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Local Chern classes

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The purpose of this paper is to give a construction of local Chern classes as conjectured by Grothendieck [6] (XIV 7.2).

The construction is given in the framework of complex vector bundles on topological spaces where it appears as a generalization of the relative Chern classes obtained from the "difference construction" in K-theory notably used by Atiyah ([1]-[4]).

It will be clear that the constructions performed work equally well in other theories, especially the etale cohomology of algebraic geometry.

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1. Introduction

By a complex $K'$ of vector bundles on a topological space $X$ we understand a finite complex of $\mathbb{C}$-vector bundles each having constant rank. By the support of $K'$ we understand the complement to the set of points $x \in X$ for which $K'_x$ is an exact complex of vector spaces.

For a space $X$, $\hat{H}'(X; \mathbb{Z})$ denotes integral cohomology in the sense of sheaf theory, $\hat{H}'(X; \mathbb{Z}) = \prod_i \hat{H}'(X; \mathbb{Z})$. For a closed subset we use interchangeably

$$H_Z(X; \mathbb{Z}) = H'(X, X-Z; \mathbb{Z})$$

for cohomology with support in $Z$. 
A theory of local Chern classes consists in assigning to a complex $K^*$ on $X$ with support in $Z$ a cohomology class

$$c^Z(K^*) \in \hat{H}^Z(X; \mathbb{Z})$$

with the following two properties

(1.1) For a continuous map $f: X \to Y$, closed subsets $Z \subseteq X$, $V \subseteq Y$ with $f(X-Z) \subseteq Y-V$ and a complex $L^*$ on $Y$ with support in $V$:

$$c^Z(f^*L^*) = f^*c^V(L^*).$$

(1.2) Let $r: \hat{H}^Z(X; Z) \to \hat{H}^V(X; Z)$ denote the canonical map. Then

$$r(c^Z(K^*)) + 1 = \prod_i c_i(K^{2i})c_i(K^{2i-1})^{-1}.$$

The main result of this paper is

**Theorem 1.3.** — *A theory of local Chern classes exists and is unique.*

As usual we introduce a local Chern character

$$ch^Z(K^*) \in \hat{H}^Z(X; \mathbb{Q})$$

with the following properties:

(1.4) **Functoriality.** — $f^*ch^V(L^*) = ch^Z(f^*L^*)$.

(1.5) $r(ch^Z(K^*)) = \sum (-1)^i ch(K^i)$.

(1.6) **Decalage.** — $ch^Z(K'[1]) = -ch^Z(K')$.

(1.7) **Additivity.** — For complexes $K^*$ and $L^*$ on $X$ with support in $Z$:

$$ch^Z(K^* \otimes L^*) = ch^Z(K^*) + ch^Z(L^*).$$

(1.8) **Multiplicativity.** — Let $K^*$ and $L^*$ be complexes on $X$ with support in $Z$ and $V$, respectively. Then

$$ch^Z_{\cap V}(K^* \otimes L^*) = ch^Z(K^*)ch^V(L^*).$$

The proof of 1.3 is given in paragraphs 2 and 3 while paragraphs 4 and 5 derives multiplicative and additive properties of $c^Z$ and $ch^Z$.

In paragraph 6 we derive Riemann-Roch formulas for the Thom class and paragraph 7 initiates applications to algebraic geometry.
In cases where $X$ is an oriented topological manifold of dimension $n$, Poincaré duality

$$H^i_\mathcal{Z}(X;\mathbb{Z})\cong H_{n-i}(\mathbb{Z};\mathbb{Z})$$

transforms our local cohomology classes into homology classes. In cases where $X$ is a smooth algebraic variety/\mathbb{C}, this should be compared with the homology classes constructed by means of MacPherson’s graph construction [5] compare [10], [14], [16].

It should also be mentioned that Illusie ([13] V.6) has constructed local Chern classes “à la Atiyah” in Hodge cohomology.

I should like to thank K. Suominen for stimulating my interest in these matters.

2. The canonical complex

Throughout this paragraph we shall work with the following data.

A topological space $X$, a sequence of vector bundles $(K^i)_{i\in\mathbb{Z}}$ on $X$ with $K^i = 0$ except for finitely many $i \in \mathbb{Z}$.

$$v_i = \text{rank } K^i.$$

We shall assume that there exists a sequence $(\lambda_i)_{i\in\mathbb{Z}}$ of integers with

$$\lambda_i + \lambda_{i+1} = v_i, \quad i \in \mathbb{Z},$$

$$\lambda_i \geq 0, \quad i \in \mathbb{Z}.$$

Put $K = \bigoplus_{i\in\mathbb{Z}} K^i$. The flag manifold whose sections are flags in $K$ of nationality $v.$ will be denoted $\text{Fl}_v$. The fixed flag defined by

$$F_i = \bigoplus_{t \leq i} K^t$$

is denoted $F_i$.

DEFINITION 2.1. — $T \subseteq \text{Fl}_v$ denote the closed subspace whose sections are flags $D$ with the property that

$$F_{i-1} \subseteq D_i \subseteq F_{i+1}, \quad i \in \mathbb{Z}.$$

The canonical projection is denoted $p: T \to X$. The restriction to $T$ of the canonical flag on $\text{Fl}_v$ will be denoted $E_i$. On $T$ we have a canonical complex $C'$ given by

$$C^i = E_i/p^*F_{i-1},$$

$$\partial^i : E_i/p^*F_{i-1} \to E_{i+1}/p^*F_i.$$
is induced by the inclusion \( E_i \subseteq E_{i+1} \). \( \partial^i+1 \partial^i = 0 \) since
\[
p^*F_i \subseteq E_i \subseteq p^*F_{i+1}, \quad i \in \mathbb{Z}.
\]
Finally \( T_\psi \) is the complement in \( T \) of the support of \( C \), and \( p_\psi: T_\psi \to X \) denotes the restriction of \( p \) to \( T_\psi \).

**Lemma 2.2.** — A section of \( T \) over \( X \) represented by a flag \( D_i \) is a section of \( T_\psi \) if and only if for all \( x \in X \):

\[
\text{rank}(D_i, x \cap F_i, x/F_i, x) = \lambda_i.
\]

**Proof.** — By definition \( D_i \) represents a section of \( T_\psi \) if and only if the complex
\[
\to D_i-1/F_i-2 \to D_i/F_i-1 \to D_i+1/F_i \to
\]
has exact fibres for all \( x \in X \). Note that \( D_i/F_i-1 \) has rank \( \nu_i \), and the lemma follows from the definition of \( (\lambda_i)_{i \in \mathbb{Z}} \).

**Theorem 2.3.** — Let \( i_\psi: T_\psi \to T \) denote the inclusion. Then
\[
i_\psi^*: \quad H^i(T; \mathbb{Z}) \to H^i(T_\psi; \mathbb{Z})
\]
is surjective.

**Proof.** — Define
\[
G_\delta = \prod Grass_{\delta_i}(K^i) \to X,
\]
where \( p_i: Grass_{\delta_i}(K^i) \to X \) is the fibre space whose sections are rank \( \lambda_i \)-subbundles of \( K^i \).
\[
f_\delta : \quad T_\psi \to G_\delta
\]
denotes the map which on the level of sections (compare 2.2) transforms
\[
D_i \mapsto (D_i \cap F_i/F_i, x)_{i \in \mathbb{Z}}.
\]

We shall first prove
\[
(2.4) \quad f_\delta^* : \quad H^i(G_\delta; \mathbb{Z}) \to H^i(T_\psi; \mathbb{Z})
\]
is an isomorphism.

We shall prove that \( f_\delta \) is a fibration with fibres of type \( A^d = (A^d: \text{affine space of dimension } d = \sum \lambda_i^2) \). For this assume \( X = P^i \). The fibre of \( f_\delta \) above \( B \in G_\delta \) consists of sequences \( (G^i)_{i \in \mathbb{Z}} \), where \( G^i \) is a \( \lambda_i+1 \)-plane in the \( \lambda_i \)-space \( B^{i+1}/B^i \) intersecting the \( \lambda_i \)-plane \( F_i/B^i \) in zero.
Next define

\[ G_v = \prod_i \text{Grass}_v(K^i \oplus K^{i+1}) \]

and maps

\[ f_v : T \to G_v, \quad D_i \mapsto (D_i/F_{i-1})_{|_{i \in \mathbb{Z}}}; \]
\[ g : G_h \to G_v, \quad B_i \mapsto (B^i \oplus B^{i+1})_{|_{i \in \mathbb{Z}}}; \]
\[ s_h : G_h \to T; \]
\[ B^i \mapsto (\oplus K^i \oplus B^i \oplus B^{i+1})_{|_{i \in \mathbb{Z}}}, \]

where in each case the transformation on the level of sections is given.

We have the following diagram

\[
\begin{array}{cccc}
T & \leftarrow & T' & \\
\downarrow f_v & & \downarrow f_h & \\
G_v & \leftarrow & G_h & \\
\end{array}
\]

with

\[ f_v s_h = g, \quad f_h s_h = 1 \]

\((f_v s_h = g)\).

Let us grant (2.5 below) that \(g^{*}\) is surjective.

\[ s_h f_h^{*} = 1 \text{ and whence by 2.4;} \]
\[ f_h^{*} s_h^{*} = 1, \text{ on the other hand;} \]
\[ s_h^{*} i_{\psi}^{*} f_v^{*} = g^{*} \text{ and whence;} \]
\[ i_{\psi}^{*} f_v^{*} = f_h^{*} g^{*}. \text{ Thus } i_{\psi}^{*} \text{ surjective.} \]

Q. E. D.

LEMMA 2.5. — The map

\[ g : \prod_i \text{Grass}_v K^i \to \prod_i \text{Grass}_v K^i \oplus K^{i+1}, \]
\[ B^i \mapsto (B^i \oplus B^{i+1})_{|_{i \in \mathbb{Z}}} \]

induces a surjective map \(g^{*}\) on integral cohomology.

Proof. — Let \(P^i\) denote the canonical \(\lambda_i\)-bundle on \(\text{Grass}_v(K^i)\). Consider

\[ H^i(\prod_i \text{Grass}_v K^i; \mathbb{Z}) \]

as a \(H^i(X; \mathbb{Z})\)-algebra. As is well known this algebra is generated by the homogeneous components of

\[ \text{pr}_i^{*} c_i(P^i), \quad i \in \mathbb{Z}. \]
Consider the composite of $g$ and the $i$'th projection

$$\prod_i \text{Grass}_i K^i \to \text{Grass}_i K^i \oplus K^{i+1}$$

to see that

$$\text{pr}_i^*c.(P^i)\text{pr}_{i+1}^*c.(P^{i+1})$$

and the inverse to that element belongs to the image of $g^*$. It is now clear by decreasing induction that $\text{pr}_i^*c.(P_i)$ and $\text{pr}_i^*c.(P_i)^{-1}$ belong to the image of $g^*$.

Q. E. D.

**Proposition 2.6.** - The $H^*(X; \mathbb{Z})$-module $H^*(T_X; \mathbb{Z})$ is finitely generated free and for any map $X' \to X$.

$$H^*(T_X; \mathbb{Z}) \otimes_{H^*(X; \mathbb{Z})} H^*(X'; \mathbb{Z}) \to H^*(T_{X} \times_X X'; \mathbb{Z})$$
is an isomorphism.

**Proof.** - By 2.4 we may replace $T_X$ by a product of Grassmannian bundles for which this is well known.

Q. E. D.

3. Construction of the local Chern class

With the notation of paragraph 2 let $(\partial_i)_{i \in \mathbb{Z}}$ be a family of linear maps $\partial^i: K^i \to K^{i+1}$ with $\partial^{i+1} \partial^i = 0, i \in \mathbb{Z}$. Define a flag $s_i(\partial^i)$ in $K = \bigoplus_i K^i$ as follows: $s_i(\partial^i)$ is the graph of the map

$$\bigoplus_{i \leq i} K^i \to \bigoplus_{i > i} K^i,$$

$$(\ldots, k_{i-1}, k_i) \mapsto (\partial^i k_i, 0^i \ldots).$$

Clearly,

$$F_{i-1} \subseteq s_i(\partial^i) \subseteq F_{i+1}, \quad i \in \mathbb{Z}.$$

Thus we may interprete $s_i(\partial^i)$ as a section of $p: T \to X$

$$s_i(\partial^i): X \to T.$$

Clearly

$$(3.1) \quad s_i(\partial^i)^*C' = (K', \partial^i).$$
Let now $Z \subseteq X$ denote a closed subset such that $\text{Supp} (K', \partial') \subseteq Z$ then
$$s.(\partial')(X-Z) \subseteq T\psi.$$

Consider the exact sequence, (2.3):
$$0 \rightarrow \hat{H}^1(T, T\psi; Z) \rightarrow \hat{H}(T; Z) \rightarrow \hat{H}^1 CL^p; Z) \rightarrow 0.$$  

The image by $i^*_\psi$ of the cohomology class
$$c.(C')^{-1} = \prod_i c.(C^{2i})^{-1} c.(C^{2i-1})^{-1} - 1$$
is zero since $C'$ is exact on $T\psi$. Let
$$\gamma_T \in \hat{H}^1(T, T\psi; Z)$$
denote the cohomology class characterized by

\begin{equation}
(3.2) \quad r_\psi(\gamma_T) + 1 = c.(C').
\end{equation}

**Definition 3.3.** — Consider the map induced by $s.(\partial')$

$$s.(\partial')^* : \hat{H}^1(T, T\psi; Z) \rightarrow \hat{H}^1(X; Z)$$

and define the local Chern class of $(K', \partial')$ supported in $Z$ by

$$c.(K', \partial') = s.(\partial')^* \gamma_T.$$

**Proof of 1.3.** — Follows from 3.1 and 3.2.

As above we consider the cohomology class

$$\gamma^\kappa_T \in \hat{H}^1(T, T\psi; Q)$$

classified by

\begin{equation}
(3.4) \quad r_\psi(\gamma^\kappa_T) = \sum_i (-1)^i \text{ch}(C^i).
\end{equation}

**Definition 3.5:**

$$\text{ch}^Z(K', \partial') = s.(\partial')^* \gamma^\kappa_T.$$

The local Chern character thus defined satisfies clearly 1.4-6. Let us remark that $\text{ch}^Z$ can be derived directly from $c.Z$ by means of the theory of $\lambda$-rings, compare paragraph 5.
4. Properties of the local Chern character

In this paragraph we shall prove the multiplication property 1.8 of $ch^2$. The proof of the additive property 1.7 is similar but simpler and will not be given. Finally, we give some variants of the additive property.

Proof of 1.8. — Let us first note that 1.8 is true if the canonical map

$$H^*_{Z\cap W}(X; Z) \to H^*(X; Z)$$

is injective. We are going to reduce the problem to this case. Let $T = T(K')$ and $S = T(L')$ with a slight abuse of notation. It will now suffice to prove that

$$H^*(T \times S; Z) \to H^*(S \times T \cup T \times S ; Z)$$

is surjective. Here and in the following all products are formed in the category of spaces $X$. $H^*(-)$ denotes integral cohomology. Let us first recall that if $Z \subseteq Y$ is a closed subset of the space $Y$ and if $U \subseteq Y$ is an open subset, then there is a canonical exact sequence

$$0 \to H^*_{Z \cap U}(X) \to H^*(X) \to H^*(U) \to H^*_{Z \cup U}(X) \to 0.$$

Put $X = S - S_\psi$ and $Y = T - T_\psi$. It follows from 2.6 that the following commutative diagram is exact [$\otimes$ is formed in the category of $H^*(X)$-modules]:

$$
\begin{array}{c}
0 \\
\downarrow \\
H^*(S_\psi) \otimes H^*_\gamma(T) \\
\downarrow \\
0 \to H^*_{X \times T}(S \times T) \to H^*(S \times T) \to H^*(S_\psi \times T) \\
\downarrow \\
0 \to H^*_{X \times T_\psi}(S \times T_\psi) \to H^*(S \times T_\psi) \to H^*(S_\psi \times T_\psi) \\
\downarrow \\
0 \\
\end{array}
$$

From this follows that

$$H^*_{X \times T}(S \times T) \to H^*_{X \times T_\psi}(S \times T_\psi)$$

is surjective by remarking that $H^*(S) \otimes H^*_\gamma(T) \to H^*(S_\psi) \otimes H^*_\gamma(T)$ is surjective, taking into account the map from $H^*(S) \otimes H^*_\gamma(T)$ into the kernel of $H^*(S \times T) \to H^*(S \times T_\psi)$. Next, apply the above long exact sequence to $(S \times T, S \times T_\psi, X \times T)$ to get the exact sequence

$$0 \to H^*_{X \times Y}(S \times T) \to H^*_{X \times T}(S \times T) \to H^*_{X \times T_\psi}(S \times T_\psi) \to 0.$$
from which we conclude that
\[ H^*_{X \times Y}(S \times T) \to H^*_{X \times T}(S \times T) \]
is injective. From the following exact sequence and 2.6
\[ \to H^*_{X \times T}(S \times T) \to H^t(S \times T) \to H^t(S_\Psi \times T) \to \]
follows that
\[ H^*_{X \times T}(S \times T) \to H^t(S \times T) \]
is injective. Compose the last two results and write still another long exact sequence to derive the result.

Q. E. D.

**Proposition 4.1.** — Let \( K' \) denote a finite double complex on the topological space \( X \). Suppose \( Z \) is a closed subset of \( X \) such that \( K'^p \), has support in \( Z \) for all \( p \in Z \). Then
\[ \text{ch}^2(\text{tot } K') = \sum (-1)^i \text{ch}^2(K'^p), \]
where \( \text{tot } K' \) denotes the total single complex associated to \( K' \).

**Proof.** — We shall first change notation and let \( K' \) denote the double indexed family of vector bundles on \( X \) underlying the above double complex. Let \( C(K') \) denote the fibre space over \( X \) whose sections are pairs \((\partial', \partial'')\) of endomorphisms of \( K' \) such that \((K', \partial', \partial'')\) form a double complex. Let \( E' \) denote the canonical double complex on \( C(K') \) and \( C_\Psi \) the complement of the support of \( \text{tot } E' \).

Consider now a fixed pair \((\partial', \partial'')\) as above and assume that \((K', 0, \partial'')\) has support in \( Z \). Consider the map of spaces/\( X \):
\[ \theta : X \times \mathbb{A}^1 \to C(K') \]
which on the section level is given by
\[ t \mapsto (K', t \partial', \partial''). \]
Clearly
\[ \theta(X-Z) \subseteq C_\Psi \]
and
\[ \theta_*^\text{tot}(E') \to \text{tot}(K', t \partial', \partial''). \]
Conclusion by (1.6), (1.7) and a simple homotopy argument.

Q. E. D.
Corollary 4.2. — Consider an exact sequence of complexes of vector bundles on $X$:

$$0 \to K \to L \to M \to 0$$

and suppose all three complexes have support in the closed subset $Z$ of $X$. Then

$$\text{ch}^Z(L) = \text{ch}^Z(K) + \text{ch}^Z(M).$$

Proof. — Consider an appropriate double complex and apply 4.1 twice.

Q. E. D.

Corollary 4.3. — Let $f: K \to L$ be a linear map of complexes on $X$ and let $K$ and $L$ have support in $Z$. If for all $x \in X$:

$$H'(f_x) : H'(K_x) \to H'(L_x)$$

is an isomorphism, then

$$\text{ch}^Z(K) = \text{ch}^Z(L).$$

Proof. — Construct the mapping cone and apply 4.2.

Q. E. D.

5. Formulas without denominators

Let $Z$ be a closed subspace of the space $X$ and consider the commutative graded ring with 1:

$$Z \oplus \hat{H}^\infty_Z(X; Z^+).$$

To this we associate

$$1 + \hat{H}^\infty_Z(X; Z)^+ = 1 + \prod_{i \geq 1} \hat{H}^{2i}_Z(X; Z)$$

which is an abelian group under cup product. Recall that $1 + \hat{H}^\infty_Z(X; Z)^+$ comes equipped with a product $\star$ with the property

$$\left(1 + x_m + \text{higher terms}\right) \star \left(1 + y_n + \text{higher terms}\right) =$$

$$1 - \frac{(n + m - 1)!}{(m - 1)! (n - 1)!} x_m y_n + \text{higher terms}$$

(5.1)

[6] (0, App. § 3).
If $K'$ is a complex on $X$ with support in $Z$, we put
\[ \tilde{c}^Z(K') = 1 + c^Z(K'), \]
\[ \tilde{c}^Z(K') \in 1 + \hat{H}^*(X; \mathbb{Z}). \]

With the notation of the corresponding formulas for $\text{ch}^Z$, 1.4-8, we have

\begin{align*}
(5.2) & \quad \tilde{c}^Z(f^*L') = f^*\tilde{c}^Y(L'), \\
(5.3) & \quad r(\tilde{c}^Z(K')) = \prod c.(K^{2i})c.(K^{2i-1})^{-1}, \\
(5.4) & \quad \tilde{c}^Z(K'[1]) = \tilde{c}^Z(K')^{-1}, \\
(5.5) & \quad \tilde{c}^Z(K' \oplus L') = \tilde{c}^Z(K')\tilde{c}^Z(L'), \\
(5.6) & \quad \tilde{c}^Z\wedge^Y(K' \otimes L') = \tilde{c}^Z(K') \star \tilde{c}^Y(L').
\end{align*}

These formulas are easily derived by the method developed in paragraph 4. From loc. cit. follows

\begin{align*}
(5.7) & \quad \text{Suppose } c^Z(K') = a_n + \text{higher terms, then } \text{ch}^Z(K') = 1/(-1)^n a_n + \text{higher terms}
\end{align*}

6. Riemann-Roch formula for the Thom class

Let $\pi: E \to X$ denote a rank $n$ vector bundle, and let $\lambda_E$ denote the canonical complex on $E$. Recall that $(\lambda_E)^i = \Lambda^i \pi^*E$. The Koszul complex, i.e. the complex dual to $\lambda_E$ will be denoted $\lambda_{\tilde{E}}$.

**Proposition 6.1.** — With the above notation

\[ (-1)^n \text{Todd}(E^*) \text{ch}^Y(\lambda_E) = \text{Todd}(E) \text{ch}^Y(\lambda_{\tilde{E}}) = \text{Thom class of } E. \]

**Proof.** — Let $\tilde{E} = \text{Proj } (E \oplus 1)$ and let $H$ denote the canonical line bundle on $\tilde{E}$. From the canonical imbedding ([1], p. 100):

\[ H^\wedge \subseteq E \oplus 1 \]

we derive the canonical section

\[ s \in \Gamma(\tilde{E}, E \otimes H \otimes H). \]

The projection of $s$ onto $E \otimes H$ will be denoted

\[ t \in \Gamma(\tilde{E}, E \otimes H). \]
The zero’s of $t$ all lie on the canonical section $X \rightarrow \tilde{E}$. Consider the commutative diagram

$$\begin{array}{ccc}
\tilde{\tau} \in \hat{H}_X(\tilde{E}; \mathbb{Q}) & \xrightarrow{\tilde{r}} & \hat{H}^*(\tilde{E}; \mathbb{Q}) \\
\downarrow & & \downarrow \\
\tau \in H_X(E; \mathbb{Q}) & \xrightarrow{r} & H^*(E; \mathbb{Q})
\end{array}$$

where $\tau$ denotes the Thom class. Let us first prove that

$$\tilde{r}(\tilde{\tau}) = c_\pi(E \otimes H).$$

For this let us note that $\tilde{r}$ is injective. Namely, $H^*(\tilde{E}; \mathbb{Q}) \rightarrow H^*(\tilde{E} - X; \mathbb{Q})$ is surjective since the restriction to $\tilde{E} - X$ of

$$1, c_1(H), \ldots, c_1(H)^{n-1}$$

form a basis for the $H^*(X; \mathbb{Q})$-module $H^*(E - X; \mathbb{Q})$. Note that the restriction of $c_\pi(E \otimes H)$ to $\tilde{E} - X$ is zero because of the section $t$. Let $\sigma \in H_X(\tilde{E})$ be such that

$$\tilde{r}(\sigma) = c_\pi(E \otimes H).$$

We shall show that $\sigma$ is the Thom class. For this it suffices to treat the case $X = P^r$. In this case $c_\pi(E \otimes H) = c_1(H)^r$ and the statement is clear.

We shall now prove the first formula. Let $\lambda^\sim$ denote the Koszul complex associated with the section $t$ of $E \otimes H$. The restriction of $\lambda^\sim$ to $E$ is $\lambda_E$. Let us recall [8], Lemma 18 that for a rank $n$ bundle $N$ we have

$$\text{ch}(\lambda_{-1} N^\sim) = c_\pi(N) \text{Todd}(N)^{-1}. \quad (6.2)$$

The formula will now follow by applying (1.5) to $\lambda^\sim$

$$\tilde{r}(\text{ch}^{X} \lambda^\sim) = \text{ch}(\lambda_{-1} \tilde{E} \otimes \tilde{H}) = c_\pi(E \otimes H) \text{Todd}(E \otimes H)^{-1},$$

$$\text{Todd}(E \otimes H)^{-1} \equiv \text{Todd}(E)^{-1} \mod c_1(H),$$

$$c_\pi(E \otimes H) c_1(H) = 0$$

as it follows from the fact that $t \in \Gamma(\tilde{E}, E \otimes H \otimes H)$ has no zeros. Whence

$$\tilde{r}(\text{ch}^{X} \lambda^\sim) = c_\pi(E \otimes H) \text{Todd}(E)^{-1}.$$

Q. E. D.

Remark. — The above formula should be considered as generalizations of formulas used in [2], [3], [4].
7. Multiplicity in algebraic geometry

In this paragraph we shall work in the framework of [7] and prove a fundamental relation 7.1 between local Chern classes and the multiplicity of local algebra [15], compare [2], 6.2.

Let $V$ denote a smooth (connected) algebraic variety over $\mathbb{C}$ and $X \subseteq V$ a closed subvariety of codimension $d$. The local fundamental class will be denoted

$$c^X \in H^{2d}_X(V;\mathbb{Z}).$$

The fundamental class of $X$, i.e. the image of $c^X$ in $H^{2d}(V;\mathbb{Z})$ will be denoted

$$\text{cl}(X) \in H^{2d}(V;\mathbb{Z}).$$

For a coherent (algebraic) sheaf $M$ on $V$ with support in $X$, $l(M)$ denotes the length of the stalk of $M$ at the generic point of $X$.

**Theorem 7.1.** — Let $E^\bullet$ denote a complex of locally free coherent (algebraic) sheaves on $V$ with $\text{Supp}(E^\bullet) \subseteq X$. Then

$$\text{ch}^X(E^\bullet) = \sum (-1)^i l(H^i E^\bullet) c^X + \text{higher terms}.$$

**Proof.** — Let $O$ denote the local ring of $V$ at the generic point of $X$, $m$ denotes the maximal ideal of $O$. Let $K_m(O)$ denote the Grothendieck group of the category of finite complexes of finitely generated free $O$-modules with homology of finite length (modulo exact complexes). We are going to define a topological character

$$l : K_m(O) \to \mathbb{Z}.$$

Recall first that if $U$ is a Zariski open subset of $V$ with $X \cap U \neq \emptyset$, then the restriction map

$$H_X^{2d}(V;\mathbb{Z}) \to H_X^{2d}(V \cap U;\mathbb{Z})$$

is an isomorphism which carries $c^X$ to $c^{X \cap U}$. From this follows that there is a character $l$ as above such that for any complex $E^\bullet$ as in the theorem

$$\text{ch}^X(E^\bullet) = l(E^\bullet) c^X + \text{higher terms}.$$

As is well known $K_m(O) \simeq \mathbb{Z}$ since $O$ is a regular local ring [6]. Thus it will suffice to find a resolution $E^\bullet$ of $O/m$ by finitely generated free sheaves with $l(E^\bullet) = 1$. Let us first consider the case $V = \mathbb{A}^d$, $X = \{0\}$. In this case we can take for $E^\bullet$ the standard Koszul complex. That $l(E^\bullet) = 1$ follows from 6.1.
In the general case choose a Zariski open set $U$ of $V$ and $f_1, \ldots, f_d \in \Gamma(U, \mathcal{O}_V)$ which defines $X \cap U$. This defines a map

$$f : \ U \rightarrow \mathbb{A}^d$$

with $f^{-1}(\{0\}) = U \cap X$. It follows that

$$f^* : H^{2d}_{(0)}(\mathbb{A}^d; \mathbb{Z}) \rightarrow H^{2d}_{X \cap U}(U; \mathbb{Z})$$

is an isomorphism. The pull-back of the complex considered before will now do the job.

Q. E. D.

**Remark 7.2.** – Taking in particular a resolution of the structure sheaf $\mathcal{O}_X$ of $X$ we obtain by means of (5.7):

$$e_d(\mathcal{O}_X) = (-1)^{d-1}(d-1)! \cdot \text{cl}(X)$$


**Remark 7.3.** Combining 7.1 and 1.8 we obtain Serre’s “alternating Tor-formula” [15] for the topological intersection number.

**REFERENCES**


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