of $\text{tot } E''$.

Consider now a fixed pair (∂', ∂'') as above and assume that $(K'', 0, \partial'')$ has support in Z . Consider the map of spaces/ X :

$$\theta : X \times \mathbb{A}^1 \rightarrow C(K'')$$

which on the section level is given by

$$t \mapsto (K'', t\partial', \partial'').$$

Clearly

$$\theta(X - Z) \subseteq C_\Psi$$

and

$$\theta_*^*(\text{tot } E'') = \text{tot}(K'', t\partial', \partial'').$$

Conclusion by (1.6), (1.7) and a simple homotopy argument.

Q. E. D.

COROLLARY 4.2. — Consider an exact sequence of complexes of vector bundles on X :

$$0 \rightarrow K^* \rightarrow L^* \rightarrow M^* \rightarrow 0$$

and suppose all three complexes have support in the closed subset Z of X . Then

$$\text{ch}^Z(L^*) = \text{ch}^Z(K^*) + \text{ch}^Z(M^*).$$

Proof. — Consider an appropriate double complex and apply 4.1 twice.

Q. E. D.

COROLLARY 4.3. — Let $f: K^* \rightarrow L^*$ be a linear map of complexes on X and let K^* and L^* have support in Z . If for all $x \in X$:

$$H^*(f_x) : H^*(K_x^*) \rightarrow H^*(L_x^*)$$

is an isomorphism, then

$$\text{ch}^Z(K^*) = \text{ch}^Z(L^*).$$

Proof. — Construct the mapping cone and apply 4.2.

Q. E. D.

5. Formulas without denominators

Let Z be a closed subspace of the space X and consider the commutative graded ring with 1:

$$Z \oplus H_Z^{ev}(X; Z^+).$$

To this we associate

$$1 + \hat{H}_Z^{ev}(X; Z)^+ = 1 + \prod_{i \geq 1} H_Z^{2i}(X; Z)$$

which is an abelian group under cup product. Recall that $1 + \hat{H}_Z^{ev}(X; Z)^+$ comes equipped with a product \star with the property

$$(5.1) \quad \begin{aligned} & (1 + x_m + \text{higher terms}) \star (1 + y_n + \text{higher terms}) = \\ & 1 - \frac{(n+m-1)!}{(m-1)!(n-1)!} x_m y_n + \text{higher terms} \end{aligned}$$

[6] (0, App. § 3).

If K' is a complex on X with support in Z , we put

$$\begin{aligned}\tilde{c}^Z(K') &= 1 + c^Z(K'), \\ \tilde{c}^Z(K') &\in 1 + \hat{H}_Z^{ev}(X; \mathbf{Z})^+.\end{aligned}$$

With the notation of the corresponding formulas for ch^Z , 1.4-8, we have

$$(5.2) \quad \tilde{c}^Z(f^*L') = f^* \tilde{c}^V(L'),$$

$$(5.3) \quad r(\tilde{c}^Z(K')) = \prod_i c.(K^{2i}) c.(K^{2i-1})^{-1},$$

$$(5.4) \quad \tilde{c}^Z(K'[1]) = \tilde{c}^Z(K')^{-1},$$

$$(5.5) \quad \tilde{c}^Z(K' \oplus L') = \tilde{c}^Z(K') \tilde{c}^Z(L'),$$

$$(5.6) \quad \tilde{c}^{Z \cap V}(K' \otimes L') = \tilde{c}^Z(K') \star \tilde{c}^V(L').$$

These formulas are easily derived by the method developed in paragraph 4. From *loc. cit.* follows

(5.7) Suppose $c^Z(K') = a_n + \text{higher terms}$, then $ch^Z(K') = 1/(-1)^{n-1} (n-1)! a_n + \text{higher terms}$

6. Riemann-Roch formula for the Thom class

Let $\pi: E \rightarrow X$ denote a rank n vector bundle, and let λ_E denote the canonical complex on E . Recall that $(\lambda_E)^i = \Lambda^i \pi^* E$. The Koszul complex, i. e. the complex dual to λ_E will be denoted $\lambda_{\check{E}}$.

PROPOSITION 6.1. — *With the above notation*

$$(-1)^n \text{Todd}(E^{\check{}}) ch^X(\lambda_E) = \text{Todd}(E) ch^X(\lambda_{\check{E}}) = \text{Thom class of } E.$$

Proof. — Let $\tilde{E} = \text{Proj}(E \oplus 1)$ and let H denote the canonical line bundle on \tilde{E} . From the canonical imbedding ([1], p. 100):

$$H^{\check{}} \subseteq E \oplus 1$$

we derive the canonical section

$$s \in \Gamma(\tilde{E}, E \otimes H \oplus H).$$

The projection of s onto $E \otimes H$ will be denoted

$$t \in \Gamma(\tilde{E}, E \otimes H).$$

The zero's of t all lie on the canonical section $X \rightarrow \tilde{E}$. Consider the commutative diagram

$$\begin{array}{ccc} \tilde{\tau} \in \hat{H}_X^*(\tilde{E}; \mathbf{Q}) & \xrightarrow{\tilde{r}} & \hat{H}^*(\tilde{E}; \mathbf{Q}) \\ \downarrow & \downarrow t & \downarrow \\ \tau \in \hat{H}_X^*(E; \mathbf{Q}) & \xrightarrow{r} & \hat{H}^*(E; \mathbf{Q}) \end{array}$$

where τ denotes the Thom class. Let us first prove that

$$\tilde{r}(\tilde{\tau}) = c_n(E \otimes H).$$

For this let us note that \tilde{r} is injective. Namely, $H^*(\tilde{E}; \mathbf{Q}) \rightarrow H^*(\tilde{E}-X; \mathbf{Q})$ is surjective since the restriction to $\tilde{E}-X$ of

$$1, c_1(H), \dots, c_1(H)^{n-1}$$

form a basis for the $H^*(X; \mathbf{Q})$ -module $H^*(E-X; \mathbf{Q})$. Note that the restriction of $c_n(E \otimes H)$ to $\tilde{E}-X$ is zero because of the section t . Let $\sigma \in H_X(\tilde{E})$ be such that

$$\tilde{r}(\sigma) = c_n(E \otimes H).$$

We shall show that σ is the Thom class. For this it suffices to treat the case $X = P^t$. In this case $c_n(E \otimes H) = c_1(H)^n$ and the statement is clear.

We shall now prove the first formula. Let λ^\sim denote the Koszul complex associated with the section t of $E \otimes H$. The restriction of λ^\sim to E is $\lambda_{\check{E}}$. Let us recall [8], Lemma 18 that for a rank n bundle N we have

$$(6.2) \quad \text{ch}(\lambda_{-1} N^\vee) = c_n(N) \text{Todd}(N)^{-1}.$$

The formula will now follow by applying (1.5) to λ^\sim

$$\begin{aligned} \tilde{r}(\text{ch}^X \lambda^\sim) &= \text{ch}(\lambda_{-1} \check{E} \otimes \check{H}) = c_n(E \otimes H) \text{Todd}(E \otimes H)^{-1}, \\ \text{Todd}(E \otimes H)^{-1} &\equiv \text{Todd}(E)^{-1} \text{ mod } c_1(H), \\ c_n(E \otimes H) c_1(H) &= 0 \end{aligned}$$

as it follows from the fact that $t \in \Gamma(\tilde{E}, E \otimes H \oplus H)$ has no zeros. Whence

$$\tilde{r}(\text{ch}^X \lambda^\sim) = c_n(E \otimes H) \text{Todd}(E)^{-1}.$$

Q. E. D.

Remark. — The above formula should be considered as generalizations of formulas used in [2], [3], [4].

7. Multiplicity in algebraic geometry

In this paragraph we shall work in the framework of [7] and prove a fundamental relation 7.1 between local Chern classes and the multiplicity of local algebra [15], compare [2], 6.2.

Let V denote a smooth (connected) algebraic variety/ \mathbb{C} and $X \subseteq V$ a closed subvariety of codimension d . The local fundamental class will be denoted

$$cl^X \in H_X^{2d}(V; \mathbb{Z}).$$

The fundamental class of X , i. e. the image of cl^X in $H^{2d}(V; \mathbb{Z})$ will be denoted

$$cl(X) \in H^{2d}(V; \mathbb{Z}).$$

For a coherent (algebraic) sheaf M on V with support in X , $l(M)$ denotes the length of the stalk of M at the generic point of X .

THEOREM 7.1. — *Let E' denote a complex of locally free coherent (algebraic) sheaves on V with $\text{Supp}(E') \subseteq X$. Then*

$$ch^X(E') = \sum (-1)^i l(H^i E') cl^X + \text{higher terms.}$$

Proof. — Let O denote the local ring of V at the generic point of X , m denotes the maximal ideal of O . Let $K_m(O)$ denote the Grothendieck group of the category of finite complexes of finitely generated free O -modules with homology of finite length (modulo exact complexes). We are going to define a topological character

$$l : K_m(O) \rightarrow \mathbb{Z}.$$

Recall first that if U is a Zariski open subset of V with $X \cap U \neq \emptyset$, then the restriction map

$$H_X^{2d}(V; \mathbb{Z}) \rightarrow H_{X \cap U}^{2d}(U; \mathbb{Z})$$

is an isomorphism which carries cl^X to $cl^{X \cap U}$. From this follows that there is a character l as above such that for any complex E' as in the theorem

$$ch^X(E') = l(E') cl^X + \text{higher terms.}$$

As is well known $K_m(O) \simeq \mathbb{Z}$ since O is a regular local ring [6]. Thus it will suffice to find a resolution E' of O/m by finitely generated free sheaves with $l(E') = 1$. Let us first consider the case $V = \mathbb{A}^d$, $X = \{0\}$. In this case we can take for E' the standard Koszul complex. That $l(E') = 1$ follows from 6.1.

In the general case choose a Zariski open set U of V and $f_1, \dots, f_d \in \Gamma(U, \mathcal{O}_V)$ which defines $X \cap U$. This defines a map

$$f : U \rightarrow \mathbb{A}^d$$

with $f^{-1}(\{0\}) = U \cap X$. It follows that

$$f^* : H_{\{0\}}^{2d}(\mathbb{A}^d; \mathbb{Z}) \rightarrow H_{X \cap U}^{2d}(U; \mathbb{Z})$$

is an isomorphism. The pull-back of the complex considered before will now do the job.

Q. E. D.

Remark 7.2. — Taking in particular a resolution of the structure sheaf \mathcal{O}_X of X we obtain by means of (5.7):

$$c_d(\mathcal{O}_X) = (-1)^{d-1} (d-1)! \text{cl}(X)$$

due to Grothendieck [11] formula 17, compare [12] (p. 53, Lemma 2).

Remark 7.3. Combining 7.1 and 1.8 we obtain Serre's "alternating Tor-formula" [15] for the topological intersection number.

REFERENCES

- [1] M. F. ATIYAH, *K-Theory*, Benjamin, New York-Amsterdam, 1967.
- [2] M. F. ATIYAH and F. HIRZEBRUCH, *Analytic Cycles on Complex Manifolds (Topology, vol. 1, 1962, p. 25-45)*.
- [3] M. F. ATIYAH and F. HIRZEBRUCH, *The Riemann-Roch Theorem for Analytic Embeddings (Topology, vol. 1, 1962, p. 151-166)*.
- [4] M. F. ATIYAH and I. M. SINGER, *The Index of Elliptic Operators III (Annals of Math., vol. 87, 1968, p. 546-604)*.
- [5] P. BAUM, W. FULTON and R. MACPHERSON, *Riemann-Roch for Singular Varieties (To appear)*.
- [6] P. BERTHELOT, A. GROTHENDIECK et L. ILLUSIE, *Théorie des intersections et théorème de Riemann-Roch (Lecture Notes in Math., Springer Verlag, Berlin-Heidelberg-New York, 1971)*.
- [7] A. BOREL et A. HAEFLIGER, *Classe d'homologie fondamentale (Bull. Soc. math. Fr., vol. 89, 1961, p. 461-513)*.
- [8] A. BOREL et J.-P. SERRE, *Le théorème de Riemann-Roch (Bull. Soc. math. Fr., vol. 86, 1958, p. 97-136)*.
- [9] G. E. BREDON, *Sheaf Theory*. McGraw-Hill, New York, 1967.
- [10] W. FULTON, *Riemann-Roch for Singular Varieties (Proc. Symp. Pure Math., XXIX, Amer. Math. Soc., Providence, Rhode Island, 1975)*.
- [11] A. GROTHENDIECK, *La théorie des classes de Chern (Bull. Soc. math. Fr., vol. 86, 1958, p. 137-154)*.

- [12] H. HIRONAKA, *Smoothing of Algebraic Cycles of Low Codimension* (*Amer. J. Math.*, vol. 90, 1968, p. 1-54).
- [13] L. ILLUSIE, *Complexe cotangent et déformations. I.* (*Lecture Notes in Math.*, n° 239, Springer, Berlin, 1971).
- [14] R. D. MACPHERSON, *Chern Classes for Singular Algebraic Varieties* (*Ann. of Math.*, vol. 100, 1974, p. 423-432).
- [15] J.-P. SERRE, *Algèbre locale et multiplicités* (*Lecture Notes in Math.*, n° 11, Springer, Berlin, 1965).
- [16] J.-L. VERDIER, *Le théorème de Riemann-Roch pour les variétés algébriques éventuellement singulières* (d'après P. Baum, W. Fulton et R. MacPherson) (*Séminaire Bourbaki*, 27^e année, 1974/1975, Exp. 464. *Lecture Notes in Math.* n° 514, Springer, Berlin, 1976).

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