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Second order linear differential systems


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SECOND ORDER LINEAR
DIFFERENTIAL SYSTEMS

BY F. NEUMAN

I. — Introduction

We shall deal with second order linear differential systems

\( y'' = Q(t) y, \)

where \( n \) by \( n \) real symmetric continuous matrices \( Q: \mathbb{R} \rightarrow \mathbb{R}^n \) satisfy

\( Q(t + \pi) = P Q(t) P^{-1} \)

for a constant orthogonal matrix \( P \). We shall derive a sufficient condition under which all solutions of \( (Q) \) comply with

\( y(t + \pi) = P y(t), \)

and we shall construct some \( (Q) \) of the property \( (1) \). If \( P = \pm I \) (I denoting the unit matrix), all solutions of \( (Q) \) are periodic or half-periodic. For the case we shall construct an example of two-dimensional system \( (Q) \) having only half-periodic solutions so that \( Q \) is not diagonalizable, i.e., it is not of the form

\[ C^{-1} \text{diag}(q_1, \ldots, q_n) C, \]

\( C \) being a real constant regular \( n \) by \( n \) matrix, and \( q_i \) are scalar functions such that all solutions of

\[ y'' = q_i(t) y \]

are half-periodic. For constructing such \( q_i \) (see [5], pp. 573-589).

Systems \( (Q) \) with solutions satisfying \( (1) \) are in close connection with investigations in differential geometry, especially with Blaschke's conjecture see [1], pp. 225-230.

The problem considered here was proposed by Professor M. Berger.
II. — Notations and basic properties

For an integer \( m \geq 0 \), let \( C^m(J, \mathbb{R}^n) \) denote the set of all matrices \( T : J \to \mathbb{R}^n \), \( J \subset \mathbb{R} \), having continuous derivatives up to and including the \( m \)-th order. \( T^* \) means the transpose of \( T \), \( \frac{d}{dt} \) denotes \( d/dt \). Throughout this paper the matrix \( Q \) in \( (Q) \) is supposed to be continuous on \( \mathbb{R} : Q \in C^0(\mathbb{R}, \mathbb{R}^{n^2}) \).

If \( Y_1 \) and \( Y_2 \) are two matrix-solutions of \( (Q) \) on \( \mathbb{R} \) such that the \( 2n \) by \( 2n \) matrix

\[
\begin{pmatrix}
Y_1 & Y_2 \\
Y_1' & Y_2'
\end{pmatrix}
\]

is regular at least at some \( t_0 \) (then it is regular on \( \mathbb{R} \)), then \( Y_1(t)C_1 + Y_2(t)C_2 \) is a general matrix-solution of \( (Q) \), \( C_1 \) and \( C_2 \) being arbitrary constant \( n \) by \( n \) matrices.

For each solution \( Y \) of \( (Q) \) with symmetric \( Q \), \( Q^* = Q \), the expression \( Y^*(t)Y'(t) - Y^*(t)Y(t) \) is a constant matrix, say \( C \). If \( C = 0 \) (the null matrix), then \( Y \) is called isotropic. For each isotropic solution \( Y \) of \( (Q) \) such that \( Y \) is regular on an interval \( J \), the matrix

\[
Y(t) \int_d^t Y^{-1}(s) Y^* - 1 (s) \, ds, \quad d \in J,
\]

is a solution of \( (Q) \) on \( J \), see e.g. [2] or [3].

**Lemma 1.** — Let \( Y \) be a solution of \( (Q) \) satisfying \( Y(a) = 0 \), \( Y'(a) \) being regular. Then there exists a neighbourhood \( V \) of \( a \) such that \( Y(t) \) is regular on \( V - \{ a \} \).

**Remark 1.** — We need not suppose the symmetry of \( Q \) for the Lemma. However, if \( Q^* = Q \), then the \( Y \) in Lemma 1 is isotropic.

**Proof.** — If such a \( V \) does not exist, there is a sequence \( \{ t_i \}_{i=1}^{\infty}, t_i \neq a, t_i \to a \) as \( i \to \infty \), such that \( \det Y(t_i) = 0 \). Because of the continuity of \( \det \) as a function of \( n^2 \) variables, we have

\[
\det Y'(a) = \det \left( \lim_{i \to \infty} [Y(t_i) - Y(a)] [t_i - a]^{-1} \right)
\]

\[
= \lim_{i \to \infty} \det \left( [Y(t_i) - Y(a)] [t_i - a]^{-1} \right)
\]

\[
= \lim_{i \to \infty} (t_i - a)^{-n} \det Y(t_i) = 0,
\]

that contradicts the regularity of \( Y'(a) \).

**Lemma 2.** — Suppose \( Q^* = Q \). Let a solution \( Y_1 \) of \( (Q) \) satisfy: \( Y_1(a) = 0 \), \( Y_1'(a) \) is regular. Let \( Y_1 \) be regular on \( (a, b) \). For

\[
Y_2(t) := Y_1(t) \int_d^t Y_1^{-1}(s) Y_1^*(s) \, ds, \quad d \in (a, b),
\]

the expression \( Y_1(t)C_1 + Y_2(t)C_2 \) is a general solution of \( (Q) \) on \( (a, b) \).
Proof. — It is sufficient to show that
\[
\begin{pmatrix}
Y_1(t); & \int d Y_1^{-1}(s) Y_1^{* -1}(s)ds \\
Y_1'(t); & \int d Y_1^{-1}(s) Y_1^{* -1}(s)ds + Y_1^{* -1}(t)
\end{pmatrix}
\]
is regular at least at some \( t_0 \in (a, b) \). For \( t_0 = d \) we get
\[
\begin{pmatrix}
Y_1(d); & 0 \\
Y_1'(d); & Y_1^{* -1}(d)
\end{pmatrix},
\]
whose determinant is \( \det Y_1(d) \). \( \det Y_1^{* -1}(d) = 1. \)

III. — Sufficient condition for \( y(t + \pi) = P y(t) \)

Suppose that a matrix-solution \( Y_1 \) of (Q), \( Q^* = Q \),
\[(2) \quad Q(t + \pi) = P Q(t) P^{-1}, \]
P being a real constant orthogonal matrix, satisfies:
\[
Y_1(a) = 0, \quad Y_1'(a) \text{ is regular},
Y_1(t) \text{ is regular on } (a, a + \pi),
Y_1(t + \pi) = P Y_1(t).
\]
Evidently \( Y_1 \in C^2(R, R^n) \), and \( a + \pi \) is the first conjugate point to \( a \), [2]. The matrix
\[
Y_2: \quad t \mapsto Y_1(t) \int d Y_1^{-1}(s) Y_1^{* -1}(s)ds, \quad d \in (a, a + \pi),
\]
is also a solution of (Q) on \( (a, a + \pi) \). Let \( \overline{Y}_2 \in C^2(R, R^n) \) denote the (unique) continuation of \( Y_2 \). Due to Lemma 2 every solution \( y \) of (Q) satisfies (1) if and only if
\[(3) \quad \overline{Y}_2(t + \pi) = P \overline{Y}_2(t) \text{ on } R. \]
Because of the uniqueness of solutions, the relation (3) holds if and only if
\[
\overline{Y}_2(a + \pi) = P \overline{Y}(a) \quad \text{and} \quad \overline{Y}_2'(a + \pi) = P \overline{Y}_2'(a).
\]
Since \( \overline{Y}_2(t) = Y_2(t) \) on \( (a, a + \pi) \), and \( \overline{Y}_2 \in C^2(R, R^n) \), there exist
\[
\lim_{t \to a} Y_2(t) = \overline{Y}_2(a), \quad \lim_{t \to a + \pi} Y_2(t) = \overline{Y}_2(a + \pi),
\]
\[
\lim_{t \to a} Y_2'(t) = \overline{Y}_2'(a), \quad \lim_{t \to a + \pi} Y_2'(t) = \overline{Y}_2'(a + \pi).
\]
Hence (3) holds iff both

$$\lim_{t \to a+\pi_-} Y_2(t) = P \lim_{t \to a_+} Y_2(t),$$

$$\lim_{t \to a+\pi_-} Y_2'(t) = P \lim_{t \to a_+} Y_2'(t).$$

Define

$$A(t) := Y_1(t) \sin^{-1}(t-a) \quad \text{for} \quad t \in (a+k\pi, a+k+1\pi),$$

$$A(t) := (-P)^k Y_1(a) \quad \text{for} \quad t = a+k\pi, \quad k = 0, \pm 1, \ldots;$$

$\sin^{-k}s$ denoting $(\sin s)^{-k}$ throughout this paper. We have

$$\lim_{t \to a+k\pi} A(t) = (-P)^k Y_1(a), \quad \lim_{t \to a+k\pi} A'(t) = 0,$$

$$\lim_{t \to a+k\pi} A''(t) = \frac{1}{3} (-P)^k (Q(a)+1) Y_1'(a).$$

Hence $A \in C^2(\mathbb{R}, \mathbb{R}^d)$, $A(t+\pi) = -PA(t)$, $A$ being regular on the whole $\mathbb{R}$. Using l'Hospital rule we get

$$\lim_{t \to a_-} Y_2(t) = \lim_{t \to a_-} A(t) \left( \frac{\int_{a}^{t} (A^*(s)A(s))^{-1} \sin^{-2}(s-a) \, ds}{\sin^{-1}(t-a)} \right)$$

$$= A(a) \lim_{t \to a_+] \frac{(A^*(t)A(t))^{-1}}{-\cos(t-a)} = -A^{*-1}(a),$$

and

$$\lim_{t \to a_+] Y_2(t) = \lim_{a \to a_+] \frac{A(a+\pi)(A^*(t)A(t))^{-1}}{-\cos(t-a)} = -PA^{*-1}(a).$$

Thus the condition (4) gives no further restriction on $A$. For (5) we have:

$$\lim_{t \to a_-} Y_2'(t) = \lim_{t \to a_-} \left\{ (A(t) \sin(t-a))' \int_{a}^{t} \frac{(A^*(s)A(s))^{-1} - (A^*(a)A(s))^{-1}}{\sin^2(s-a)} \, ds \right.$$

$$+ (A(t) \sin(t-a))' (A^*(a)A(a))^{-1} [\ctg(d-a) - \ctg(t-a)] + A^{*-1}(a) \sin^{-1}(t-a) \left\} \right.$$

$$= A(a) \int_{a}^{d} \frac{(A^*(s)A(s))^{-1} - (A^*(a)A(a))^{-1}}{\sin^2(s-a)} \, ds + A^{*-1}(a) \ctg(d-a),$$

because of

$$\lim_{t \to a_-} [- (A(t) \sin(t-a))' (A^*(a)A(a))^{-1} \ctg(t-a) + A^{*-1}(a) \sin^{-1}(t-a)] = 0.$$
Analogously

\[
\lim_{t \to a^+} Y'_2(t) = PA(a) \int_{a}^{a+n} \frac{(A^*(s)A(s))^{-1} - (A^*(a)A(a))^{-1}}{\sin^2(s-a)} \, ds + PA^{-1}(a) \cotg(d-a).
\]

Due to our conditions on \(A\) the expression

\[
\frac{(A^*(s)A(s))^{-1} - (A^*(a)A(a))^{-1}}{\sin^2(s-a)}
\]

has limits both for \(t \to a\) and for \(t \to a + \pi\), hence the above definite integrals are well defined and we may equivalently rewrite the condition (5) as

\[
\int_{a}^{a+n} \frac{(A^*(t)A(t))^{-1} - (A^*(a)A(a))^{-1}}{\sin^2(t-a)} \, dt = 0.
\]

Let us summarize our considerations in:

**Theorem.** — Let \(Q^* = Q, a \in \mathbb{R}, Y_1\) be a matrix-solution of \((Q)\) such that \(Y_1(a) = 0, Y'_1(a)\) is regular, \(Y_1(t + \pi) = PY_1(t)\) for an orthogonal constant matrix \(P, Y_1\) being regular on \((a, a + \pi)\) (or equivalently, \(a + \pi\) being the 1st conjugate point to \(a\)).

Then

\[
Y_1(t) = A(t) \sin(t - a),
\]

where

\[
A \in \mathbb{C}^2(\mathbb{R}, \mathbb{R}^r), \quad A \text{ is regular on } \mathbb{R},
\]

(7) \[A(t + \pi) = -PA(t), \quad A(a) = Y'_1(a), \quad A'(a) = 0,\]

and

(8) \[Q(t) = A''(t)A^{-1}(t) + 2A'(t)A^{-1}(t) \cotg(t - a) - 1.\]

Moreover, every solution \(y\) of \((Q)\) satisfies (1) if and only if (6) holds.

**Remark 2.** \(A'(t)A^{-1}(t) \cotg(t - a)\) in (8) is continuous by defining its value at \(a + k\pi\) as \(P^kA''(a)A^{-1}(a)P^{-k}\).

**Remark 3.** — We may always take \(Y_1\) normalized by \(Y'_1(a) = 1\) that gives \(A(a) = 1\) and

\[
\int_{a}^{a+n} \frac{(A^*(t)A(t))^{-1} - 1}{\sin^2(t-a)} \, dt = 0
\]

instead of (6).

**IV. — Constructions**

In the first part of the paragraph we shall use the condition (9) for constructing some differential systems \((Q)\) with all solutions satisfying (1).
In the second part we shall construct a two-dimensional differential system (Q) with all solutions satisfying

\[ y(t + \pi) = -y(t), \]

[i.e. \( P = -I \) in (1)], the system \( Q \) being non diagonalizable, i.e., \( Q \) being not of the form \( C^{-1} \text{diag}(q_1, \ldots, q_n)C \) for a regular constant matrix \( C \).

For both the parts relation (8) with a suitable \( A \) satisfying (7) and (9) will be considered. If such an \( A \) is taken, the only one requirement we need to guarantee is the symmetry of \( Q \). In can easily be checked that for

\[ S(t) := A'(t)A^{-1}(t) \]

the relation (8) reads

\[ Q(t) = S'(t) + S^2(t) + 2S(t)\text{ctg}(t - a) - I. \]

Compare with formulae in [5].

We shall prove:

**Lemma 3.** \( Q = Q^* \) if and only if \( S = S^* \).

**Proof.** \((\Leftarrow)\) If \( S = S^* \) then (10) gives \( Q = Q^* \).

\((\Rightarrow)\) For \( Q = Q^* \), the solution \( Y(t) := A(t)\sin(t - a) \) [hence \( Y(a) = Y^*(a) = 0 \)] is isotropic:

\[ Y^*Y' - Y*'Y = 0, \]

or

\[ (A^*A' - A*'A)\sin^2(t - a) = 0. \]

Because of continuity of \( A' \) we get \( A^*A' - A*'A = 0 \), or \( A^*A^-1A*'A = (A'A^-1)^* \). ■

As a sufficient condition for \( Q \) being not diagonalizable we shall use the following two Lemmas:

**Lemma 4.** — Let \( Q = Q^* \) and \( Q \) be diagonalizable, i.e. \( Q(t) = C^{-1}D(t)C \), where \( D(t) = \text{diag}(d_1(t), \ldots, d_n(t)) \). Then for \( R(t) := (A^*(t)A(t))^{-1} \) the matrix \( R'R^{-1}R'' \) is symmetric.

**Proof.** — Let \( Z \) be a solution of

\[ Z'' = \text{diag}(d_1(t), \ldots, d_n(t)).Z \]

determined by \( Z(a) = 0, Z'(a) = 1 \). Then

\[ Z(t) = \text{diag}(z_1(t), \ldots, z_n(t)), \]

where

\[ z_1''(t) = d_1(t)z_1(t), \]

\[ z_1(a) = 0, \quad z_1'(a) = 1. \]
Put $Y(t) := C^{-1} Z(t) C$.

Then

$$Y(a) = 0, \quad Y'(a) = 1,$$

and

$$Y'' = C^{-1} D(t) Z C = C^{-1} D(t) C Y = Q(t) Y.$$

For $Y(t) = A(t) \sin(t - a)$ we have $A(t) = C^{-1} \delta(t) C$, where $C$ is a regular constant matrix and $\delta$ is a diagonal matrix.

According to Lemma 3 it holds $A^* A' = A' A$. Hence

$$R' R^{-1} = -(A^* A)^{-1} (A^* A') = -A^{-1} A^{-1} (A^* A + A^* A') = -2 A^{-1} A^{-1} (A^* A') = -2 A^{-1} A' = -2 C^{-1} \delta^{-1} \delta' C,$$

i.e. $R' R^{-1}$ is diagonalizable.

Thus it commutes with its derivative

$$(R' R^{-1})(R' R^{-1})' = (R' R^{-1})' (R' R^{-1}),$$

or

$$R' R^{-1} (R'' R^{-1} - (R' R^{-1})^2) = (R'' R^{-1} - (R' R^{-1})^2) (R' R^{-1}).$$

We get $R' R^{-1} R'' = R'' R^{-1} R'$. Because of symmetricity of $R = (A^* A)^{-1}$,

$$R' R^{-1} R'' = (R' R^{-1} R'')^*.$$

**Lemma 5.** — Let $R(t) = \begin{pmatrix} u_1(t) & u_2(t) \\ u_2(t) & u_3(t) \end{pmatrix}$ be a 2 by 2 regular real symmetric matrix of the class $C^2(J, R^2)$. Then $R' R^{-1} R''$ is symmetric on $J$ if and only if

$$\det \begin{pmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1(t) & u_2(t) & u_3(t) \\ u_1(t) & u_2(t) & u_3(t) \end{pmatrix} = W(u_1, u_2, u_3) = 0 \text{ on } J.$$

**Proof.** — Let $\Delta := \det R$. Then

$$R^{-1} = \Delta^{-1} \begin{pmatrix} u_3 & -u_2 \\ -u_2 & u_1 \end{pmatrix},$$

$$R' R^{-1} R'' = \Delta^{-1} \begin{pmatrix} u_1' u_3 - u_2 u_2' & -u_1' u_2 + u_1 u_2' \\ u_2' u_3 - u_2 u_3' & -u_2 u_2' + u_4 u_3' \end{pmatrix} \begin{pmatrix} u_1'' & u_2'' \\ u_2'' & u_3'' \end{pmatrix},$$

and $R' R^{-1} R''$ is symmetric if and only if

$$u_1' u_2' u_3 - u_2 u_2' u_2' - u_1' u_2 u_3' + u_1 u_2' u_3' = u_1' u_2' u_3 - u_1' u_2 u_3' - u_2 u_2' u_2' + u_1 u_2' u_3,'$$
or
$$u_1 (u'_2 u'_3 - u'_2 u'_3) - u_2 (u'_1 u'_3 - u'_1 u'_3) + u_3 (u'_1 u'_2 - u'_1 u'_2) = 0,$$

or \( W(u_1, u_2, u_3) = 0 \). \( \blacksquare \)

PART I. — We are going to construct a system \((Q)\) with all solutions satisfying (1) for an orthogonal constant matrix \(P\).

Let a symmetric matrix \(M \in \mathbb{C}^2(\mathbb{R}, \mathbb{R}^2)\) be periodic,

$$M(t + \pi) = M(t), \quad \text{and} \quad \int_0^\pi M(t) dt = 0.$$

Moreover, let the eigenvalues of \(M\) be greater than \(-1\). Then the matrix \(M(t) \sin^2 t + I\) has only positive eigenvalues. Let \(N(t)\) denote the symmetric square root with only positive eigenvalues of the symmetric matrix \((I + M(t) \sin^2 t)^{-1}\). Then \(N \in \mathbb{C}^2(\mathbb{R}, \mathbb{R}^2)\), \(\det N(t)\) is always positive,

$$N(t + \pi) = N(t), \quad N^*(t) = N(t), \quad N(0) = I, \quad N'(0) = 0,$$

and

$$\int_0^\pi N^{-2}(t) - I \sin^2 t dt = \int_0^\pi M(t) dt \neq 0.$$

We put \(A(t) := B(t) N(t)\), where \(B \in \mathbb{C}^2(\mathbb{R}, \mathbb{R}^2)\) is an orthogonal matrix. With respect to Lemma 3 we are looking for such a \(B\), that \(S := A' A^{-1}\) is symmetric. Hence we need

$$0 = S - S^* = (BN)'(BN)^{-1} - (BA)^{-1} (BA)^* = 2 B' B^{-1} + B' B^{-1} + B (N' N^{-1} - (N' N^{-1})*) B^{-1},$$

because of orthogonality of \(B\) and skew-symmetricity of \(B'B^{-1}\), see e. g. [4]. We get

$$B' = B. \frac{1}{2} (N' N^{-1} - (N' N^{-1})*).$$

Since \(1/2 (N' N^{-1} - (N' N^{-1})*) \in \mathbb{C}^1(\mathbb{R}, \mathbb{R}^2)\) is skew-symmetric, \(B\) is orthogonal for every \(t\) if it is orthogonal at some \(t_0\).

By taking \(B(0) = I\) we have \(B \in \mathbb{C}^2(\mathbb{R}, \mathbb{R}^2)\) and orthogonal for every \(t\). Then \(S = S^*\) and also \(Q = Q^*\) due to lemma 3. For \(A = B N\) we get

$$\int_0^\pi \frac{(A^*(t) A(t))^{-1} - I}{\sin^2 t} dt = \int_0^\pi \frac{N^{-2}(t) - I}{\sin^2 t} dt = 0.$$

Evidently \(A \in \mathbb{C}^2(\mathbb{R}, \mathbb{R}^2)\), \(A(0) = N(0) = I, A'(0) = B'(0) + N'(0) = 0\), and \(A\) is regular on \(\mathbb{R}\). Moreover, since \(N\) is periodic, the system (10) is also periodic and due to Floquet Theory, there exists a regular real constant matrix \(C\) such that \(B(t + \pi) = C B(t)\) for all \(t\). Because of orthogonality of \(B\), \(C\) is also orthogonal. Hence

$$A(t + \pi) = B(t + \pi) N(t + \pi) = C B(t) N(t) = C A(t).$$

For \(P := -C\) we have

$$A(t + \pi) = -PA(t) \quad \text{for all} \quad t.$$
Let us summarize our construction. \( M \in \mathbb{C}^2 (\mathbb{R}, \mathbb{R}^2) \) is symmetric, periodic with all eigenvalues \( > -1 \), and \( \int_0^T M(t) \, dt = 0 \). \( N \) is the symmetric square root of \((I + M(t) \sin^2 t)^{-1}\) with only positive eigenvalues. \( B \) is a solution of (10) with \( B(0) = I \). Thus (9) is satisfied for \( A := BN, a = 0 \), and \( Q \) defined by (8) is symmetric. Also \( P := -B(t + \pi)B^{-1}(t) \) is a constant real orthogonal matrix and \( A(t + \pi) = -PA(t) \).

Due to Theorem 1, all solutions of the system (Q) with \( Q \) given by (8) satisfy (1).

**PART II.** - Now we are going to specify the matrix \( P \) in (1), namely we take \( P = -I \). The aim of this part is to construct a two-dimensional system (Q) with non-diagonalizable \( Q \) having only half-periodic solutions, \( y(t + \pi) = -y(t) \).

Again we use Theorem 1 and relation (8) for constructing \( Q \). We are looking for \( A \) of the form

\[
A(t) = \begin{pmatrix}
H(t) & D(t) \\
G(t) & 0
\end{pmatrix}
\]

where periodic \( H, D, G \in \mathbb{C}^2 (\mathbb{R}, \mathbb{R}^2) \),

\[
D(t) = \begin{pmatrix}
d_1(t) & 0 \\
0 & d_2(t)
\end{pmatrix}
\]

is diagonal,

\[
G(t) = \begin{pmatrix}
\cos \alpha(t) & \sin \alpha(t) \\
-\sin \alpha(t) & \cos \alpha(t)
\end{pmatrix}, \quad H(t) = \begin{pmatrix}
\cos \beta(t) & \sin \beta(t) \\
-\sin \beta(t) & \cos \beta(t)
\end{pmatrix}
\]

are orthogonal 2 by 2 matrices such that

\[
H(0) = I, \quad H'(0) = 0; \quad D(0) = I; \quad D'(0) = 0;
\]

\[
G(0) = I, \quad G'(0) = 0;
\]

that is satisfied by

\[
\alpha, \beta, d_i \in \mathbb{C}^2 (\mathbb{R}, \mathbb{R}),
\]

(11) \( \alpha(0) = 0, \quad \alpha'(0) = 0, \quad \beta(0) = 0, \quad \beta'(0) = 0, \quad d_i(0) = 1, \quad d_i'(0) = 0; \quad i = 1, 2. \)

With respect to Lemma 3 we need \( A^*A = A^*A \), or

\[
D(H^*H' - H'^*H)D = GG^*D^2 - D^2 G' G^*,
\]

or

\[
2 \beta'(t) \left( \begin{array}{cc} 0 & d_1 d_2 \\ -d_1 d_2 & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & -d_1^2 - d_2^2 \\ d_1^2 + d_2^2 & 0 \end{array} \right) \alpha'(t),
\]
or equivalently

(12) \[ 2 \beta d_1 d_2 + \alpha (d_1^2 + d_2^2) = 0 \quad \text{on } \mathbb{R}. \]

Consider now (9) for \( a = 0 \):

\[
\int_0^\pi \frac{(A^*(t) A(t))^{-1} - I}{\sin^2 t} dt = \int_0^\pi (G^* D^2 G - I) \sin^{-2} t dt
\]

\[
= \int_0^\pi \left[ (d_1^2 - 1) \cos^2 \alpha + (d_2^2 - 1) \sin^2 \alpha + \frac{1}{2} (d_1^2 - d_2^2) \sin^2 \alpha
\right]
\]

\[
\times \left[ (d_1^2 - 1) \sin^2 \alpha + (d_2^2 - 1) \cos^2 \alpha \right] \sin^{-2} t dt.
\]

Let

\[
f_i \in C^2(\mathbb{R}, \mathbb{R}), \quad i = 1, 2,
\]

and

(13) \[
\begin{aligned}
&f_i(t + \pi) = f_i(t), \\
&f_i(\pi/2 + t) = -f_i(\pi/2 - t), \quad \text{or} \quad f_i(t) = -f_i(\pi - t), \\
&|f_i(t)| < 1, \\
&f_i(0) = 0, \quad f'_i(0) = 0.
\end{aligned}
\]

Then \( d_i := (1 + f_i(t))^{-1/2} \) satisfy

(14) \[
\begin{aligned}
d_i \in C^2(\mathbb{R}, \mathbb{R}), \\
d_i(t) > 0, \quad d_i(0) = 1, \quad d'_i(0) = 0, \\
d_i(t + \pi) = d_i(t), \\
d_i^{(-2)}(t) - 1 = -(d_i^{(-2)}(\pi - t) - 1), \quad i = 1, 2.
\end{aligned}
\]

Hence

\[
\int_0^\pi (d_i^{(-2)}(t) - 1) \frac{\cos^2 \alpha(t)}{\sin^2 t} dt
\]

\[
= \int_0^\pi (d_i^{(-2)}(t) - 1) \frac{\cos^2 \alpha(t)}{\sin^2 t} dt + \int_0^\pi (d_i^{(-2)}(\pi - t) - 1) \frac{\cos^2 \alpha(\pi - t)}{\sin^2 (\pi - t)} dt = 0
\]

if

(15) \[ \alpha(t) = \alpha(\pi - t). \]
Similarly

\[ \int_0^\alpha (d_1^{-2}(t) - 1) \frac{\sin^2 \alpha(t)}{\sin^2 t} dt = 0 \]

and

\[ \int_0^\alpha (d_1^{-2}(t) - d_2^{-2}(t)) \frac{\sin 2 \alpha(t)}{\sin^2 t} dt = \int_0^\alpha [(d_1^{-2}(t) - 1) - (d_2^{-2}(t) - 1)] \frac{\sin 2 \alpha(t)}{\sin^2 t} dt = 0, \]

because of \( \sin 2 \alpha (\pi - t) = \sin 2 \alpha (t) \).

Let us see for conditions on \( \alpha \) and \( \beta \). If

\[
\begin{cases}
  \alpha \in C^2(R, R), & \alpha(t + \pi) = \alpha(t), \\
  \alpha(t) = \alpha(\pi - t) & [\text{see (15)}], \\
  \alpha(0) = 0, & \alpha'(0) = 0 & [\text{see (11)}],
\end{cases}
\]

then \( G \in C^2(R, R) \) is periodic, \( G(0) = 1, G'(0) = 0 \). The same remains true for \( G \) if instead of \( \alpha \) the function \( k \alpha \) is taken, \( k \) being a constant.

Due to (12):

\[ \beta(t) = -\int_0^t \frac{\alpha'(s)}{2} \left( \frac{d_1(s)}{d_2(s)} + \frac{d_2(s)}{d_1(s)} \right) ds, \]

and hence

\[ \beta \in C^2(R, R), \]

\[ \beta(0) = \beta'(0) = 0, \]

and because of periodicity of \( \alpha, d_1, d_2 \) also

\[ \beta(t + \pi) = \beta(t) - k_0, \]

where

\[ k_0 = \int_0^\alpha \frac{\alpha'(s)}{2} \left( \frac{d_1(s)}{d_2(s)} + \frac{d_2(s)}{d_1(s)} \right) ds. \]

If \( k_0 = 0 \), then \( H \in C^2(R, R^2) \) is periodic, and that is what we need.

If \( k_0 \neq 0 \), then take \((2 \pi/k_0) \alpha(t)\) instead of \( \alpha(t) \).

Then \( \beta(t + \pi) = \beta(t) - 2 \pi, \) and \( H \in C^2(R, R^2) \) is periodic.

Since again \( \beta(0) = \beta'(0) = 0 \), we have \( H(0) = 1, H'(0) = 0. \)

It remains to look for conditions of non-diagonalization of \( Q \). According to Lemma 5 it would be sufficient to have

\[ R = (A^* A)^{-1} = \begin{pmatrix} \frac{u_1}{u_2} & \frac{u_2}{u_3} \end{pmatrix}. \]
such that $W(u_1, u_2, u_3)$, Wronskian of $u_1, u_2, u_3$, be different from zero. Since

$$R = G^* D^{-2} G = \begin{bmatrix}
    d_1^{-2} \cos^2 \alpha + d_2^{-2} \sin^2 \alpha & \frac{1}{2} (d_1^{-2} - d_2^{-2}) \sin 2 \alpha \\
    \frac{1}{2} (d_1^{-2} - d_2^{-2}) \sin 2 \alpha & d_1^{-2} \sin^2 \alpha + d_2^{-2} \cos^2 \alpha
\end{bmatrix},$$

if

(17) $d_1$ and $d_2$ have different positive constant values

on some subinterval $(c, d)$ of $(\pi/4, \pi/3)$,

then the Wronskian of

$$d_1^{-2} \cos^2 \alpha + d_2^{-2} \sin^2 \alpha, \quad \frac{1}{2} (d_1^{-2} - d_2^{-2}) \sin 2 \alpha, \quad d_1^{-2} \sin^2 \alpha + d_2^{-2} \cos^2 \alpha$$

on the interval $(c, d)$ has the value $(\alpha'(t))^3$. $W(y_1, y_2, y_3)$, where

$$y_1(t) = d_1^{-2} \cos^2 t + d_2^{-2} \sin^2 t,$$
$$y_2(t) = \frac{1}{2} (d_1^{-2} - d_2^{-2}) \sin 2 t,$$
$$y_3(t) = d_1^{-2} \sin^2 t + d_2^{-2} \cos^2 t,$$

$d_1^{-2} \neq d_2^{-2}$ being constants, are three linearly independent solutions of $y''' + 4 y' = 0$, having $c_1 + c_2 \sin 2 t + c_3 \cos 2 t$ as its general solution. Hence $W(y_1, y_2, y_3) \neq 0$ and if $\alpha$ besides of above restrictions complies with

(18) $\alpha'(t) \neq 0$ on $(c, d),$

then our $Q$ is not diagonalizable.

We summarize our considerations. Let $f_1$ satisfy (13), $f_1$ and $f_2$ being different constants on $(c, d) \subset (\pi/4, \pi/3)$, then $d_1(t) := (1 + f_1(t))^{-1/2}$ comply with (14), and (17). Take $\alpha$ satisfying (16) and (18). If

$$k_0 = \int_0^t \frac{\alpha'}{2} \left( \frac{d_1}{d_2} + \frac{d_2}{d_1} \right) ds \neq 0,$$

take $(2 \pi/k_0) \alpha(t)$ instead of the $\alpha(t)$. Define

$$\beta(t) := - \int_0^t \frac{\alpha'(s)}{2} \left( \frac{d_1(s)}{d_2(s)} + \frac{d_2(s)}{d_1(s)} \right) ds.$$
Using $\alpha$, $\beta$, and $d$, we get periodic matrices $G$, $H$, and $D$. For $A := HDG$ we define $Q$ by means of (8). This $Q$ is symmetric [Lemma 3 and relation (12)], non-diagonalizable [Lemma 5 and conditions (17) and (18)]. Our $A$ complies with Theorem 1 for $P = -1$ [i.e. $A(t + \pi) = A(t)$] and satisfies relation (9) with $a = 0$. Hence all solutions of (Q) satisfy $y(t + \pi) = -y(t)$.

Remark 4. — Having a two-dimensional second order non-diagonalizable system (Q) with all solutions satisfying $y(t + \pi) = -y(t)$, we may construct a non-diagonalizable system of the same property for any dimension $n (n > 2)$ simply by extending the second order system (Q) by adding $n - 2$ equations $y_i^\prime = -y_i$, $i = 3, \ldots, n$.

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ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE