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Annales scientifiques de l’É.N.S. 4e série, tome 12, n° 1 (1979), p. 33-45

<http://www.numdam.org/item?id=ASENS_1979_4_12_1_33_0>
CLOSEDNESS OF REGULAR 1-FORMS
ON ALGEBRAIC SURFACES (1)

By Niels O. Nygaard

Introduction

Let $X/k$ be a proper, smooth surface over a perfect field $k$. If $k$ has characteristic 0 it follows from Hodge theory and the Lefshetz principle that all regular 1-forms on $X$ are closed, i.e. that the differential

$$d : H^0(X, \Omega^1_{X/k}) \to H^0(X, \Omega^2_{X/k}),$$

vanishes.

In characteristic $p > 0$ the situation is more complicated indeed Mumford [11] and more recently Raynaud have constructed surfaces with regular 1-forms which are not closed (2). It therefore becomes interesting to look for conditions on $X$ that will ensure the closedness of regular 1-forms. We relate this question to an invariant defined and studied by Artin and Mazur in [1], the formal Brauer group, $Br_X$, specially we show that if $Br_X$ is pro-representable by a $p$-divisible formal group (Barsotti-Tate group) then all the regular 1-forms are closed, and indeed the whole Hodge to de Rham spectral sequence degenerates at $E_1$. In a subsequent paper [13] we shall further develop the techniques employed in the proof of the above statement, and show how these can be used to prove the Rydakov-Shafarevitch theorem, that $K3$ surfaces have no global vector fields.

We also consider a smooth family of surfaces $f : X \to S$ over an irreducible base scheme of characteristic $p$, here we show that if there is just one fiber $X_s$ with $p$-divisible formal Brauer group then the differential

$$d : f_* \Omega^1_{X/S} \to f_* \Omega^2_{X/S},$$

is zero.

(1) This work was supported in part by the Danish Research Council.
(2) Examples have also been constructed by W. Lang [17].

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1. Some properties of the slope spectral sequence

For the construction and the basic properties of the slope spectral sequence we refer to Bloch [3]. Bloch’s construction has been generalized and the restrictions on the relation between the dimension and characteristic has been removed (Illusie [9]), so the restriction in Bloch’s paper will be ignored.

The notation will be as in [3]; the proof of the properties listed below will appear in [9].

Let F, V and d denote respectively the Frobenius, the Verschiebung and the differential in the pro-complex $C_{\cdot X}$, then:

1. (1.1) $F V = V F = p$.
2. (1.2) $d F = F d$, $V d = p d V$.
3. (1.3) $F d V = d$.
4. (1.4) F, V and p are injective as maps of pro-sheaves i.e. the transition maps in the pro-system of kernels are 0.
5. (1.5) Let $n = \dim X$ then F is an automorphism of the pro-sheaf $C^n_{\cdot X}$.

2. Surfaces over a perfect field

In this section we show that if the formal Brauer group of $X/k$ is pro-representable by a $p$-divisible formal group then the Hodge to de Rham spectral sequence degenerates at $E_1$. If we further assume that $H^2_{\text{cris}}(X/W)$ is torsion free then the Hodge symmetry

$$h^{i,j} = \dim_k H^i(X, \Omega^j_{X/k}) = \dim_k H^i(X, \Omega^j_{X/k}) = h^{i,j},$$

holds as well.

The following proposition has also been proved by Berthelot (private communication) using results of Mazur and Messing.
(2.1) **Proposition.** — Let $X/k$ be a smooth proper variety over a perfect field $k$ of characteristic $p > 0$. Assume that $H^2_{\text{crys}}(X/W)$ is torsion free, then the Picard scheme $\text{Pic}(X)$ is reduced.

**Proof.** — Consider the exact sequence of Zariski sheaves on $X$:

$$0 \to \mathcal{W}_r(\mathcal{O}_X) \to \mathcal{W}_{r+1}(\mathcal{O}_X) \to \mathcal{O}_X \to 0,$$

which gives rise to an exact sequence of finite length $W(k)$-modules

$$\to H^1(X, \mathcal{W}_r(\mathcal{O}_X)) \to H^1(X, \mathcal{W}_{r+1}(\mathcal{O}_X)) \to H^1(X, \mathcal{O}_X) \to H^{1+1}(X, \mathcal{W}_r(\mathcal{O}_X)),$$

and hence (using Mittag-Leffler) an exact sequence of $W(k)$-modules

$$\to H^1(X, \mathcal{W}(\mathcal{O}_X)) \to H^1(X, \mathcal{W}(\mathcal{O}_X)) \to H^1(X, \mathcal{O}_X) \to H^{1+1}(X, \mathcal{W}(\mathcal{O}_X)).$$

By [12], p. 196, $\text{Pic}(X)$ is reduced if and only if the connecting homomorphism, in the exact sequence above, vanishes, this is equivalent to

$$H^1(X, \mathcal{W}(\mathcal{O}_X)) \to H^1(X, \mathcal{O}_X),$$

being surjective.

Define the pro-complex $^*C_{-X}$ by

$$^*C_{-X} = 0 \to C^0_{-X} \to C^1_{-X} \to \ldots \to C^{\dim X}_{-X} \to 0,$$

since

$$dF = (p F) dd \quad \text{by (1.2),}$$

this is indeed a complex.

Now define

$$\tilde{V} : C_{-X} \to ^*C_{-X},$$

by

$$\tilde{V}^i : C^i_{-X} \to ^*C^i_{-X} = \begin{cases} V : C^0_{-X} \to C^0_{-X} & \text{if } i = 0, \\ \text{id} : C^i_{-X} \to C^i_{-X} & \text{if } i > 0. \end{cases}$$

It is clear by (1.3) that $\tilde{V}$ is a map of complexes, and since $C^0_{-X} \simeq \mathcal{W}.(\mathcal{O}_X)$ we get an exact sequence of pro-complexes

$$0 \to C^i_{-X} \to ^*C^i_{-X} \to \mathcal{O}_X \to 0.$$
Passing to hypercohomology we obtain an exact sequence of pro-modules

$$H^i(X, C^\cdot X) \to H^i(X, *C^\cdot X) \to H^i(X, \mathcal{O}_X) \to H^{i+1}(X, C^\cdot X).$$

Since $H^i(X, C^\cdot X)$ has finite length over $\mathcal{O}(k)$ for all $i, j, r ([3], II, Prop. (1.1)) it follows from the hypercohomology spectral sequences that $H^i(X, C)$ and $H^i(X, *C)$ are pro-systems of modules of finite lengths so by Mittag-Leffler we get an exact sequence

$$H^i_{\text{crys}}(X/W) \to \lim H^i(X, *C^\cdot X) \to H^i(X, \mathcal{O}_X) \to H^{i+1}_{\text{crys}}(X/W).$$

Since $H^2_{\text{crys}}(X/W)$ is assumed torsion free the connecting homomorphism

$$H^1(X, \mathcal{O}_X) \to H^2_{\text{crys}}(X/W),$$

in the exact sequence above vanishes, i.e.

$$\lim H^1(X, C^\cdot X) \to H^1(X, \mathcal{O}_X).$$

is surjective.

We have a commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \to & C^\cdot X & \xrightarrow{q} & *C^\cdot X & \to & \mathcal{O}_X & \to & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{O}_X & \to & \mathcal{O}_X & \to & 0.
\end{array}
$$

hence a commutative diagram

$$
\begin{array}{cccccc}
H^1_{\text{crys}}(X/W) & \to & \lim H^1(X, *C^\cdot X) & \to & H^1(X, \mathcal{O}_X) & \to & 0 \\
\downarrow & & \downarrow & & \| & & \\
H^1(X, \mathcal{O}_X) & \xrightarrow{q} & H^1(X, \mathcal{O}_X) & \to & H^1(X, \mathcal{O}_X).
\end{array}
$$

it follows that

$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X),$$

is surjective as desired.

The next proposition was pointed out by the referee, the proof is based on an idea by Deligne.

(2.2) Proposition. — Assume that the differentials in the $E_1$ term of the slope spectral sequence vanish then it degenerates at $E_1$.

Proof. — We show by induction that the differentials in the $E_t$ term vanish so assume that the differentials in the $E_t$ terms $t = 1, \ldots, s - 1$ are zero.
This implies that $E_{i,j}^n = H^j(X, C_X^i)$ for all $i, j$ so we must show that
\[ d : H^j(X, C_X^i) \to H^{j-s+1}(X, C_X^{i+s}) \]
vanishes.

Consider the commutative diagram of pro-complexes
\[
\begin{array}{cccc}
C^0_X & \to & \cdots & \to \ C^i_X \to \cdots & \to \ C^{i+s}_X \to \cdots & \to \ C^{\dim X}_X \to 0, \\
\uparrow^p & & & & \uparrow^v & & \downarrow^\pi \\
C^0_{-1,X} & \to & \cdots & \to \ C^i_{-1,X} \to \cdots & \to \ C^{i+s}_{-1,X} \to \cdots & \to \ C^{\dim X}_{-1,X} \to 0, \\
\downarrow^\delta & & & & \downarrow^s & & \downarrow^m \\
C^i_{-1,X} & \to & \cdots & \to \ C^{i+s}_{-1,X} \to \cdots & \to \ C^{\dim X-0}_{-1,X} \\
\end{array}
\]

Consider the hyper cohomology sequences then we have a commutative diagram
\[
\begin{array}{cccc}
E^i_j(C_X) & \to & \cdots & \to \ E^i_s, j-s+1(C_X), \\
\uparrow^v & & & & \uparrow^\delta \\
E^i_j(C_{-1,X}) & \to & \cdots & \to \ E^i_s, j-s+1(C_{-1,X}), \\
\downarrow^\delta & & & & \downarrow^\pi \\
E^i_j(\mathcal{C}) & \to & \cdots & \to \ E^i_s, j-s+1(\mathcal{C}), \\
\end{array}
\]

Passing to the limit we get a commutative diagram
\[
(2.3)
\begin{array}{cccc}
E^i_j & \to & \cdots & \to \ E^i_s, j-s+1, \\
\downarrow^v & & & & \uparrow^\delta \\
E^i_j(\mathcal{C}) & \to & \cdots & \to \ E^i_s, j-s+1(\mathcal{C}), \\
\downarrow^\delta & & & & \downarrow^\pi \\
E^i_j & \to & \cdots & \to \ E^i_s, j-s+1. \\
\end{array}
\]

If the differentials in the preceding terms vanish $\pi$ and $\delta$ are identities so we have a commutative diagram
\[
\begin{array}{cccc}
H^j(X, C_X) & \to & \cdots & \to \ H^{j-s+1}(X, C_X^{i+s}), \\
\uparrow^v & & & & \downarrow^m \\
H^j(X, C_X) & \to & \cdots & \to \ H^{j-s+1}(X, C_X^{i+s}). \\
\end{array}
\]

By iteration we get
\[ d_s = p^m F^s d_s V^n \text{ for all } n, \text{ hence,} \]
\[ \text{Im } d_s = \bigcap_n p^n H^{j-s+1}(X, C_X^{i+s}) = 0. \]

(2.4) Theorem. — Let $X/k$ be a surface, proper and smooth over $k$ with $k$ perfect of characteristic $p > 0$, then the slope spectral sequence degenerates at $E_1$ if and only if $H^2(X, W^1(\mathcal{O}_X))$ is a finitely generated $W^k(k)$ module.
Proof. Assume that the slope spectral sequence degenerates at $E_1$ then $H^2(X, \mathcal{W}(\mathcal{O}_X))$ is a quotient of $H^2_{\text{tors}}(X/W)$ hence is finitely generated. The proof of the other implication rests on the following Lemma.

(2.5) **Lemma.** Let $d : L \to M$ be a linear map of $\mathcal{W}(k)$ modules. Let $F$ (resp. $V$) be a $\sigma$-linear (resp. $\sigma^{-1}$-linear) endomorphism of $M$ (resp. $L$) [this means $F(\lambda x) = \lambda^n F(x)$ and $V(\lambda y) = \lambda^{-1} V(y)$ where $\lambda \in \mathcal{W}(k)$ and $\sigma$ denotes the frobenius endomorphism of $\mathcal{W}(k)$]. Assume that $L$ and $M$ are topological $\mathcal{W}(k)$ modules, $d$ is continuous, $M$ is separated and the topology on $L$ is weaker than the $V$-topology (i.e. the topology defined by the submodules $\{ V^n L \}$), assume moreover that $F d V = d$. Then if the chains

\[ \ker d \subseteq \ker F d \subseteq \ldots \subseteq \ker F^n d \subseteq \ldots \subseteq L, \]
\[ \text{Im } d \subseteq \text{Im } F d \subseteq \ldots \subseteq \text{Im } F^n d \subseteq \ldots \subseteq M, \]

stabilize one has $d = 0$.

**Proof.** Assume that both chains are stable at the $n$'th level. Let $x \in \ker F^n d$, then $0 = F^n d x = F^{n+1} d V x$ so $V x \in \ker F^{n+1} d = \ker F^n d$ i.e. $\ker F^n d$ is stable under $V$ and so $\ker F^n d$ hence $dx = F^n d V^n x = 0$ and it follows that

\[ \ker d = \ker F d = \ldots = \ker F^n d = \ldots \subseteq L. \]

Now the commutative diagram

\[
\begin{array}{ccc}
L/\ker d & \xrightarrow{F d} & \text{Im } F^n d, \\
\downarrow V & & \downarrow \\
L/\ker d & \xrightarrow{V F d} & \text{Im } F^{n+1} d.
\end{array}
\]

shows that $V$ induces an automorphism on $L/\ker d$ which is equivalent to $\ker d$ being dense in the $V$-topology. Since the original topology on $L$ is weaker than the $V$-topology, $\ker d$ is also dense in the original topology. But $d$ is continuous and $M$ is separated hence $\ker d$ is also closed and so $\ker d = L$.

Let us go back to the proof of the Theorem. By (2.2) it is enough to show that the differentials in the $E_1$ term vanish. The $E_1$ term looks as below:

\[
\begin{align*}
H^2(X, \mathcal{W}(\mathcal{O}_X)) & \xrightarrow{d_{1}^{1}} H^2(X, \mathcal{O}_X) \xrightarrow{d_{1}^{2}} H^2(X, \mathcal{O}_X), \\
H^1(X, \mathcal{W}(\mathcal{O}_X)) & \xrightarrow{d_{1}^{1}} H^1(X, \mathcal{O}_X) \xrightarrow{d_{1}^{2}} H^1(X, \mathcal{O}_X), \\
H^0(X, \mathcal{W}(\mathcal{O}_X)) & \xrightarrow{d_{1}^{0}} H^0(X, \mathcal{O}_X) \xrightarrow{d_{1}^{1}} H^0(X, \mathcal{O}_X).
\end{align*}
\]

Let us first show that the differentials in the bottom row are 0. This follows from the fact (1.4) that $p$ is injective on $H^0(X, \mathcal{O}_X)$ i.e. these modules are torsion free and the slope spectral sequence degenerates at $E_1$ modulo torsion ([3], III (3.2)). Next consider the differentials

\[
d_{1}^{0,i} : H^i(X, \mathcal{W}(\mathcal{O}_X)) \to H^i(X, \mathcal{O}_X), \quad i = 1, 2.
\]
The modules have separated and complete topologies being limits of the discrete spaces $H^i(X, \mathcal{W}_r(\mathcal{O}_X))$ and $H^i(X, \mathcal{C}_r^i)$, clearly $d^{0,1}_i$ is continuous. The relation $F d^{0,1}_i V = d^{0,1}_i$ is satisfied by (1.3) and the exact sequences

$$H^i(X, \mathcal{W}_r(\mathcal{O}_X)) \xrightarrow{\varphi} H^i(X, \mathcal{W}_r(\mathcal{C}_x)) \to H^i(X, \mathcal{W}_r(\mathcal{O}_X)).$$

show that the $\varphi$-topology is finer than the limit topology on $H^i(X, \mathcal{W}_r(\mathcal{O}_X))$ (they are actually identical), so by (2.5) we only have to show that the chains

$$\ker d^{0,1}_i \subset \ker F d^{0,1}_i \subset \ldots \subset \ker F^n d^{0,1}_i \subset \ldots \subset H^i(X, \mathcal{W}_r(\mathcal{C}_x)).$$

$$\operatorname{Im} d^{0,1}_i \subset \operatorname{Im} F d^{0,1}_i \subset \ldots \subset \operatorname{Im} F^n d^{0,1}_i \subset \ldots \subset H^i(X, \mathcal{C}_r^i),$$

stabilize. Now $H^i(X, \mathcal{W}_r(\mathcal{O}_X))$ is finitely generated, for $i = 2$ it is the assumption and for $i = 1$ it is always true as proved in [15], Proposition 4 so the first chain stabilizes.

For the second we have

$$\operatorname{Im} F^n d^{0,1}_i \subset \ker d^{1,1}_i \quad \text{for all } n,$$

so

$$\operatorname{Im} F^n d^{0,1}_i / \operatorname{Im} d^{0,1}_i \subset \ker d^{1,1}_i / \operatorname{Im} d^{1,1}_i = E_2^{1,1},$$

and we have $E_2^{1,1} = E_\infty^{1,1}$ (since $\dim X = 2$) which is a subquotient of $H^i_{\operatorname{cris}}(X/W)$ hence finitely generated so the chain

$$\operatorname{Im} F d^{0,1}_i / \operatorname{Im} d^{0,1}_i \subset \ldots \subset \operatorname{Im} F^n d^{0,1}_i / \operatorname{Im} d^{0,1}_i \subset \ldots \subset E_\infty^{1,1},$$

stabilizes which shows that the second chain is stable.

For the differential

$$d^{1,2} : H^2(X, \mathcal{C}_r^2) \to H^3(X, \mathcal{C}_r^2),$$

we use the fact that $F$ is an automorphism of $H^2(X, \mathcal{C}_r^2)$ to conclude that the chain of kernels stabilize, namely

$$\ker d^{1,2} = \ker F^n d^{1,2} \quad \text{for all } n.$$

The chain of images stabilizes because $H^2(X, \mathcal{C}_r^2) / \operatorname{Im} d^{1,2} = E_2^{2,2} = E_\infty^{2,2}$ is finitely generated. To conclude that $d^{1,2} = 0$ we only need to show that the $\varphi$-topology is finer than the limit topology, this follows however from the commutative diagram

$$\begin{array}{cccc}
H^1(X, \mathcal{C}_r^i) & \xrightarrow{\varphi} & H^i(X, \mathcal{C}_r^i), \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H^i(X, \mathcal{C}_r^i).
\end{array}$$
The only differential left is
\[ d_{1}^{1} : H^{1}(X, C^{1}_{\mathcal{X}}) \rightarrow H^{1}(X, C^{2}_{\mathcal{X}}). \]

The chain of kernels stabilizes for the same reasons as above, and in order to show that the chain of images stabilizes it is enough to show that \( H^{1}(X, C^{2}_{\mathcal{X}})/\text{Im} \, d_{1}^{1} \) is finitely generated, this follows however from the exact sequence
\[
E_{2}^{0, 2} \rightarrow H^{1}(X, C^{2}_{\mathcal{X}}) / \text{Im} \, d_{1}^{1} = E_{2}^{2, 1} \rightarrow E_{3}^{2, 1} = E_{\infty}^{2, 1},
\]
because \( E_{2}^{0, 2} \subset H^{2}(X, \mathcal{W}(\mathcal{O}_{X})) \) is finitely generated. This concludes the proof of the Theorem.

**Remark.** — Some parts of the proof of (2.4) goes through without assuming that \( H^{2}(X, \mathcal{W}(\mathcal{O}_{X})) \) is finitely generated or that \( X \) is a surface; in particular that
\[ H^{1}(X, \mathcal{W}(\mathcal{O}_{X})) = E_{\infty}^{0, 1}, \]
and
\[ H^{0}(X, C^{1}_{\mathcal{X}}) = E_{\infty}^{1, 0}, \]
there results an exact sequence
\[
0 \rightarrow H^{0}(X, C^{1}_{\mathcal{X}}) \rightarrow H^{1}_{\text{crys}}(X/W) \rightarrow H^{1}(X, \mathcal{W}(\mathcal{O}_{X})) \rightarrow 0,
\]
since \( H^{0}(X, C^{1}_{\mathcal{X}}) \) is torsion free by (1.4) and \( H^{1}(X, \mathcal{W}(\mathcal{O}_{X})) \) is torsion free by [15], p. 32, we deduce the well known fact that \( H^{1}_{\text{crys}}(X/W) \) is torsion free.

(2.6) **Corollary.** — Let \( X/k \) be a smooth proper variety and assume that \( \text{Pic}(X) \) is reduced then the differential
\[ d_{1}^{0} : H^{1}(X, \mathcal{O}_{X}) \rightarrow H^{1}(X, \Omega^{1}_{X/k}). \]
vanishes.

**Proof.** — Let \((E, d)\) denote the slope spectral sequence and \((E', d')\) the Hodge to de Rham spectral sequence. Since \( C_{1, X} \approx \Omega^{1}_{X/k} ([3], II (3.1)) \) we have a map of spectral sequences
\[(E, d) \rightarrow (E', d'), \]
in particular a commutative diagram
\[
\begin{array}{ccc}
H^{1}(X, \mathcal{W}(\mathcal{O}_{X})) & \overset{d_{1}^{0}}{\rightarrow} & H^{1}(X, C^{1}_{\mathcal{X}}), \\
\downarrow & & \downarrow \\
H^{1}(X, \mathcal{O}_{X}) & \overset{d_{1}^{0}}{\rightarrow} & H^{1}(X, \Omega^{1}_{X}).
\end{array}
\]
By the remark above the horizontal map on top is zero, and the left hand vertical map is surjective since \( \text{Pic}(X) \) is reduced, hence the corollary.
(2.6) Has also been proved by T. Oda in his Harvard thesis [14].

(2.7) COROLLARY. — Let \( X/k \) be a smooth proper surface. Assume that \( \text{Br}^*_X \) is pro-represented by a \( p \)-divisible formal group then the Hodge to de Rham spectral sequence degenerates at \( E_1 \).

Proof. — By [1], Corollary (4.3), the (covariant) Dieudonné module of \( \text{Br}^*_X \) is \( H^2(X, \mathcal{W}(\mathcal{O}_X)) \), so \( \text{Br}^*_X \) \( p \)-divisible implies that \( H^2(X, \mathcal{W}(\mathcal{O}_X)) \) is finitely generated and free [10], and hence by (2.4) the slope spectral sequence degenerates at \( E_1 \).

Since \( H^2(X, \mathcal{W}(\mathcal{O}_X)) \) is free

\[
H^1(X, \mathcal{W}(\mathcal{O}_X)) \rightarrow H^1(X, \mathcal{O}_X),
\]
is surjective.

\[
H^2(X, \mathcal{W}(\mathcal{O}_X)) \rightarrow H^2(X, \mathcal{O}_X),
\]
is surjective because \( H^3(X, \mathcal{W}(\mathcal{O}_X)) = 0 \), and

\[
H^0(X, \mathcal{W}(\mathcal{O}_X)) \rightarrow H^0(X, \mathcal{O}_X),
\]
because \( H^1(X, \mathcal{W}(\mathcal{O}_X)) \) is free ([15], p. 32) it follows that

\[
d^0_{1,i} : H^i(X, \mathcal{O}_X) \rightarrow H^i(X, \Omega^1_{X/k}),
\]
is zero \( i = 0, 1, 2 \), by Serre duality the rest of the differentials in the \( E_1 \) term vanish. A similar argument shows that the higher differentials vanish as well.

(2.8) PROPOSITION. — With the assumptions of (2.7) assume further that \( H^2_{\text{cris}}(X/W) \) is torsion free then:

(i) \( \dim_k H^i_{\text{DR}}(X/k) = \dim_k H^i_{\text{cris}}(X/W) \otimes K, i = 0, 1, 2, 3, 4 \), where \( K \) is the fraction field of \( \mathcal{W}(k) \);

(ii) \( h^{i,j} = \dim_k H^i(X, \Omega^j_{X/k}) = \dim_k H^i(X, \Omega^j_{X/k}) = h^{i,j} \).

Proof. — (i) follows from the exact sequences:

\[
0 \rightarrow H^i_{\text{cris}}(X/W) \otimes k \rightarrow H^i_{\text{DR}}(X/k) \rightarrow \text{Tor}^W_{i+1}(H^i_{\text{cris}}(X/W), k) \rightarrow 0.
\]

plus the fact that \( H^2_{\text{cris}}(X/W) \) is also torsion free (by Poincaré duality).

To prove (ii) it is enough to show

\[
h^{0,1} = h^{1,0}.
\]
the other equalities then follow from Serre duality.

\[ h^{0,1} = \dim_k H^1(X, \mathcal{O}_X) = \dim \operatorname{Pic}^0(X), \]

since \( \operatorname{Pic}^0(X) \) is reduced.

\[ \dim H^1_{\text{Dr}}(X/k) = h^{0,1} + h^{1,0}, \quad \text{by (2.7)} \]

and

\[ \dim_k H^1_{\text{crys}}(X/W) \otimes K = 2 \dim \operatorname{Pic}^0(X), \]

and the equality follows from (i).

3. Surfaces over an irreducible scheme

In this section we consider a smooth proper \( S \)-scheme \( f : X \to S \) with geometrically irreducible fibers of dimension 2; \( S \) an irreducible \( \mathbb{F}_p \)-scheme such that \( f^* (\mathcal{O}_S) = (\mathcal{O}_S) \).

(3.1) Lemma. — Let \( A \) be a local domain of characteristic \( p \) with maximal ideal \( \mathfrak{m} \) and residue field \( k \). Let \( \hat{A} \) be the completion at \( \mathfrak{m} \) and \( L \) the fraction field of \( \hat{A} \). Assume that \( G = \text{Spf } A[[t_1, \ldots, t_n]] \) is a connected formal Lie group such that \( G_{\hat{A}} \) is \( p \)-divisible, then the formal Lie group \( G_L \) is \( p \)-divisible.

Proof. — Let the power series \( f_1, \ldots, f_n \) define multiplication by \( p \) in \( G \), then \( \ker p : G_L \to G_L \) is represented by \( L[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n) \) and it is enough to show that this is a finite dimensional \( L \)-vectorspace ([6], p. 47). Since \( G_{\hat{A}} = \text{Spf } \hat{k}[[t_1, \ldots, t_n]] \) is \( p \)-divisible \( G_{\hat{A}/\hat{\mathfrak{m}}} = \text{Spf } A/\mathfrak{m}^r [[t_1, \ldots, t_n]] \) is \( p \)-divisible for all \( r \geq 1 \) ([6], p. 62) so \( A/\mathfrak{m}^r [[t_1, \ldots, t_n]]/(f_1, \ldots, f_n) \) is a finitely generated \( A/\mathfrak{m}^r \)-module. Let \( e_1, \ldots, e_s \in A[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n) \) such that

\[ \{ e_1, \ldots, e_s \} \subset k[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n), \]

is a set of generators, it follows from Nakayama’s Lemma that

\[ \{ \tilde{e}_1, \ldots, \tilde{e}_s \} \subset A/\mathfrak{m}^r [[t_1, \ldots, t_n]]/(f_1, \ldots, f_n), \]

generates for all \( r \geq 1 \).

Let \( M \) be the \( A \)-module generated by \( \{ e_1, \ldots, e_s \} \) then

\[ M/\mathfrak{m}^r M = A/\mathfrak{m}^r [[t_1, \ldots, t_n]]/(f_1, \ldots, f_n), \]

\[ \hat{A} [[t_1, \ldots, t_n]]/(f_1, \ldots, f_n) = \lim_{r \to \infty} A/\mathfrak{m}^r [[t_1, \ldots, t_n]]/(f_1, \ldots, f_n) = \lim_{r \to \infty} M/\mathfrak{m}^r M = \hat{M}. \]

Since \( M \) is finitely generated \( \hat{M} = M \otimes \hat{A} \) is finitely generated over \( \hat{A} \) so

\[ \hat{A} [[t_1, \ldots, t_n]]/(f_1, \ldots, f_n). \]
and hence
\[ L[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n). \]
is finitely generated.

(3.2) Theorem. — Assume that there is a closed point \( s_0 \in S \) such that the geometric fibre \( Y = X_{s_0} \) has \( p \)-divisible formal Brauer group, then the differential
\[ d : f_* \Omega^1_{X/S} \to f_* \Omega^2_{X/S}. \]
is zero.

Proof. — By the smoothness of \( f \), \( f_* \Omega^2_{X/S} \) is a locally free sheaf on \( S \) so the set
\[ F = \{ s \in S \mid d_s : (f_* \Omega^1_{X/S})_s \to (f_* \Omega^2_{X/S})_s \text{ is zero } \} \]
is a closed set. We are going to show that the generic points is in \( F \).

\( \text{Pic}^0 X/S \) is representable by [7], Theorem (3.1) and since we have assumed that \( Br_Y \) is pro-representable by a \( p \)-divisible formal group \( H^2(Y, \mathcal{U}((\mathcal{O}_y))) \) is free so \( H^1(Y, \mathcal{O}_y) \to H^2(Y, \mathcal{U}((\mathcal{O}_y))) \) is zero hence \( \text{Pic}^0 (Y) \) is smooth ([12], p. 196).

By [8], Theorem (3.5) there is a non-empty open set \( s_0 \in \mathcal{U} \subset S \) such that \( \text{Pic}^0 X/\mathcal{U} \) is smooth and hence \( Br_{X/\mathcal{U}} \) is representable by a formal group which is formally smooth since the fibre dimension is 2 ([1], Cor. (4.1)).

Let \( \{ G[n] \}_n \) be the inductive system of locally free finite groups associated to the formal Lie group \( Br_{X/\mathcal{U}} \) ([6], Prop. (2.6)). Locally on \( \mathcal{U} \) each \( G[n] \) is isomorphic to \( \text{Spec} \mathcal{O}_y[t_1, \ldots, t_d]/(t_1^{r_1}, \ldots, t_d^{r_d}) \) where \( d \) is the rank of the conormal bundle of \( Br_{X/\mathcal{U}} \) ([6], Prop. (2.1)).

We can assume \( S = \text{Spec} R \) where \( R = \mathcal{O}_{S, s_0} \), hence over \( \text{Spec} R \), \( Br_{X/R} \) is isomorphic to \( \lim G[n] \) with each
\[ G[n] \simeq \text{Spec} R[t_1, \ldots, t_d]/(t_1^{r_1}, \ldots, t_d^{r_d}). \]

Since \( X/R \) is smooth the functor \( Br_{X/R} \) is isomorphic to the sheaf \( R^2 f_* \mathcal{G}_m \) on the big etale site of \( \text{Spec} R \) ([1], Prop. (1.7)). By general theorems about sheaf cohomology ([16], Prop. (5.1)) this implies that the formal Brauer group commutes with all base changes. In terms of the inductive system this means that
\[ Br_{X,Y} = \lim G[n] \otimes_R T \simeq \lim \text{Spec} \mathcal{O}_Y[t_1, \ldots, t_d]/(t_1^{r_1}, \ldots, t_d^{r_d}). \]
for every \( R \)-scheme \( T \).

Let \( \eta \) be the fraction field of \( R \) and \( L \) the fractional field of \( R \), \( \kappa \) is the residue field. By assumption
\[ Br_Y = \lim L[t_1, \ldots, t_d]/(t_1^{r_1}, \ldots, t_d^{r_d}). \]
is $p$-divisible hence (3.1) gives that

$$\text{Br}^{-}_{X_{\mathbb{E}}} = \lim_{\text{Spec} L[t_{1}, \ldots, t_{d}]/(t_{1}^{p}, \ldots, t_{d}^{p}),}$$

is $p$-divisible and so also $\text{Br}^{-}_{X_{\mathbb{E}}}$ is $p$-divisible.

(2.4) Then implies that

$$d : H^{0}(X_{\mathbb{L}}, \Omega^{1}_{X_{\mathbb{E}}}) \to H^{0}(X_{\mathbb{L}}, \Omega^{2}_{X_{\mathbb{E}}}),$$

is zero, and by faithfully flat descent

$$d : H^{0}(X_{\eta}, \Omega^{1}_{X_{\mathbb{E}}}) \to H^{0}(X_{\eta}, \Omega^{2}_{X_{\mathbb{E}}}),$$

is zero which shows $\text{spec} \eta \in F$.

(3.3) COROLLARY. — With the assumptions of (3.2) assume that all the sheaves $R^{j}f_{*}\Omega^{k}_{X/S}$ are locally free on $S$ then the spectral sequence

$$E^{1}_{r} = R^{j}f_{*}\Omega^{k}_{X/S} \Rightarrow H^{*}_{\text{DR}}(X/S),$$

degenerates at $E_{1}$.

Proof. — In this case the set

$$F = \{ s \in S | (R^{j}f_{*}\Omega^{k}_{X/S})_{s} \Rightarrow H^{*}_{\text{DR}}(X/S)_{s} \text{ degenerates at } E_{1} \},$$

is closed and the Proof of (3.2) shows that $F$ contains the generic point.

REFERENCES


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(Manuscrit reçu le 27 février 1978,
révisé le 19 juin 1978.)