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ON LENS SPACES WHICH ARE ISOSPECTRAL BUT NOT ISOMETRIC

By Akira IKEDA

Introduction

A compact connected Riemannian manifold of positive constant curvature 1 with cyclic fundamantal group is called a lens space. The purpose of this paper is to show that there are many pairs of lens spaces which are isospectral but not isometric and also to give explicit examples of lens spaces which are isospectral to each other but not even homotopy equivalent.

In this paper, we consider only compact Riemannian manifolds. One of the most important differential operators arising from the Riemannian geometry is Laplacian Δ acting on the space of smooth functions on a Riemannian manifold. It is well known that the operator Δ has a discrete spectrum consisting non-negative eigenvalues with finite multiplicities. Riemannian manifolds M and N are said to be isospectral to each other if their spectra are identical. The fundamental problem of the spectrum of Δ is;

(0.1) Does the spectrum on a Riemannian manifold M determine the Riemannian structure on M?

The famous negative example for the above problem was constructed by Milnor [9]; there exist two flat 16-dimensional tori which are isospectral but not isometric. Recently, negative examples for the above problem were constructed for certain compact Riemannian surfaces by Vignéras [13]. These examples are diffeomorphic and also homeomorphic examples. The negative examples for the above problem (0.1) which we shall give in this paper are not even homeomorphic. Moreover, we shall give examples which are isospectral but not even homotopy equivalent. This shows that the spectrum of Laplacian on a Riemannian manifold M does not determine the topological structure on M in general and does not determine even the homotopy class of M.

Remark 1. — It is known that the following Riemannian manifolds are completely characterized by their spectra as Riemannian manifolds:

(i) the standard n-dimensional sphere ($n \leqq 6$) and the standard n-dimensional real projective space ($n \leqq 6$) ([1], [12]);

(ii) compact connected 3-dimensional Riemannian manifolds of positive constant curvature ([6], [7], [11], [14]);

(iii) compact connected 5-dimensional Riemannian manifolds of positive constant curvature whose fundamental groups are non-cyclic [8].

Remark 2. — From the computations of Ray for analytic torsion of lens spaces in [10], we know the following: let L_1 and L_2 be $(2n-1)$-dimensional lens spaces with the same fundamental groups of order q, Z_q. For each irreducible unitary representation α of Z_q, one has the associated locally flat line bundles $E_1 \to L_1$, $E_2 \to L_2$. Suppose that the corresponding Laplacians $\Delta_{1,p}$, $\Delta_{2,p}$ acting on each p-forms ($0 \leqq p \leqq 2n-1$) with values these bundles are isospectral for all p. Then they are isometric.

1. Preliminaries

Let q be a positive integer. We denote by Z_q the ring of residues modulus q of integers and K_q the multiplicative group of residues modulus q of integers prime to q. The order of K_q is denoted by $\varphi(q)$, the Euler's function. Put $q_0 = \varphi(q)/2$. Throughout this paper we always assume $q_0 \geqq 4$. For any positive integer n with $2 \leqq n \leqq (q_0 - 2)$, the set $\tilde{I}(q, n)$ is the set of n-tuples $(p_1, \ldots, p_n)$ of integers prime to q, and put

$$(1.1) \qquad \tilde{I}_0(q, n) = \{(p_1, \ldots, p_n) \in \tilde{I}(q, n) \mid p_i \not\equiv \pm p_j (\bmod q)\ 1 \leqq i < j \leqq n\}.$$

We introduce an equivalence relation in $\tilde{I}(q, n)$ as follows: $(p_1, \ldots, p_n)$ is equivalent to $(s_1, \ldots, s_n)$ if and only if there is a number l and there are numbers $e_i \in \{-1, 1\}$ such that $(p_1, \ldots, p_n)$ is a permutation of $(e_1 l s_1, \ldots, e_n l s_n)$ (mod q).

This equivalence relation also defines an equivalence relation in $\tilde{I}_0(q, n)$. The quotient sets of $\tilde{I}(q, n)$ and $\tilde{I}_0(q, n)$ by the relation are denoted by $I(q, n)$ and $I_0(q, n)$, respectively.

Throughout this paper we use the same notation for an element of $\tilde{I}(q, n)$ [resp. $\tilde{I}_0(q, n)$] and its equivalence class in $I(q, n)$ [resp. $I_0(q, n)$]. Put $k = q_0 - n$. We shall define the map ω of $I_0(q, n)$ into $I_0(q, k)$:

$$(1.2) \qquad \omega: \quad I_0(q, n) \to I_0(q, k),$$

as follows: for any element $(p_1, \ldots, p_n) \in \tilde{I}_0(q, n)$, we choose an element $(q_1, \ldots, q_k) \in \tilde{I}_0(q, k)$ such that the set of integers

$$\{p_1, -p_1, \ldots, p_n, -p_n, q_1, -q_1, \ldots, q_k, -q_k\}$$

forms a complete set of incongruent residues prime to q. Then we define

$$(1.3) \qquad \omega((p_1, \ldots, p_n)) = (q_1, \ldots, q_k).$$

It is easy to see that the map ω is well defined and gives a one to one onto mapping of $\mathrm{I}_0(q, n)$ onto $\mathrm{I}_0(q, k)$:

$$(1.4)\qquad \omega:\ \mathrm{I}_0(q, n) \simeq \mathrm{I}_0(q, k).$$

For a finite set A, we denote by $|\mathrm{A}|$ the number of elements of A. For non-negative integers a and b with $a \geqq b$, we put

$$\binom{a}{b}=1 \quad \text{if } ab=0, \qquad = \frac{a(a-1)\dots(a-b+1)}{b(b-1)\dots 2.1} \quad \text{otherwise.}$$

PROPOSITION 1.1. — *Let* $\mathrm{I}_0(q, n)$ *be as in the above. Then*

$$|\mathrm{I}_0(q, n)| \geqq \frac{1}{q_0}\binom{q_0}{n}.$$

Proof. — Let $\tilde{\mathrm{I}}'_0(q, n)$ be the subset of $\tilde{\mathrm{I}}_0(q, n)$ such that

$$\tilde{\mathrm{I}}'_0(q, n)=\{(p_1, \dots, p_n)\in\tilde{\mathrm{I}}_0(q, n) \mid 1=p_1< \dots <p_n<q/2\}.$$

Then it is easy to see that any element of $\tilde{\mathrm{I}}_0(q, n)$ has an equivalence element in $\tilde{\mathrm{I}}'_0(q, n)$. On the other hand, for any equivalence class in $\mathrm{I}_0(q, n)$, the number of elements in $\tilde{\mathrm{I}}'_0(q, n)$ which belong to that class is at most n. Hence we have

$$|\mathrm{I}_0(q, n)| \geqq \frac{1}{n}|\tilde{\mathrm{I}}'_0(q, n)| = \frac{1}{n}\binom{q_0-1}{n-1} = \frac{1}{q_0}\binom{q_0}{n},$$

which proves the Proposition.

Q.E.D.

Let q be a positive integer and γ a primitive q-th root of 1. We denote by $\mathrm{Q}(\gamma)$ the q-th cyclotomic field over the rational number field Q and denote by $\Phi_q(z)$ the q-th cyclotomic polynomial. We assume q is an odd prime. Then we have

$$q_0=(q-1)/2$$

and

$$\Phi_q(z)=\sum_{t=0}^{q-1} z^t.$$

We shall define the map $\Psi_{q,k}$ of $\mathrm{I}_0(q, k)$ into $\mathrm{Q}(\gamma)[z]$. For any equivalence class in $\mathrm{I}_0(q, k)$, we take an element $(p_1, \dots, p_k)$ of $\tilde{\mathrm{I}}_0(q, k)$ which belongs to that class. Then the polynomial

$$\sum_{l=1}^{q-1}\prod_{i=1}^{k}(z-\gamma^{p_i l})(z-\gamma^{-p_i l})$$

in $\mathrm{Q}(\gamma)[z]$ is independent of the choice of the elements which belong to the class. Hence we can define the map $\Psi_{q,k}$ by

$$(1.5)\qquad \Psi_{q,k}((p_1, \dots, p_k))=\sum_{l=1}^{q-1}\prod_{i=1}^{k}(z-\gamma^{p_i l})(z-\gamma^{-p_i l}).$$

PROPOSITION 1.2. — *If we put*

$$\Psi_{q,k}((p_1, \ldots, p_k)) = \sum_{i=0}^{2k} (-1)^i a_i z^{2k-i},$$

then we have:

(i) $a_i = a_{2k-i}$;

(ii) $a_0 = (q-1)$;

(iii) $a_1 = -2k$;

(iv) $a_2 = k(q-2k+1)$.

Proof. — Since we have $(z-\gamma^{p_i l})(z-\gamma^{-p_i l}) = (\gamma^{p_i l} z - 1)(\gamma^{-p_i l} z - 1)$: (i) is easy to see, (ii) is clear. On the other hand,

$$a_1 = \sum_{l=1}^{q-1} \sum_{i=1}^{k} \gamma^{\pm p_i l} = 2 \sum_{i=1}^{k} \sum_{l=1}^{q-1} \gamma^{p_i l} = -2k,$$

and

$$a_2 = \sum_{l=1}^{q-1} \Big(\sum_{1 \leqq i < j \leqq k} \gamma^{\pm(p_i+p_j)l} + \sum_{1 \leqq i, j \leqq k} \gamma^{(p_i-p_j)l} \Big) = -2\binom{k}{2} - 2\binom{k}{2} + k(q-1) = k(q-2k+1).$$

Thus (iii) and (iv) have been shown.

Q.E.D.

Set

$$\text{(1.6)} \qquad J(q, k) = \Psi_{q,k}(I_0(q, k)).$$

COROLLARY 1.3. — *Let q be an odd prime not less than 11. Then we have*

$$|J(q, 2)| = 1.$$

Proof. — Let q be an odd prime not less than 11, then $q_0 \geqq 4$. Hence, $I_0(q, 2)$ and $J(q, 2)$ can be defined. On the other hand, for any $(p_1, p_2) \in I_0(q, 2)$, the polynomial $\Psi_{q,2}((p_1, p_2))$ is degree 4. By Proposition 1.2, its coefficients are independent of the choice of elements in $I_0(q, 2)$. Thus we have $|J(q, 2)| = 1$.

Q.E.D.

Let p_1, p_2, p_3 be integers with $(p_1, p_2, p_3) \in \tilde{I}_0(q, 3)$ and s_1, s_2, s_3, s_4 be integers with $(s_1, s_2, s_3, s_4) \in \tilde{I}_0(q, 4)$. We define the sets $A_q(p_1, p_2, p_3)$ and $A_q(s_1, s_2, s_3, s_4)$ by

$$A_q(p_1, p_2, p_3) = \{(\lambda, u) : p_1 + \lambda p_2 + \mu p_3 \equiv 0 \pmod{q}, \lambda, \mu \in \{-1, 1\}\},$$

$$A_q(s_1, s_2, s_3, s_4) = \{(\lambda, \mu, \nu) : s_1 + \lambda s_2 + \mu s_3 + \nu s_4 \equiv 0 \pmod{q}, \lambda, \mu, \nu \in \{-1, 1\},$$

respectively.

LEMMA 1.4. — *For any element $(p_1, p_2, p_3) \in \tilde{I}_0(q, 3)$, we have*

$$|A_q(p_1, p_2, p_3)| \leqq 1.$$

Proof. — If one of the integers $p_1 \pm p_2 \pm p_3$ is congruent to zero (mod q), we may assume $p_1+p_2+p_3 \equiv 0 \pmod q$, if necessary, changing the sign of p_2 or p_3. Then $(p_1+p_2+p_3)-(p_1+p_2-p_3) \equiv 2p_3 \pmod q$. Hence, $p_1+p_2-p_3 \not\equiv 0 \pmod q$. In the same way, we can see the integers $p_1-p_2+p_3$ and $p_1-p_2-p_3$ are not congruent to zero (mod q). Hence we have proved the Lemma.

Q.E.D.

LEMMA 1.5. — *For any element* $(p_1, p_2, p_3, p_4) \in \tilde{\mathrm{I}}_0(q, 4)$, *we have*

(i) $$|\mathrm{A}_q(p_1, p_2, p_3, p_4)| \leqq 1;$$

(ii) $$\sum_{1 \leqq i_1 < i_2 < i_3 \leqq 4} |\mathrm{A}_q(p_{i_1}, p_{i_2}, p_{i_3})| \leqq 2.$$

Proof. — (i) can be obtained in the same way as Lemma 1.4. We shall prove (ii). Suppose

$$\sum_{1 \leqq i_1 < i_2 < i_3 \leqq 4} |\mathrm{A}_q(p_{i_1}, p_{i_2}, p_{i_3})| \geqq 3.$$

Then we may assume

$$|\mathrm{A}_q(p_1, p_2, p_3)| = |\mathrm{A}_q(p_1, p_2, p_4)| = |\mathrm{A}_q(p_1, p_3, p_4)| = 1.$$

We may also assume, changing the sign of p_2 or p_3 if necessary,

$$p_1+p_2+p_3 \equiv 0 \pmod q. \tag{1.7}$$

Since $p_3 \not\equiv \pm p_4 \pmod q$, we have $p_1-p_2+p_4 \equiv 0 \pmod q$ or $p_1-p_2-p_4 \equiv 0 \pmod q$. In the same reason as above, we may assume

$$p_1-p_2+p_4 \equiv 0 \pmod q. \tag{1.8}$$

Since $p_2 \not\equiv \pm p_4$, we have

$$p_1-p_3+p_4 \equiv 0 \pmod q, \tag{1.9}$$

or

$$p_1-p_3-p_4 \equiv 0 \pmod q. \tag{1.9'}$$

From (1.8) and (1.9), we have $p_2 \equiv p_3 \pmod q$, which is a contradiction. On the other hand, from (1.7), (1.8) and (1.9′), we have $3p_1 \equiv 0 \pmod q$. Since $\tilde{\mathrm{I}}_0(3,4)$ is empty, we have a contradiction. Thus we have proved the Lemma.

Q.E.D.

Let q be an odd prime and p_1, p_2, p_3, p_4 integers prime to q such that $(p_1, p_2, p_3, p_4) \in \tilde{\mathrm{I}}_0(q, 4)$. Fix a primitive q-th root of 1, say γ. We shall compute the coefficients a_3, a_4 of the polynomial $\Psi_{q,4}((p_1, p_2, p_3, p_4))$:

$$a_3 = \sum_{l=1}^{q-1} \sum_{1 \leqq i_1 < i_2 < i_3 \leqq 4} \gamma^{(\pm p_{i_1} \pm p_{i_2} \pm p_{i_3})l} + 3 \sum_{l=1}^{q-1} \sum_{i=1}^{4} \gamma^{\pm p_i l}$$

$$= 2 \sum_{l=1}^{q-1} \sum_{1 \leqq i_1 < i_2 < i_3 \leqq 4} \gamma^{(p_{i_1} \pm p_{i_2} \pm p_{i_3})l} + 6 \sum_{l=1}^{q-1} \sum_{i=1}^{4} \gamma^{p_i l}$$

$$= 2(q-1) \sum_{1 \leqq i_1 < i_2 < i_3 \leqq 4} \left| \mathrm{A}_q(p_{i_1}, p_{i_2}, p_{i_3}) \right|$$

$$-2\Big(16 - \sum_{1 \leqq i_1 < i_2 < i_3 \leqq 4} \left| \mathrm{A}_q(p_{i_1}, p_{i_2}, p_{i_3}) \right|\Big) - 24$$

$$= 2q \sum_{1 \leqq i_1 < i_2 < i_3 \leqq 4} \left| \mathrm{A}_q(p_{i_1}, p_{i_2}, p_{i_3}) \right| - 56,$$

and

$$a_4 = 2 \sum_{l=1}^{q-1} \gamma^{(p_1 \pm p_2 \pm p_3 \pm p_4)l} + 4 \sum_{l=1}^{q-1} \sum_{1 \leqq i < j \leqq 4} \gamma^{(p_i \pm p_j)l} + \sum_{l=1}^{q-1} \binom{4}{2}$$

$$= 2q \left| \mathrm{A}_q(p_1, p_2, p_3, p_4) \right| + 6q - 70.$$

Hence we have:

PROPOSITION 1.6. — *Let* $(p_1, p_2, p_3, p_4), (s_1, s_2, s_3, s_4) \in \mathrm{I}_0(q, 4)$. *Then we have*

$$\Psi_{q,4}((p_1, p_2, p_3, p_4)) \doteq \Psi_{q,4}((s_1, s_2, s_3, s_4)),$$

if and only if

(i) $$\sum_{1 \leqq i_1 < i_2 < i_3 \leqq 4} \left| \mathrm{A}_q(p_{i_1}, p_{i_2}, p_{i_3}) \right| = \sum_{1 \leqq i_1 < i_2 < i_3 \leqq 4} \left| \mathrm{A}_q(s_{i_1}, s_{i_2}, s_{i_3}) \right|$$

and

(ii) $$\left| \mathrm{A}_q(p_1, p_2, p_3, p_4) \right| = \left| \mathrm{A}_q(s_1, s_2, s_3, s_4) \right|.$$

In the same way as above, we have:

PROPOSITION 1.7. — *Let* $(p_1, p_2, p_3), (s_1, s_2, s_3) \in \mathrm{I}_0(q, 3)$. *Then we have*

$$\Psi_{q,3}((p_1, p_2, p_3)) = \Psi_{q,3}((s_1, s_2, s_3))$$

if and only if

$$\left| \mathrm{A}_q(p_1, p_2, p_3) \right| = \left| \mathrm{A}_q(s_1, s_2, s_3) \right|.$$

From Lemma 1.4, 1.5 and Proposition 1.6, 1.7, we have:

PROPOSITION 1.9. — (i) *let q be a prime not less than* 11, *then we have* $\left| \mathrm{J}(q, 3) \right| \leqq 2$; (ii) *let q be a prime not less than* 13 *then we have* $\left| \mathrm{J}(q, 4) \right| \leqq 6$.

2. Lens spaces, their generating functions and certain topological conditions

Let q be a positive integer and $p_1, p_2, \ldots, p_n$ integers prime to q. We denote by g the orthogonal matrix given by

$$g=\begin{pmatrix} R(p_1/q) & & 0 \\ & \ddots & \\ 0 & & R(p_n/q) \end{pmatrix},$$

where

$$R(\theta)=\begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}.$$

Then g generates the cyclic subgroup $G=\{g^k\}_{k=1}^{q}$ of order q of the orthogonal group $O(2n)$ of degree $2n$. By the definition (*see* [6]), the lens space

$$L(q : p_1, \ldots, p_n)=S^{2n-1}/G,$$

is a Riemannian manifolds of positive constant curvature 1.

On lens spaces, the following Theorems are known.

THEOREM 2.1 (*cf.* [2], [6]). — *Let* $L=L(q : p_1, \ldots, p_n)$ *and* $L'=L(q : s_1, \ldots, s_n)$ *be lens spaces. Then the following assertions are equivalent:*

1. L *is isometric to* L′;
2. L *is diffeomorphic to L′;*
3. L *is homeomorphic to* L′;
4. *there is a number* l *and there are numbers* $e_i \in \{-1, 1\}$ *such that* $(p_1, \ldots, p_n)$ *is a permutation of* $(e_1 ls_1, \ldots, e_n ls_n)$ (mod q).

THEOREM 2.2 (*cf.* [2], [3]). — *Let* L *and* L′ *be as in the above Theorem. Then* L *is homotopy equivalent to* L′ *if and only if there are numbers* l *and* $e \in \{-1, 1\}$ *such that* $s_1 \ldots s_n \equiv el^n p_1 \ldots p_n$ (mod q).

LEMMA 2.3. — *If* $\varphi(q)$ *is prime to* n, *then all the* $(2n-1)$*-dimensional lens spaces with fundamental group of order* q *are mutually homotopy equivalent to each other.*

Proof. — If $\varphi(q)$ is prime to n, then the homomorphism $g \to g^n$ of K_q into itself is an isomorphism. Now, the Lemma follows directly from Theorem 2.2.

Q.E.D.

Let Δ be the Laplacien acting on the space of smooth functions on $L(q : p_1, \ldots, p_n)$. Then each eigenvalue of Δ is of the form $k(k+2n-2)$ $(k=0, 1, 2, \ldots)$. We denote by $E_{k(k+2n-2)}$ the eigenspace of Δ with eigenvalue $k(k+2n-2)$. In [6] (*see* also [7], [8]), we introduced the generating function associated to the spectrum of Δ on $L(q : p_1, \ldots, p_n)$. In this paper, the generating function is denoted by $F_q(z : p_1, \ldots, p_n)$;

$$F_q(z : p_1, \ldots, p_n)=\sum_{k=0}^{\infty} (\dim E_{k(k+2n-2)})z^k. \tag{2.1}$$

First note

(2.2) The lens space $L(q : p_1, \ldots, p_n)$ is isospectral to $L(q : s_1, \ldots, s_n)$ if and only if their generating functions are identical (*see* [6]). And next, we observe that the generating function is a meromorphic function and more strongly it is a rational function;

$$F_q(z : p_1, \ldots, p_n) = \frac{1}{q} \sum_{l=1}^{q} \frac{1-z^2}{\prod_{i=1}^{n} (z-\gamma^{p_i l})(z-\gamma^{-p_i l})}, \tag{2.3}$$

where γ is a primitive q-th root of 1 (*see* [6]).

Let $\tilde{\mathscr{L}}(q, n)$ be the family of all the $(2n-1)$-dimensional lens spaces with fundamental group of order q, and let $\tilde{\mathscr{L}}_0(q, n)$ the subfamily of $\tilde{\mathscr{L}}(q, n)$ defined by

$$\tilde{\mathscr{L}}_0(q, n) = \{ L(q : p_1, \ldots, p_n) \in \tilde{\mathscr{L}}(q, n) \mid p_i \not\equiv \pm p_j \pmod q \ (1 \leqq i < j \leqq n) \}.$$

The set of isometry classes of $\tilde{\mathscr{L}}_0(q, n)$ [resp. $\tilde{\mathscr{L}}(q, n)$] is denoted by $\mathscr{L}_0(q, n)$ [resp. $\mathscr{L}(q, n)$]. Then by Theorem 2.1, the map $L(q : p_1, \ldots . p_n) \to (p_1, \ldots, p_n)$ of $\tilde{\mathscr{L}}_0(q, n)$ [resp. $\tilde{\mathscr{L}}(q, n)$] onto $\tilde{I}_0(q, n)$ [resp. $\tilde{I}(q, n)$] induces a one to one corresponding between $\mathscr{L}_0(q, n)$ and $I_0(q, n)$ [resp. $\mathscr{L}(q, n)$ and $I(q, n)$]. Together this fact with Proposition 1.1, we have:

PROPOSITION 2.4. — *Retaining the notations in the above, we have*

$$|\mathscr{L}_0(q, n)| \geqq \frac{1}{q_0} \binom{q_0}{n}. \tag{2.4}$$

PROPOSITION 2.5. — *Let* $L(q : p_1, \ldots, p_n)$ *be a lens space belonging to* $\tilde{\mathscr{L}}_0(q, n)$, $k = q_0 - n$ *and let* ω *be the map of* $I_0(q, n)$ *onto* $I_0(q, k)$ *defined in* 1. *Assume* q *is an odd prime. Then we have*

$$F_q(z : p_1, \ldots, p_n) = \frac{1}{q} \left\{ \frac{1-z^2}{(1-z)^{2n}} + \frac{\Psi_{q,k}(\omega((p_1, \ldots, p_n)))(1-z^2)}{\Phi_q(z)} \right\}. \tag{2.5}$$

Proof. — We choose integers $q_1, \ldots, q_k$ such that the set of integers

$$\{p_1, -p_1, \ldots, p_n, -p_n, q_1, -q_1, \ldots, q_k, -q_k\}$$

forms a complete set of residues prime to q. Then for any $l \not\equiv 0 \pmod q$, we have

$$1 \Big/ \prod_{i=1}^{n} (z-\gamma^{p_i l})(z-\gamma^{-p_i l}) = \prod_{i=1}^{k} (z-\gamma^{q_i l})(z-\gamma^{-q_i l}) / \Phi_q(z).$$

Now, the Proposition follows directly from the formula (2.3).

Q.E.D.

From this Proposition and (2.2), we have:

PROPOSITION 2.6. — *Let* $L=L(q:p_1, \ldots, p_n)$ *and* $L'=L(q:s_1, \ldots, s_n)$ *be lens spaces belonging to* $\mathscr{L}_0(q, n)$. *Assume q is an odd prime. Then* L *is isospectral to* L′ *if and only if*

$$\Psi_{q,k}(\omega((p_1, \ldots, p_n)))=\Psi_{q,k}(\omega((s_1, \ldots, s_n))),$$

where $k=q_0-n$.

3. Lens spaces which are isospectral but not isometric

We have the following diagram from the results in 1 and 2,

$$\mathscr{L}_0(q, n) \simeq I_0(q, n) \underset{\omega}{\simeq} I_0(q, k) \xrightarrow[\Psi_{q,k}]{} Q(\gamma)[z], \tag{3.1}$$

where q is an odd prime, $q_0=\varphi(q)/2$, k, $n \geqq 2$ and $k+n=q_0$. Hence by proposition 2.6, to obtain lens spaces which are isospectral but not isometric, we must seek the integers q, n such that $\Psi_{q,k}$ is not injective.

THEOREM 3.1. — (i) *let q be a prime not less than* 11. *Then there exist at least two* $(q-6)$-*dimensional lens spaces with fundamental groups of order q which are isospectral but not isometric;*

(ii) *let q be a prime not less than* 13. *Then there exist at least two* $(q-8)$-*dimensional lens spaces with fundamental groups of order q which are isospectral but not isometric;*

(iii) *let q be a prime not less than* 17. *Then there exist at least two* $(q-10)$-*dimensional lens spaces with fundamental groups of order q which are isospectral but not isometric.*

Proof. — (i) let q be a prime with $q \geqq 11$. *Then* $q_0=\varphi(q)/2=(q-1)/2 \geqq 5$. Put $k=2$ and $n=q_0-2$. Then $2n-1=q-6$. To prove (i), it suffices to show that the map $\Psi_{q,2}$ is not injective. By Proposition 2.4 and Corollary 1.3, we have

$$|\mathscr{L}_0(q, n)| \geqq \frac{1}{q_0}\binom{q_0}{2}=(q_0-1)/2 \geqq 2>1=|J(q, 2)|.$$

This means that $\Psi_{q,2}$ is not injective;

(ii) let q be a prime with $q \geqq 13$. *Then* $q_0 \geqq 6$. But $k=3$ and $n=q_0-3$. Then $2n-1=q-8$, On the other hand,

$$|\mathscr{L}_0(q, n)| \geqq \frac{1}{q_0}\binom{q_0}{3}=(q_0-1)(q_0-2)/6 \geqq 20/6>2 \geqq |J(q, 3)|,$$

which implies $\Psi_{q,3}$ is not injective;

(iii) let q be a prime and $q \geqq 17$. Then $q_0 \geqq 8$. Put $k=4$ and $n=q_0-4$. Then $2n-1=q-10$. Since

$$|\mathscr{L}_0(q, n)| \geqq \frac{1}{q_0}\binom{q_0}{4} > 6 \geqq |\mathrm{J}(q, 4)|,$$

the map $\Psi_{q,4}$ is not injective, which implies (iii). Now the Theorem is completed.

Q.E.D.

Remark. — By Theorem 2.1, these lens spaces, which are isospectral but not isometric, are neither diffeomorphic nor homeomorphic to each other.

4. Examples

Let q be a positive integer ($q \geqq 2$) such that the multiplicative group K_q is cyclic. We shall give a combinatorial method to determine all non-isometric classes of lens spaces with fundamental group of order q. It is known from the elementary number theory that K_q is cyclic if and only if $q=2, 4, p^\alpha$ or $2p^\alpha$, where p is an odd prime and $\alpha \geqq 1$. For such a q, an integer r is said to be a primitive root of q if the residue class (mod q) of r is a generator of K_q.

In what follows, we always assume that K_q is cyclic and $q_0=\varphi(q)/2 \geqq 4$. We fix a primitive root of q, say r. Since K_q is cyclic of order $\varphi(q)$, we have

$$r^{q_0} \equiv -1 \pmod{q}. \tag{4.1}$$

Take an integer n with $2 \leqq n \leqq q_0-2$. Put $k=q_0-n$. From (4.1), we have:

LEMMA 4.1. — *For any* $(p_1, \ldots, p_n) \in \tilde{\mathrm{I}}(q, n)$, *there are integers* $a_1, \ldots, a_n$ *with* $0=a_1 \leqq a_2 \leqq \ldots \leqq a_n < q_0$ *such that* $(p_1, \ldots, p_n)$ *is equivalent to* $(1, r^{a_2}, \ldots, r^{a_n})$.

LEMMA 4.2. — *Let* $(r^{a_1}, r^{a_2}, \ldots, r^{a_n})$ *and* $(r^{b_1}, r^{b_2}, \ldots, r^{b_n})$ *be elements in* $\tilde{\mathrm{I}}(q, n)$. *Suppose the sequences* $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$ *are non-decreasing with* $a_1=b_1=0$ *and* $a_n, b_n < q_0$. *Then* $(r^{a_1}, \ldots, r^{a_n})$ *is equivalent to* $(r^{b_1}, \ldots, r^{b_n})$ *if and only if there are numbers* n_0 *and* b *satisfying the condition that whenever* $k \equiv k'+n_0 \pmod{n}$, *we have* $a_k \equiv b+b_{k'} \pmod{q_0}$.

Proof. — Since $r^{q_0} \equiv -1 \pmod{q}$, $(r^{a_1}, \ldots, r^{a_n})$ is equivalent to $(r^{b_1}, \ldots, r^{b_n})$ if and only if there is an integer b such that $(a_1, \ldots, a_n)$ is a permutation of $(b+b_1, \ldots, b+b_n)$ (mod q_0). Now, the Lemma follows from the assumption that the sequences $\{a_k\}_{k=1}^n$, $\{b_k\}_{k=1}^n$ are non-decreasing.

Q.E.D.

For any non-decreasing sequence of n-integers $\{a_k\}_{k=1}^n$ with $a_1=0$ and $a_k<q_0$ ($k=2, \ldots, n$), we define the sequence of non-negative integers $\{c_k\}_{k=1}^n$ by

$$\begin{cases} c_k=a_{k+1}-a_k & \text{if } 1 \leqq k < n, \\ c_n=q_0-a_n+a_1. \end{cases} \tag{4.2}$$

Then we have $\sum_{k=1}^{n} c_k = q_0$ and $c_n > 0$. Conversely, for any sequence $\{c_k\}_{k=1}^{n}$ of non-negative n-integers with $\sum_{k=1}^{n} c_k = q_0$ and $c_n > 0$, we define the non-decreasing sequence of n-integers $\{a_k\}_{k=1}^{n}$ by

$$(4.3) \qquad \begin{cases} a_1 = 0, \\ a_k = \sum_{1 \leqq k' < k} c_{k'} \quad \text{if} \quad 1 < k \leqq n. \end{cases}$$

Then we have $a_k < q_0$ $(k = 1, \ldots, n)$.

Let $\tilde{\mathrm{C}}(q, n)$ [resp. $\tilde{\mathrm{C}}_0(q, n)$] be the set of non-negative (resp. strictly positive) sequences $\{c_k\}_{k=1}^{n}$ of n-integers with $\sum_{k=1}^{n} c_k = q_0$ and $c_n > 0$. We define two sequences $\{c_k\}_{k=1}^{n}$ and $\{c'_k\}_{k=1}^{n}$ in $\tilde{\mathrm{C}}(q, n)$ [resp. $\tilde{\mathrm{C}}_0(q, n)$] are equivalent if there is a number k_0 such that whenever $k \equiv k' + k_0 \pmod n$, we have $c_k = c'_{k'}$. The set of equivalence classes of $\tilde{\mathrm{C}}(q, n)$ [resp. $\tilde{\mathrm{C}}_0(q, n)$] is denoted by $\mathrm{C}(q, n)$ [resp. $\mathrm{C}_0(q, n)$]. For the lens space $\mathrm{L}(q: r^{a_1}, \ldots, r^{a_n})$ with $0 = a_1 \leqq a_2 \leqq \ldots \leqq a_n < q_0$, let $\{c_k\}_{k=1}^{n}$ be the sequence defined as in (4.2). Then we can see easily that the correspondence

$$(4.4) \qquad \mathrm{L}(q: r^{a_1}, r^{a_2}, \ldots, r^{a_n}) \to \{c_k\}_{k=1}^{n},$$

defines a one to one corresponding between $\mathscr{L}(q, n)$ and $\mathrm{C}(q, n)$ [resp. $\mathscr{L}_0(q, n)$ and $\mathrm{C}_0(q, n)$].

By using the above one to one corresponding between $\mathscr{L}_0(q, n)$ and $\mathrm{C}_0(q, n)$, and by applying Corollary 1.3, Proposition 1.6 and 1.7, we can obtain many explicit examples of lens spaces which are isospectral but not isometric. Here, we shall give three examples, containing non-homotopy equivalent ones.

(I) 5-dimensional examples ($n = 3$).

Case (i): $q = 11$.

Then $q_0 = 5$, $k = 2$ *and* $r = 2$.

$\{c_1, c_2, c_3\}$	$\mathrm{L}(q: r^{a_1}, r^{a_2}, r^{a_3})$
$\{1, 1, 3\}$	$\mathrm{L}(11: 1, 2, 2^2)$
$\{1, 2, 2\}$	$\mathrm{L}(11: 1, 2, 2^3)$

Since $k = 2$, these lens spaces are isospectral. On the other hand, $(\varphi(q), n) = (10, 3) = 1$. Hence, by Lemma 2.3, the lens space $\mathrm{L}(11: 1, 2, 2^2)$ is homotopy equivalent to $\mathrm{L}(11: 1, 2, 2^3)$.

Case (ii): $q = 13$.

Then $q_0=6$, $k=3$ and $r=2$.

$\{c_1, c_2, c_3\}$	$L(q: r^{a_1}, r^{a_2}, r^{a_3})$	(q_1, q_2, q_3)	$\lvert A_q(q_1, q_2, q_3)\rvert$
$\{1, 1, 4\}$	$L(13: 1, 2, 2^2)$	$(2^3, 2^4, 2^5)$	0
$\{2, 1, 3\}$	$L(13: 1, 2^2, 2^3)$	$(2, 2^4, 2^5)$	0
$\{1, 2, 3\}$	$L(13: 1, 2, 2^3)$	$(2^2, 2^4, 2^5)$	1
$\{2, 2, 2\}$	$L(13: 1, 2^2, 2^4)$	$(2, 2^3, 2^5)$	1

where $(q_1, q_2, q_3)=\omega((r^{a_1}, r^{a_2}, r^{a_3}))$.

By Proposition 1.7 and 2.6, the lens spaces $L(13: 1, 2, 2^2)$ and $L(13: 1, 2^2, 2^3)$ are isospectral to each other, and also the lens spaces $L(13: 1, 2, 2^3)$, $L(13: 1, 2^2, 2^4)$ are isospectral. But these isospectral lens spaces are non-homotopy equivalent to each other. This fact follows easily from the facts $r^n \equiv 2^3 \equiv 8 \pmod{13}$ and $(2^3)^2 \equiv -1 \pmod{13}$, and Theorem 2.2.

(II) A 7-dimensional example ($n=4$).

Case $q=13$.

Then $q_0=6$, $k=2$ and $r=2$.

$\{c_1, c_2, c_3, c_4\}$	$L(q: r^{a_1}, r^{a_2}, r^{a_3}, r^{a_4})$
$\{1, 1, 1, 3\}$	$L(13: 1, 2, 2^2, 2^3)$
$\{1, 1, 2, 2\}$	$L(13: 1, 2, 2^2, 2^4)$
$\{1, 2, 1, 2\}$	$L(13: 1, 2, 2^3, 2^4)$

In this case we see these lens spaces are mutually isospectral but non-homotopy equivalent to each other.

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