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CLOSEDNESS OF REGULAR 1-FORMS ON ALGEBRAIC SURFACES (1)

By Niels O. NYGAARD

Introduction

Let X/k be a proper, smooth surface over a perfect field k. If k has characteristic 0 it follows from Hodge theory and the Lefshetz principle that all regular 1-forms on X are closed, i. e. that the differential

$$d: \quad H^0(X, \Omega^1_{X/k}) \to H^0(X, \Omega^2_{X/k}),$$

vanishes.

In characteristic $p>0$ the situation is more complicated indeed Mumford [11] and more recently Raynaud have constructed surfaces with regular 1-forms which are not closed (2). It therefore becomes interesting to look for conditions on X that will ensure the closedness of regular 1-forms. We relate this question to an invariant defined and studied by Artin and Mazur in [1], the formal Brauer group, $\widehat{Br}_X$, specially we show that if $\widehat{Br}_X$ is pro-representable by a p-divisible formal group (Barsotti-Tate group) then all the regular 1-forms are closed, and indeed the whole Hodge to de Rham spectral sequence degenerates at E_1. In a subsequent paper [13] we shall further develop the techniques employed in the proof of the above statement, and show how these can be used to prove the Rydakov-Shafarevitch theorem, that K 3 surfaces have no global vector fields.

We also consider a smooth family of surfaces $f: X \to S$ over an irreducible base scheme of characteristic p, here we show that if there is just one fiber X_s with p-divisible formal Brauer group then the differential

$$d: \quad f_* \Omega^1_{X/S} \to f_* \Omega^2_{X/S},$$

is zero.

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(2) Examples have also been constructed by W. Lang [17].

Contents

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1. Some properties of the slope spectral sequence

For the construction and the basic properties of the slope spectral sequence we refer to Bloch [3]. Bloch's construction has been generalized and the restrictions on the relation between the dimension and characteristic has been removed (Illusie [9]), so the restriction in Bloch's paper will be ignored.

The notation will be as in [3]; the proof of the properties listed below will appear in [9].

Let F, V and d denote respectively the Frobenius, the Verschiebung and the differential in the pro-complex $C^{\cdot}_{.,X}$, then:

(1.1) $FV = VF = p$.

(1.2) $dF = pFd$, $Vd = pdV$.

(1.3) $FdV = d$.

(1.4) F, V and p are injective as maps of pro-sheaves i.e. the transition maps in the pro-system of kernels are 0.

(1.5) Let $n = \dim X$ then F is an automorphism of the pro-sheaf $C^{n}_{.,X}$.

2. Surfaces over a perfect field

In this section we show that if the formal Brauer group of X/k is pro-representable by a p-divisible formal group then the Hodge to de Rham spectral sequence degenerates at E_1. If we further assume that $H^2_{crys}(X/W)$ is torsion free then the Hodge symmetry

$$h^{i,j} = \dim_k H^j(X, \Omega^i_{X/k}) = \dim_k H^i(X, \Omega^j_{X/k}) = h^{j,i},$$

holds as well.

The following proposition has also been proved by Berthelot (private communication) using results of Mazur and Messing.

(2.1) PROPOSITION. — *Let X/k be a smooth proper variety over a perfect field k of characteristic $p>0$. Assume that $H^2_{crys}(X/W)$ is torsionfree, then the Picard scheme $\underline{\mathrm{Pic}}(X)$ is reduced.*

Proof. — Consider the exact sequence of Zariski sheaves on X:

$$0 \to \mathscr{W}_r(\mathcal{O}_X) \xrightarrow{V} \mathscr{W}_{r+1}(\mathcal{O}_X) \to \mathcal{O}_X \to 0,$$

which gives rise to an exact sequence of finite length $W(k)$-modules

$$\to H^i(X, \mathscr{W}_r(\mathcal{O}_X)) \xrightarrow{V} H^i(X, \mathscr{W}_{r+1}(\mathcal{O}_X)) \to H^i(X, \mathcal{O}_X) \to H^{i+1}(X, \mathscr{W}_r(\mathcal{O}_X)),$$

and hence (using Mittag-Leffler) an exact sequence of $W(k)$-modules

$$\to H^i(X, \mathscr{W}(\mathcal{O}_X)) \xrightarrow{V} H^i(X, \mathscr{W}(\mathcal{O}_X)) \to H^i(X, \mathcal{O}_X) \to H^{i+1}(X, \mathscr{W}(\mathcal{O}_X)).$$

By [12], p. 196, $\underline{\mathrm{Pic}}(X)$ is reduced if and only if the connecting homomorphism, in the exact sequence above, vanishes, this is equivalent to

$$H^1(X, \mathscr{W}(\mathcal{O}_X)) \to H^1(X, \mathcal{O}_X),$$

being surjective.

Define the pro-complex $^*C^{\cdot}_{.,X}$ by

$$^*C^{\cdot}_{.,X} = 0 \to C^0_{.+1,X} \xrightarrow{Fd} C^1_{.,X} \xrightarrow{d} \ldots C^{\dim X}_{.,X} \to 0,$$

since

$$dF\,d = (p\,F)\,dd \quad \text{by (1.2)},$$

this is indeed a complex.

Now define

$$\tilde{V}: \quad C^{\cdot}_{.,X} \to {}^*C^{\cdot}_{.,X},$$

by

$$\tilde{V}^i: \quad C^i_{.,X} \to {}^*C^i_{.,X} = \begin{cases} V: C^0_{.,X} \to C^0_{.+1,X} & \text{if } i=0, \\ \mathrm{id}: C^i_{.,X} \to C^i_{.,X} & \text{if } i>0. \end{cases}$$

It is clear by (1.3) that $\tilde{V}$ is a map of complexes, and since $C^0_{.,X} \simeq \mathscr{W}.(\mathcal{O}_X)$ we get an exact sequence of pro-complexes

$$0 \to C^{\cdot}_{.,X} \xrightarrow{\tilde{V}} {}^*C^{\cdot}_{.,X} \to \mathcal{O}_X \to 0.$$

Passing to hypercohomology we obtain an exact sequence of pro-modules

$$\mathbf{H}^i(\mathrm{X}, \mathrm{C}^{\cdot}_{.,\mathrm{X}}) \to \mathbf{H}^i(\mathrm{X}, {}^*\mathrm{C}^{\cdot}_{.,\mathrm{X}}) \to \mathrm{H}^i(\mathrm{X}, \mathcal{O}_\mathrm{X}) \to \mathbf{H}^{i+1}(\mathrm{X}, \mathrm{C}^{\cdot}_{.,\mathrm{X}}).$$

Since $\mathrm{H}^j(\mathrm{X}, \mathrm{C}^i_{r,\mathrm{X}})$ has finite length over $\mathscr{W}(k)$ for all i, j, r ([3], III, Prop. (1.1)) it follows from the hypercohomology spectral sequences that $\mathbf{H}^j(\mathrm{X}, \mathrm{C}^{\cdot}_{.})$ and $\mathbf{H}^j(\mathrm{X}, {}^*\mathrm{C}^{\cdot}_{.})$ are pro-systems of modules of finite lengths so by Mittag-Leffler we get an exact sequence

$$\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) \to \varprojlim \mathbf{H}^i(\mathrm{X}, {}^*\mathrm{C}^{\cdot}_{.,\mathrm{X}}) \to \mathrm{H}^i(\mathrm{X}, \mathcal{O}_\mathrm{X}) \to \mathrm{H}^{i+1}_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}).$$

Since $\mathrm{H}^2_{\mathrm{crys}}(\mathrm{X}/\mathrm{W})$ is assumed torsion free the connecting homomorphism

$$\mathrm{H}^1(\mathrm{X}, \mathcal{O}_\mathrm{X}) \to \mathrm{H}^2_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}),$$

in the exact sequence above vanishes, i. e.

$$\varprojlim \mathbf{H}^1(\mathrm{X}, \mathrm{C}_{.,\mathrm{X}}) \to \mathrm{H}^1(\mathrm{X}, \mathcal{O}_\mathrm{X}),$$

is surjective.

We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to & \mathrm{C}^{\cdot}_{.,\mathrm{X}} & \xrightarrow{\tilde{\mathrm{V}}} & {}^*\mathrm{C}^{\cdot}_{.,\mathrm{X}} & \to & \mathcal{O}_\mathrm{X} & \to 0 \\
& \downarrow & & \downarrow & & \| & \\
0 \to & \mathscr{W}_{.}(\mathcal{O}_\mathrm{X}) & \xrightarrow{\mathrm{V}} & \mathscr{W}_{.+1}(\mathrm{X}) & \to & \mathcal{O}_\mathrm{X} & \to 0.
\end{array}$$

hence a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{H}^1_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) & \to & \varprojlim \mathbf{H}^1(\mathrm{X}, {}^*\mathrm{C}^{\cdot}_{.,\mathrm{X}}) & \to & \mathrm{H}^1(\mathrm{X}, \mathcal{O}_\mathrm{X}) & \to 0 \\
\downarrow & & \downarrow & & \| & \\
\mathrm{H}^1(\mathrm{X}, \mathscr{W}(\mathcal{O}_\mathrm{X})) & \xrightarrow{\mathrm{V}} & \mathrm{H}^1(\mathrm{X}, \mathscr{W}(\mathcal{O}_\mathrm{X})) & \to & \mathrm{H}^1(\mathrm{X}, \mathcal{O}_\mathrm{X}), &
\end{array}$$

it follows that

$$\mathrm{H}^1(\mathrm{X}, \mathscr{W}(\mathcal{O}_\mathrm{X})) \to \mathrm{H}^1(\mathrm{X}, \mathcal{O}_\mathrm{X}),$$

is surjective as desired.

The next proposition was pointed out by the referee, the proof is based on an idea by Deligne.

(2.2) PROPOSITION. — *Assume that the differentials in the* E_1 *term of the slope spectral sequence vanish then it degenerates at* E_1.

Proof. — We show by induction that the differentials in the E_s term vanish so assume that the differentials in the E_t terms $t=1, \ldots, s-1$ are zero.

This implies that $E_s^{i,j} = H^j(X, C_X^i)$ for all i, j so we must show that

$$d: \quad H^j(X, C_X^i) \to H^{j-s+1}(X, C_X^{i+s}),$$

vanishes.

Consider the commutative diagram of pro-complexes

$$\begin{array}{ccccccccccccccc}
C^{\cdot}_{\cdot,X} & = 0 \to & C^0_{\cdot,X} & \xrightarrow{d} \dots \to & C^i_{\cdot,X} & \xrightarrow{d} & C^{i+1}_{\cdot,X} & \xrightarrow{d} \dots \xrightarrow{d} & C^{i+s}_{\cdot,X} & \to \cdot \to & C^{\dim X}_{\cdot,X} & \to 0, \\
 & & \uparrow p^i V & & \uparrow V & & \| & & \| & & \| & \\
\tilde{C}^{\cdot}_{\cdot,X} & = 0 \to & C^0_{\cdot-1,X} & \xrightarrow{d} \dots \to & C^i_{\cdot-1,X} & \xrightarrow{dV} & C^{i+1}_{\cdot,X} & \xrightarrow{d} \dots \xrightarrow{d} & C^{i+s}_{\cdot,X} & \to \cdot \to & C^{\dim X}_{\cdot,X} & \to 0, \\
 & & \| & & \| & & \downarrow F & & \downarrow p^s F & & \downarrow p^{(\dim X - i)} F & \\
C^{\cdot}_{\cdot-1,X} & = 0 \to & C^0_{\cdot-1,X} & \to \dots \to & C^i_{\cdot-1,X} & \to & C^{i+1}_{\cdot-1,X} & \to \dots \to & C^{i+s}_{\cdot-1,X} & \to \sim \to & C^{\dim X}_{\cdot-1,X} & \to 0.
\end{array}$$

Consider the hyper cohomology sequences then we have a commutative diagram

$$\begin{array}{ccc}
E_s^{i,j}(C^{\cdot}_{\cdot,X}) & \xrightarrow{d_s} & E_s^{i+s,j-s+1}(C^{\cdot}_{X}), \\
\uparrow V & & \uparrow \pi \\
E_s^{i,j}(\tilde{C}^{\cdot}_{\cdot,X}) & \xrightarrow{d_s} & E_s^{i+s,j-s+1}(\tilde{C}^{\cdot}_{\cdot,X}), \\
\downarrow \delta & & \downarrow p^s F \\
E_s^{i,j}(C^{\cdot}_{\cdot-1,X}) & \xrightarrow{d_s} & E_s^{i+s,j-s+1}(C^{\cdot}_{\cdot-1,X}).
\end{array}$$

Passing to the limit we get a commutative diagram

(2.3)
$$\begin{array}{ccc}
E_s^{i,j} & \xrightarrow{d_s} & E_s^{i+s,j-s+1}, \\
\downarrow V & & \uparrow \pi \\
E_s^{i,j}(\tilde{C}) & \xrightarrow{d_s} & E_s^{i+s,j-s+1}(\tilde{C}), \\
\downarrow \delta & & \downarrow p^s F \\
E_s^{i,j} & \xrightarrow{d_s} & E_s^{i+s,j-s+1}.
\end{array}$$

If the differentials in the preceding terms vanish π and δ are identities so we have a commutative diagram

$$\begin{array}{ccc}
H^j(X, C_X^i) & \xrightarrow{d_s} & H^{j-s+1}(X, C_X^{i+s}), \\
\uparrow V & & \downarrow p^s F \\
H^j(X, C_X^i) & \xrightarrow{d_s} & H^{j-s+1}(X, C_X^{i+s}).
\end{array}$$

By iteration we get

$$d_s = p^{sn} F^n d_s V^n \quad \text{for all } n, \text{ hence,}$$

$$\operatorname{Im} d_s \subset \bigcap_n p^n H^{j-s+1}(X, C_X^{i+s}) = 0.$$

(2.4) THEOREM. — *Let X/k be a surface, proper and smooth over k with k perfect of characteristic $p>0$, then the slope spectral sequence degenerates at E_1 if and only if $H^2(X, \mathscr{W}(\mathcal{O}_X))$ is a finitely generated $\mathscr{W}(k)$ module.*

Proof. — Assume that the slope spectral sequence degenerates at E_1 then $H^2(X, \mathscr{W}(\mathscr{O}_X))$ is a quotient of $H^2_{crys}(X/W)$ hence is finitely generated. The proof of the other implication rests on the following Lemma.

(2.5) LEMMA. — *Let $d : L \to M$ be a linear map of $\mathscr{W}(k)$ modules. Let* F (resp. V) *be a* σ-*linear* (resp. σ^{-1}-linear) *endomorphism of* M (resp. L) [*this means* $F(\lambda x)=\lambda^{\sigma} F(x)$ *and* $V(\lambda y)=\lambda^{\sigma^{-1}} V(y)$ *where* $\lambda \in \mathscr{W}(k)$ *and* σ *denotes the frobenius endomorphism of* $\mathscr{W}(k)$]. *Assume that* L *and* M *are topological* $\mathscr{W}(k)$ *modules, d is continuous,* M *is separated and the topology on* L *is weaker than the* V-*topology* (i.e. *the topology defined by the submodules* $\{V^n L\}$), *assume moreover that* $F d V = d$. *Then if the chains*

$$\ker d \subset \ker F d \subset \ldots \subset \ker F^n d \subset \ldots \subset L,$$

$$\operatorname{Im} d \subset \operatorname{Im} F d \subset \ldots \subset \operatorname{Im} F^n d \subset \ldots \subset M,$$

stabilize one has $d=0$.

Proof. — Assume that both chains are stable at the n'th level. Let $x \in \operatorname{Ker} F^n d$, then $0=F^n dx=F^{n+1} dVx$ so $Vx \in \ker F^{n+1} d = \ker F^n d$ i.e. $\ker F^n d$ is stable under V and so $V^n x \in \ker F^n d$ hence $dx = F^n d V^n x = 0$ and it follows that

$$\ker d = \ker F d = \ldots = \ker F^n d = \ldots \subset L.$$

Now the commutative diagram

$$\begin{array}{ccc} L/\ker d & \xrightarrow[\sim]{F^n d} & \operatorname{Im} F^n d, \\ \downarrow V & & \| \\ L/\ker d & \xrightarrow[\sim]{F^{n+1} d} & \operatorname{Im} F^{n+1} d, \end{array}$$

shows that V induces an automorphism on $L/\ker d$ which is equivalent to $\ker d$ being dense in the V-topology. Since the original topology on L is weaker than the V-topology, $\ker d$ is also dense in the original topology. But d is continuous and M is separated hence $\ker d$ is also closed and so $\ker d = L$.

Let us go back to the proof of the Theorem. By (2.2) it is enough to show that the differentials in the E_1 term vanish. The E_1 term looks as below:

$$\begin{array}{ccccc} H^2(X, \mathscr{W}(\mathscr{O}_X)) & \xrightarrow{d_1^{0,2}} & H^2(X, C_X^1) & \xrightarrow{d_1^{1,2}} & H^2(X, C_X^2), \\ H^1(X, \mathscr{W}(\mathscr{O}_X)) & \xrightarrow{d_1^{0,1}} & H^1(X, C_X^1) & \xrightarrow{d_1^{1,1}} & H^1(X, C_X^2), \\ H^0(X, \mathscr{W}(\mathscr{O}_X)) & \xrightarrow{d_1^{0,0}} & H^0(X, C_X^1) & \xrightarrow{d_1^{1,0}} & H^0(X, C_X^2). \end{array}$$

Let us first show that the differentials in the bottom row are 0. This follows from the fact (1.4) that p is injective on $H^0(X, C_X^i)$ i.e. these modules are torsion free and the slope spectral sequence degenerates at E_1 modulo torsion ([3], III (3.2)). Next consider the differentials

$$d_1^{0,i} : \; H^i(X, \mathscr{W}(\mathscr{O}_X)) \to H^i(X, C_X^1), \qquad i=1, 2.$$

The modules have separated and complete topologies being limits of the discrete spaces $H^i(X, \mathscr{W}_r(\mathscr{O}_X))$ and $H^i(X, C^1_{r,X})$, clearly $d_1^{0,i}$ is continuous. The relation $F d_1^{0,i} V = d_1^{0,i}$ is satisfied by (1.3) and the exact sequences

$$H^i(X, \mathscr{W}(\mathscr{O}_X)) \xrightarrow{V^r} H^i(X, \mathscr{W}(\mathscr{O}_X)) \to H^i(X, \mathscr{W}_r(\mathscr{O}_X)),$$

show that the V-topology is finer than the limit topology on $H^i(X, \mathscr{W}(\mathscr{O}_X))$ (they are actually identical), so by (2.5) we only have to show that the chains

$$\ker d_1^{0,i} \subset \ker F d_1^{0,i} \subset \ldots \subset \ker F^n d_1^{0,i} \subset \ldots \subset H^i(X, \mathscr{W}(\mathscr{O}_X)),$$
$$\operatorname{Im} d_1^{0,i} \subset \operatorname{Im} F d_1^{0,i} \subset \ldots \subset \operatorname{Im} F^n d_1^{0,i} \subset \ldots \subset H^i(X, C^1_X),$$

stabilize. Now $H^i(X, \mathscr{W}(\mathscr{O}_X))$ is finitely generated, for $i=2$ it is the assumption and for $i=1$ it is always true as proved in [15], Proposition 4 so the first chain stabilizes.

For the second we have

$$\operatorname{Im} F^n d_1^{0,i} \subset \ker d_1^{1,i} \quad \text{for all } n,$$

so

$$\operatorname{Im} F^n d_1^{0,i} / \operatorname{Im} d_1^{0,i} \subset \ker d_1^{1,i} / \operatorname{Im} d_1^{1,i} = E_2^{1,i},$$

and we have $E_2^{1,i} = E_\infty^{1,i}$ (since $\dim X = 2$) which is a subquotient of $H^{i+1}_{\text{crys}}(X/W)$ hence finitely generated so the chain

$$\operatorname{Im} F d_1^{0,i} / \operatorname{Im} d_1^{0,i} \subset \ldots \subset \operatorname{Im} F^n d_1^{0,i} / \operatorname{Im} d_1^{0,i} \subset \ldots \subset E_\infty^{1,i},$$

stabilizes which shows that the second chain is stable.

For the differential

$$d_1^{1,2} : \quad H^2(X, C^1_X) \to H^2(X, C^2_X),$$

we use the fact that F is an automorphism of $H^2(X, C^2_X)$ to conclude that the chain of kernels stabilize, namely

$$\ker d_1^{1,2} = \ker F^n d_1^{1,2} \quad \text{for all } n.$$

The chain of images stabilizes because $H^2(X, C^2_X)/\operatorname{Im} d_1^{1,2} = E_2^{2,2} = E_\infty^{2,2}$ is finitely generated. To conclude that $d_1^{1,2} = 0$ we only need to show that the V-topology is finer than the limit topology, this follows however from the commutative diagram

$$\begin{array}{ccc} H^1(X, C^i_X) & \xrightarrow{V^r} & H^j(X, C^i_X), \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & H^j(X, C^i_{r,X}). \end{array}$$

The only differential left is

$$d_1^{1,1}: \quad H^1(X, C_X^1) \to H^1(X, C_X^2).$$

The chain of kernels stabilizes for the same reasons as above, and in order to show that the chain of images stabilizes it is enough to show that $H^1(X, C_X^2)/\mathrm{Im}\, d_1^{1,1}$ is finitely generated, this follows however from the exact sequence

$$E_2^{0,2} \xrightarrow{d_2^{0,2}} H^1(X, C_X^2)/\mathrm{Im}\, d_1^{1,1} = E_2^{2,1} \to E_3^{2,1} = E_\infty^{2,1},$$

because $E_2^{0,2} \subset H^2(X, \mathscr{W}(\mathscr{O}_X))$ is finitely generated. This concludes the proof of the Theorem.

Remark. – Some parts of the proof of (2.4) goes through without assuming that $H^2(X, \mathscr{W}(\mathscr{O}_X))$ is finitely generated or that X is a surface; in particular that

$$H^1(X, \mathscr{W}(\mathscr{O}_X)) = E_\infty^{0,1},$$

and

$$H^0(X, C_X^1) = E_\infty^{1,0},$$

there results an exact sequence

$$0 \to H^0(X, C_X^1) \to H^1_{\text{crys}}(X/W) \to H^1(X, \mathscr{W}(\mathscr{O}_X)) \to 0,$$

since $H^0(X, C_X^1)$ is torsion free by (1.4) and $H^1(X, \mathscr{W}(\mathscr{O}_X))$ is torsion free by [15], p. 32, we deduce the well known fact that $H^1_{\text{crys}}(X/W)$ is torsion free.

(2.6) Corollary. – *Let X/k be a smooth proper variety and assume that* $\underline{\mathrm{Pic}}$ (X) *is reduced then the differential*

$$d_1^{0,1}: \quad H^1(X, \mathscr{O}_X) \to H^1(X, \Omega^1_{X/k}),$$

vanishes.

Proof. – Let (E, d) denote the slope spectral sequence and (E′, d′) the Hodge to de Rham spectral sequence. Since $C^{\cdot}_{1,X} \simeq \Omega^{\cdot}_{X/k}$ ([3], II (3.1)) we have a map of spectral sequences

$$(E, d) \to (E', d'),$$

in particular a commutative diagram

$$\begin{array}{ccc} H^1(X, \mathscr{W}(\mathscr{O}_X)) & \xrightarrow{d_1^{0,1}} & H^1(X, C_X^1), \\ \downarrow & & \downarrow \\ H^1(X, \mathscr{O}_X)) & \xrightarrow{d_1'^{0,1}} & H^1(X, \Omega_X^1). \end{array}$$

By the remark above the horizontal map on top is zero, and the left hand vertical map is surjective since $\underline{\mathrm{Pic}}$ (X) is reduced, hence the corollary.

(2.6) Has also been proved by T. Oda in his Harvard thesis [14].

(2.7) COROLLARY. — *Let X/k be a smooth proper surface. Assume that $\hat{Br}_X$ is pro-represented by a p-divisible formal group then the Hodge to de Rham spectral sequence degenerates at* E_1.

Proof. — By [1], Corollary (4.3), the (covariant) Dieudonné module of $\hat{Br}_X$ is $H^2(X, \mathscr{W}(\mathscr{O}_X))$ so $\hat{Br}_X$ p-divisible implies that $H^2(X, \mathscr{W}(\mathscr{O}_X))$ is finitely generated and free [10], and hence by (2.4) the slope spectral sequence degenerates at E_1.

Since $H^2(X, \mathscr{W}(\mathscr{O}_X))$ is free

$$H^1(X, \mathscr{W}(\mathscr{O}_X)) \to H^1(X, \mathscr{O}_X),$$

is surjective.

$$H^2(X, \mathscr{W}(\mathscr{O}_X)) \to H^2(X, \mathscr{O}_X),$$

is surjective because $H^3(X, \mathscr{W}(\mathscr{O}_X))=0$, and

$$H^0(X, \mathscr{W}(\mathscr{O}_X)) \to H^0(X, \mathscr{O}_X),$$

because $H^1(X, \mathscr{W}(\mathscr{O}_X))$ is free ([15], p. 32) it follows that

$$d_1^{0,i}: \quad H^i(X, \mathscr{O}_X) \to H^i(X, \Omega^1_{X/k}),$$

is zero $i=0, 1, 2$, by Serre duality the rest of the differentials in the E_1 term vanish. A similar argument shows that the higher differentials vanish as well.

(2.8) PROPOSITION. — *With the assumptions of* (2.7) *assume further that* $H^2_{crys}(X/W)$ *is torsion free then:*

(i) $\dim_k H^i_{DR}(X/k) = \dim_K H^i_{crys}(X/W) \otimes K$, $i=0, 1, 2, 3, 4$, *where* K *is the fraction field of* $\mathscr{W}(k)$;

(ii) $h^{i,j} = \dim_k H^j(X, \Omega^i_{X/k}) = \dim_k H^i(X, \Omega^j_{X/k}) = h^{j,i}$.

Proof. — (i) follows from the exact sequences:

$$0 \to H^i_{crys}(X/W) \otimes k \to H^i_{DR}(X/k) \to \mathrm{Tor}_1^{W(k)}(H^{i+1}_{crys}(X/W), k) \to 0,$$

plus the fact that $H^3_{crys}(X/W)$ is also torsion free (by Poincaré duality).

To prove (ii) it is enough to show

$$h^{0,1} = h^{1,0},$$

the other equalities then follow from Serre duality.

$$h^{0,1}=\dim_k H^1(X, \mathcal{O}_X)=\dim \underline{\mathrm{Pic}}^0(X),$$

since $\underline{\mathrm{Pic}}^0(X)$ is reduced.

$$\dim H^1_{DR}(X/k)=h^{0,1}+h^{1,0}, \qquad \text{by (2.7)}$$

and

$$\dim_K H^1_{crys}(X/W)\otimes K=2 \dim \underline{\mathrm{Pic}}^0(X),$$

and the equality follows from (i).

3. Surfaces over an irreducible scheme

In this section we consider a smooth proper S-scheme $f: X \to S$ with geometrically irreducible fibers of dimension 2; S an irreducible $\mathbf{F}_p$-scheme such that $f_* \mathcal{O}_X=\mathcal{O}_S$.

(3.1) LEMMA. — *Let* A *be a local domain of characteristic p with maximal ideal* $\mathcal{M}$ *and residue field k. Let* $\hat{A}$ *be the completion at* $\mathcal{M}$ *and* L *the fraction field of* $\hat{A}$. *Assume that* $G=\mathrm{Spf}\, A[[t_1, \ldots, t_n]]$ *is a connected formal Lie group such that* $G_{\bar{k}}$ *is p-divisible, then the formal Lie group* G_L *is p-divisible.*

Proof. — Let the power series $f_1, \ldots, f_n$ define multiplication by p in G, then $\ker p: G_L \to G_L$ is represented by $L[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n)$ and it is enough to show that this is a finite dimensional L-vectorspace ([6], p. 47). Since $G_{\bar{k}}=\mathrm{Spf}\, \bar{k}[[t_1, \ldots, t_n]]$ is p-divisible $G_{A/\mathcal{M}^r}=\mathrm{Spf}\, A/\mathcal{M}^r[[t_1, \ldots, t_n]]$ is p-divisible for all $r \geqq 1$ ([6], p. 62) so $A/\mathcal{M}^r[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n)$ is a finitely generated $A/\mathcal{M}^r$-module. Let $e_1, \ldots, e_s \in A[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n)$ such that

$$\{\bar{e}_1, \ldots, \bar{e}_s\} \subset k[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n),$$

is a set of generators, it follows from Nakayama's Lemma that

$$\{\bar{e}_1, \ldots, \bar{e}_s\} \subset A/\mathcal{M}^r[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n),$$

generates for all $r \geqq 1$.

Let M be the A-module generated by $\{e_1, \ldots, e_s\}$ then

$$M/\mathcal{M}^r M=A/\mathcal{M}^r[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n),$$

$$\hat{A}[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n)=\varprojlim A/\mathcal{M}^r[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n)=\varprojlim M/\mathcal{M}^r M=\hat{M}.$$

Since M is finitely generated $\hat{M}=M\otimes\hat{A}$ is finitely generated over $\hat{A}$ so

$$\hat{A}[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n),$$

and hence

$$L[[t_1, \ldots, t_n]]/(f_1, \ldots, f_n),$$

is finitely generated.

(3.2) THEOREM. — *Assume that there is a closed point $s_0 \in S$ such that the geometric fibre $Y = X_{\bar{s}_0}$ has p-divisible formal Brauer group, then the differential*

$$d: \ f_* \Omega^1_{X/S} \to f_* \Omega^2_{X/S},$$

is zero.

Proof. — By the smoothness of f, $f_* \Omega^2_{X/S}$ is a locally free sheaf on S so the set

$$F = \{ s \in S \mid d_s : (f_* \Omega^1_{X/S})_s \to (f_* \Omega^2_{X/S})_s \text{ is zero} \},$$

is a closed set. We are going to show that the generic points is in F.

$\underline{\mathrm{Pic}}^0\, X/S$ is representable by [7], Theorem (3.1) and since we have assumed that Br_Y is pro-representable by a p-divisible formal group $H^2(Y, \mathscr{W}(\mathcal{O}_Y))$ is free so $H^1(Y, \mathcal{O}_Y) \to H^2(Y, \mathscr{W}(\mathcal{O}_Y))$ is zero hence $\underline{\mathrm{Pic}}^0(Y)$ is smooth ([12], p. 196).

By [8], Theorem (3.5) there is a non-empty open set $s_0 \in \mathscr{U} \subset S$ such that $\underline{\mathrm{Pic}}^0\, X/\mathscr{U}$ is smooth and hence $\hat{\mathrm{Br}}_{X/\mathscr{U}}$ is representable by a formal group which is formally smooth since the fibre dimension is 2 ([1], Cor. (4.1)).

Let $\{ G[n] \}_n$ be the inductive system of locally free finite groups associated to the formal Lie group $\hat{\mathrm{Br}}_{X/\mathscr{U}}$ ([6], Prop. (2.6)). Locally on $\mathscr{U}$ each $G[n]$ is isomorphic to Spec $\mathcal{O}_{\mathscr{U}}[t_1, \ldots, t_d]/(t_1^{p^n}, \ldots, t_d^{p^n})$ where d is the rank of the conormal bundle of $\hat{\mathrm{Br}}_{X/\mathscr{U}}$ ([6], Prop. (2.1)).

We can assume $S = \mathrm{Spec}\, R$ where $R = \mathcal{O}_{S, s_0}$, hence over Spec R, $\hat{\mathrm{Br}}_{X/R}$ is isomorphic to $\underline{\lim}\, G[n]$ with each

$$G[n] \simeq \mathrm{Spec}\, R[t_1, \ldots, t_d]/(t_1^{p^n}, \ldots, t_d^{p^n}).$$

Since X/R is smooth the functor $\hat{\mathrm{Br}}_{X/R}$ is isomorphic to the sheaf $R^2 f_* \hat{\mathbf{G}}_m$ on the big etale site of Spec R ([1], Prop. (1.7)). By general theorems about sheaf cohomology ([16], Prop. (5.1)) this implies that the formal Brauer group commutes with all base changes. In terms of the inductive system this means that

$$\hat{\mathrm{Br}}_{X_T, T} \simeq \varinjlim G[n] \otimes_R T \simeq \varinjlim \mathrm{Spec}\, \mathcal{O}_T[t_1, \ldots, t_d]/(t_1^{p^n}, \ldots, t_d^{p^n}),$$

for every R-scheme T.

Let η be the fraction field of R and L the fractional field of $\hat{R}$, $\varkappa$ is the residue field. By assumption

$$\hat{\mathrm{Br}}_Y = \varinjlim \bar{\varkappa}[t_1, \ldots, t_d]/(t_1^{p^n}, \ldots, t_d^{p^n}),$$

is p-divisible hence (3.1) gives that

$$\mathrm{Br}_{\mathrm{X_L}}^{\wedge} = \varinjlim \operatorname{Spec} \mathrm{L}[t_1, \ldots, t_d]/(t_1^{p^n}, \ldots, t_d^{p^n}),$$

is p-divisible and so also $\mathrm{Br}_{\mathrm{X}_{\bar{\mathrm{L}}}}^{\wedge}$ is p-divisible.

(2.4) Then implies that

$$d: \quad \mathrm{H}^0(\mathrm{X}_{\bar{\mathrm{L}}}, \Omega^1_{\mathrm{X}_{\bar{\mathrm{L}}}}) \to \mathrm{H}^0(\mathrm{X}_{\bar{\mathrm{L}}}, \Omega^2_{\mathrm{X}_{\bar{\mathrm{L}}}}),$$

is zero, and by faithfully flat descent

$$d: \quad \mathrm{H}^0(\mathrm{X}_\eta, \Omega^1_{\mathrm{X}_\eta}) \to \mathrm{H}^0(\mathrm{X}_\eta, \Omega^2_{\mathrm{X}_\eta}),$$

is zero which shows spec $\eta \in \mathrm{F}$.

(3.3) COROLLARY. — *With the assumptions of* (3.2) *assume that all the sheaves* $\mathrm{R}^j f_* \Omega^i_{\mathrm{X/S}}$ *are locally free on* S *then the spectral sequence*

$$\mathrm{E}_1^{i,j} = \mathrm{R}^j f_* \Omega^i_{\mathrm{X/S}} \quad \Rightarrow \quad \mathrm{H}^*_{\mathrm{DR}}(\mathrm{X/S}),$$

degenerates at E_1.

Proof. — In this case the set

$$\mathrm{F} = \{ s \in \mathrm{S} \mid (\mathrm{R}^j f_* \Omega^i_{\mathrm{X/S}})_s \Rightarrow \mathrm{H}^*_{\mathrm{DR}}(\mathrm{X/S})_s \text{ degenerates at } \mathrm{E}_1 \},$$

is closed and the Proof of (3.2) shows that F contains the generic point.

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