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HYPERSURFACES OF EINSTEIN MANIFOLDS

BY NORIHITO KOISO (1)

0. Introduction and results

Let $(\overline{M}, \overline{g})$ be an Einstein manifold of dimension $n+1$ $(n \geqq 2)$. We consider certain classes of hypersurfaces in $(\overline{M}, \overline{g})$. First, let (M, g) be a totally umbilical hypersurface in $(\overline{M}, \overline{g})$, i.e., we assume that the second fundamental form α satisfies $\alpha = fg$ for some function f on M. If we know completely the curvature tensor of $(\overline{M}, \overline{g})$, we can get much information on (M, g). For example, if $(\overline{M}, \overline{g})$ is a symmetric space, then (M, g) is also a locally symmetric space, and so the classification of such pairs $[(\overline{M}, \overline{g}), (M, g)]$ reduces to Lie group theory (*see* Chen [4] (2), Chen and Nagano [5], Naitoh [10]). But if we know nothing about $(\overline{M}, \overline{g})$, we can *only* say that (M, g) has constant scalar curvature. In fact, we will prove the following.

THEOREM A. — *Let* (M, g) *be a real analytic riemannian manifold with constant scalar curvature. Then, there exists an Einstein manifold* $(\overline{M}, \overline{g})$ (which may be non-complete) such that (M, g) *is isometrically imbedded into* $(\overline{M}, \overline{g})$ *as a totally geodesic hypersurface.*

This Theorem means also that there exist many examples of totally geodesic Einstein hypersurfaces in Einstein manifolds. But, if we assume that $(\overline{M}, \overline{g})$ is complete (or compact), the situation changes drastically. In fact, we will show the following.

THEOREM B. — *Let* (M, g) *be a totally umbilical Einstein hypersurface in a complete Einstein manifold* $(\overline{M}, \overline{g})$. *Then the only possible cases are:*

(*a*) *g has positive Ricci curvature. Then g and $\overline{g}$ have constant sectional curvature;*

(*b*) *$\overline{g}$ has negative Ricci curvature. If $\overline{M}$ is compact or $(\overline{M}, \overline{g})$ is homogeneous, then g and $\overline{g}$ have constant sectional curvature;*

(*c*) *g and $\overline{g}$ have zero Ricci curvature. If $(\overline{M}, \overline{g})$ is simply connected, then $(\overline{M}, \overline{g})$ decomposes as $(\tilde{M}, \tilde{g}) \times \mathbf{R}$, where $(\tilde{M}, \tilde{g})$ is a totally geodesic hypersurface in $(\overline{M}, \overline{g})$ which contains (M, g).*

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(2) Theorem 1 is not true as stated, but Theorem 2 is true. *See* Proof of Proposition 15 in Naitoh [10].

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To prove this Theorem, we need essentially a result of D. M. DeTurck and J. L. Kazdan according to which all Einstein metrics are real analytic. In other words, the manifold $(\overline{M}, \overline{g})$ in Theorem A is uniquely defined by (M, g) (Prop. 4). If we apply Proposition 4 to a Kähler-Einstein manifold $(\overline{M}, \overline{g})$, we can get much information on (M, g) and $(\overline{M}, \overline{g})$, even without assuming anything on (M, g), since in this situation, the Gauss-Codazzi equations imply many properties of (M, g).

THEOREM C. — *Let* $(\overline{M}, \overline{g})$ *be a simply connected complete Kähler-Einstein manifold with Ricci curvature* $\overline{e}$. *If there exists a totally geodesic real hypersurface* (M, g) *in* $(\overline{M}, \overline{g})$, *then there exists a totally geodesic complex hypersurface* $(\tilde{M}, \tilde{g})$ *in* $(\overline{M}, \overline{g})$, *and* $(\overline{M}, \overline{g})$ *decomposes as* $(\overline{M}, \overline{g})=(\tilde{M}, \tilde{g})\times(S, \overline{e})$, *where* $(S, \overline{e})$ *means the simply connected and complete Riemann surface of constant Ricci curvature* $\overline{e}$. *In this decomposition,* M *is contained in* $\tilde{M}\times \operatorname{Im}\gamma$, *where* γ *is a geodesic in* S.

Remark that Theorem C holds locally even if $(\overline{M}, \overline{g})$ is not complete. Next, let (M, g) be an orientable minimal hypersurface in an orientable manifold $(\overline{M}, \overline{g})$. By Corollary 3.6.1 in Simons [11], if $\overline{g}$ has positive Ricci curvature, then there is no orientable compact stable minimal hypersurface in $(\overline{M}, \overline{g})$. By a similar method, we will show.

THEOREM D. — *Let* $(\overline{M}, \overline{g})$ *be an orientable Einstein manifold with zero Ricci curvature. Then all orientable compact stable minimal hypersurfaces without singularity are totally geodesic.*

Combining with Theorem C, we will get.

COROLLARY E. — *Let* $(\overline{M}, \overline{g})$ *be a Kähler-Einstein manifold with zero Ricci curvature and without local factor* $\mathbb{C}$. *Then there is no orientable compact stable minimal real hypersurface without singularity.*

Remark that we do not assume in Theorem A, B, C that (M, g) is complete. The paper is organized as follows: In **1**, we derive some fundamental formulae and prove Theorem D. In **2**, we consider the real case and prove Theorem A and Theorem B. In **3**, we consider the Kähler case and prove Theorem C and Corollary E. The author would like to express his sincere gratitude to Professors J.-P. Bourguignon and R. Michel. Theorem A is an answer to a question of R. Michel and Corollary E is a generalization of a remark of J.-P. Bourguignon.

1. Preliminary and propositions

Let $(\overline{M}, \overline{g})$ be an Einstein manifold of dimension $n+1\geqq 3$ and M a hypersurface in $(\overline{M}, \overline{g})$ with induced metric g. In this paper, riemannian manifolds are not assumed to be complete, unless otherwise stated. The second fundamental form α is given by:

$$\alpha(X, Y)N=\overline{D}_X Y-D_X Y,$$

where N is the unit normal vector field, X and Y are vector fields on M, and D (resp. $\overline{D}$) is the covariant derivative of (M, g) [resp. $(\overline{M}, \overline{g})$]. The following formulae are known as the Gauss-Godazzi equations:

$$\overline{R}(X, Y; Z, U)=R(X, Y; Z, U)+\alpha(X, U)\alpha(Y, Z)-\alpha(X, Z)\alpha(Y, U),$$

$$\overline{R}(X, Y; Z, N)=(D_Y\alpha)(X, Z)-(D_X\alpha)(Y, Z),$$

where R (resp. $\overline{R}$) is the curvature tensor of (M, g) [resp. $(\overline{M}, \overline{g})$] and the sign convention is taken in such a way that $R(X, Y; X, Y)\geqq 0$ for the standard sphere. Set:

$$\overline{R}(X, N; Y, N)=\beta(X, Y).$$

Then, the Ricci tensor $\overline{r}$ of $(\overline{M}, \overline{g})$ is given by:

$$\overline{r}(X, Y)=r(X, Y)+\alpha^2(X, Y)-\mu\alpha(X, Y)+\beta(X, Y),$$

$$\overline{r}(X, N)=(d\mu)(X)+(\delta\alpha)(X),$$

$$\overline{r}(N, N)=\operatorname{tr}\beta,$$

where r is the Ricci tensor of (M, g), μ is the mean curvature defined by $\mu=\operatorname{tr}\alpha$, and α^2 and $\delta\alpha$ are defined by:

$$(\alpha^2)_{ij}=\alpha_i^k\alpha_{kj},$$

$$(\delta\alpha)_i=-D^k\alpha_{ki}.$$

Since $\overline{g}$ is an Einstein metric, i.e., $\overline{r}=\overline{eg}$ for some real number $\overline{e}$, we see that:

$$\overline{e}g=r+\alpha^2-\mu\alpha+\beta, \tag{1.1.a}$$

$$0=d\mu+\delta\alpha, \tag{1.1.b}$$

$$\overline{e}=\operatorname{tr}\beta, \tag{1.1.c}$$

and so:

$$(n-1)\overline{e}=u+\operatorname{tr}\alpha^2-\mu^2, \tag{1.2}$$

where u is the scalar curvature of (M, g). Thus it is easy to check the following.

PROPOSITION 1. — *If* (M, g) *is a minimal hypersurface* (*i.e.*, $\mu=0$) *of* an *Einstein manifold* $(\overline{M}, \overline{g})$, *then* $u\leqq(n-1)\overline{e}$. *Equality holds if and only if* (M, g) *is a totally geodesic hypersurface in* $(\overline{M}, \overline{g})$.

PROPOSITION 2. — *If* (M, g) *is a totally umbilical hypersurface of an Einstein manifold* $(\overline{M}, \overline{g})$, *i.e.*, $\alpha=fg$ *for some* $f\in C^\infty(M)$, *then* f *is constant and* $u\geqq(n-1)\overline{e}$. *Equality holds if and only if* (M, g) *is a totally geodesic hypersurface in* $(\overline{M}, \overline{g})$.

Proof. — By (1.1.b), $0=d\operatorname{tr}(fg)+\delta(fg)=(n-1)df$, so f is constant. Since $\mu=nf$ and $\operatorname{tr}\alpha^2=nf^2$, the latter half is obvious by (1.2).

Q.E.D.

Without any further property of β, we cannot proceed any more. To answer the question "What is the meaning of β?" we consider a one-parameter family of hypersurfaces in $(\overline{M}, \overline{g})$. Denote by i and i_t the mappings: $M \times \mathbf{R} \to \overline{M}$ and $M \to \overline{M}$, defined by:

$$i(x, t) = \exp_x t N, \qquad i_t(x) = i(x, t).$$

Then there is an open set R of $M \times \mathbf{R}$ containing $M \times \{0\}$ such that $g_t = i_t^* \overline{g}$ is a riemannian metric on $\{x \in M; (x, t) \in R\}$. We identify $\overline{M}$ with its image R (locally) and we see that $g_t + dt^2$ coincides with $\overline{g}$. In fact, N extends as the vector field d/dt, whose integral curves are geodesics in $(\overline{M}, \overline{g})$, and:

$$\frac{d}{dt}\overline{g}(X, N) = \overline{g}(\overline{D}_N X, N) + \overline{g}(X, \overline{D}_N N) = \overline{g}(\overline{D}_X N, N) = \frac{1}{2} X(\overline{g}(N, N)) = 0,$$

where we identify $X \in T_x M$ with the vector field along the geodesic $i_t(x)$ defined by $X(i_t(x)) = i_{t*} X$. We derive the relation between g', g'' and α, β, where $'$ means the derivative with respect to t:

$$\begin{aligned} g'(X, Y) &= (\overline{g}(X, Y))' = \overline{g}(\overline{D}_N X, Y) + \overline{g}(X, \overline{D}_N Y) \\ &= X(\overline{g}(N, Y)) - \overline{g}(N, \overline{D}_X Y) + Y(\overline{g}(X, N)) - g(\overline{D}_Y X, N) = -2\alpha(X, Y), \\ \beta(X, Y) &= \overline{g}(\overline{R}(X, N) Y, N) = \overline{g}(\overline{D}_{[X, N]} Y - \overline{D}_X \overline{D}_N Y + \overline{D}_N \overline{D}_X Y, N) \\ &= -\overline{g}(\overline{D}_X \overline{D}_Y N, N) + (\overline{g}(\overline{D}_X Y, N))' - \overline{g}(\overline{D}_X Y, \overline{D}_N N) \\ &= -X(\overline{g}(\overline{D}_Y N, N)) + \overline{g}(\overline{D}_Y N, \overline{D}_X N) + (\alpha(X, Y))'. \end{aligned}$$

Here, $\overline{g}(\overline{D}_Y N, N) = 0$ and $\overline{g}(\overline{D}_Y N, X) = -\alpha(X, Y)$. Thus we get:

$$g' = -2\alpha, \tag{1.3}$$

$$\beta = \alpha^2 - (1/2) g''. \tag{1.4}$$

The Einstein equation becomes:

$$\begin{aligned} \overline{e} g &= r + (1/2)(g')^2 - (1/4)(\operatorname{tr} g') g' - (1/2) g'', \\ 0 &= -(1/2) d \operatorname{tr} g' - (1/2) \delta g', \\ \overline{e} &= -(1/2) \operatorname{tr} g'' + (1/4) \operatorname{tr} (g')^2. \end{aligned}$$

We conclude that:

$$g'' = -2\overline{e} g + 2r - (1/2)(\operatorname{tr} g') g' + (g')^2, \tag{1.5.a}$$

$$d \operatorname{tr} g' + \delta g' = 0, \tag{1.5.b}$$

$$\operatorname{tr} (g')^2 - (\operatorname{tr} g')^2 = 4(n-1)\overline{e} - 4u. \tag{1.5.c}$$

Remark that these equations hold on R, where r, tr, $(\ )^2$, δ and u are defined by g_t. We shall solve this equation in 2.

Before developing this equation, we point out some facts related to Proposition 1. Assume that M is compact without boundary and that i_0 is a *stable* minimal immersion. (Here, stable means: the second derivative of volume is non-negative for any variation.) Then, if the unit normal vector field N is globally defined on M:

$$0 \leqq \left(\int_M v_g \right)''_{t=0} = -\frac{1}{2} \int_M \operatorname{tr}(g')^2 v_g + \frac{1}{2} \int_M \operatorname{tr} g'' v_g + \frac{1}{4} \int_M (\operatorname{tr} g')^2 v_g,$$

where v_g denotes the volume element of g. By (1.3) and (1.4), we see that:

$$0 \leqq \int_M (-2(\alpha, \alpha) - (\operatorname{tr} \beta - \operatorname{tr} \alpha^2)) v_g = -\int_M (\operatorname{tr} \alpha^2 + \bar{e}) v_g.$$

Here, $\operatorname{tr} \alpha^2 + \bar{e} = n\bar{e} - u$ by (1.2), and we get:

PROPOSITION 3. — *If* (M, g) *is compact without boundary and immersed in an Einstein manifold* ($\overline{\mathrm{M}}$, $\bar{g}$) *as a stable minimal hypersurface with trivial normal bundle then:*

$$\int_M u v_g \geqq n\bar{e} \operatorname{Vol}(M, g).$$

Moreover, if $\bar{e}=0$, *then* $u=0$ *and* (M, g) *is totally geodesic.*

Proof. — The integral inequality is obvious. If $\bar{e}=0$, then $\int_M u v_g \geqq 0$. But Proposition 1 implies $u \leqq 0$, so $u=0$. Then the equality in Proposition 1 holds, so (M, g) is totally geodesic.

Q.E.D.

Proof of Theorem D. — It is obtained as a corollary of Proposition 3.

Q.E.D.

Remark 4. — In Theorem D, if $\overline{\mathrm{M}}$ is simply connected, then the assumption that M is orientable is not necessary. In fact, Lemma 4.5 and Theorem 4.6 in Hirsch [8] says that all compact hypersurfaces in a simply connected manifold are orientable.

2. Solution of (1.5) – real case

Consider equation (1.5). Theorem 5.2 in DeTurck and Kazdan [6] says that all Einstein metrics are real analytic with respect to harmonic coordinates. This implies that the solution of (1.5) is unique for given initial data $g=g_0$ and $g'=h$, as long as g_t is positive definite. Moreover, we get the following global uniquness property.

PROPOSITION 5. — *Let* (M, g) *be a real analytic hypersurface of a simply connected and complete Einstein manifold* ($\overline{\mathrm{M}}$, $\bar{g}$) *with second fundamental form* α. *Assume that there is another simply connected and complete Einstein manifold* ($\overline{\mathrm{M}}_1$, $\bar{g}_1$) *such that* (M, g) *is imbedded* *into* $(\overline{M}_1, \bar{g}_1)$ *as a real analytic hypersurface with the same second fundamental form* α. *Then* $(\overline{M}, \bar{g})$ *and* $(\overline{M}_1, \bar{g}_1)$ *are isometric with one another.*

Proof. — By the uniqueness Theorem 5.4 in DeTurck and Kazdan [6].

Q.E.D.

Conversely, by Cauchy-Kovalevski's existence Theorem, we can solve (1.5.*a*) locally for any real analytic initial data, since the Ricci tensor r is expressed in terms of the derivatives up to the second order of the metric tensor g.

PROPOSITION 6. — *Let* (M, g) *be a real analytic riemannian manifold and* α *a real analytic symmetric bilinear form on* M *which satisfies* $d\,\mathrm{tr}\,\alpha+\delta\alpha=0$ *and* $\mathrm{tr}\,\alpha^2-(\mathrm{tr}\,\alpha)^2=(n-1)\bar{e}-u$. *Then, there exists an Einstein manifold* $(\overline{M}, \bar{g})$ *with* $\bar{r}=\bar{e}\bar{g}$ *in which* (M, g) *is imbedded as a hypersurface with second fundamental form* α.

Proof. — There exists a unique real analytic solution g_t of (1.5.*a*) with initial data $g_0=g$ and $g'_0=-2\alpha$. We must check that this solution satisfies (1.5.*b*) and (1.5.*c*). By standard tensor calculus, we see using (1.5.*a*) that:

$$(\mathrm{tr}\,g')'=-2n\bar{e}+2u-(1/2)(\mathrm{tr}\,g')^2,$$

$$(\delta g')'=(1/4)\,d\,\mathrm{tr}\,(g')^2-(1/2)(\mathrm{tr}\,g')\,\delta g'-du,$$

$$(\mathrm{tr}\,(g')^2)'=-4\bar{e}\,\mathrm{tr}\,g'-(\mathrm{tr}\,g')\,\mathrm{tr}\,(g')^2+4(r, g'),$$

$$u'=\Delta\,\mathrm{tr}\,g'+\delta\delta\,g'-(r, g') \quad (\textit{see}\ \text{Berger [1] (2.11)}).$$

Therefore:

$$(d\,\mathrm{tr}\,g'+\delta g')' = -(1/2)(\mathrm{tr}\,g')(d\,\mathrm{tr}\,g'+\delta g')+(1/4)\,d(\mathrm{tr}\,(g')^2-(\mathrm{tr}\,g')^2+4u),$$

$$(\mathrm{tr}\,(g')^2-(\mathrm{tr}\,g')^2+4u)' = 4\,\delta(d\,\mathrm{tr}\,g'+\delta g')-(\mathrm{tr}\,g')(\mathrm{tr}\,(g')^2-(\mathrm{tr}\,g')^2+4u-4(n-1)\bar{e}).$$

Thus analyticity implies that (1.5.*b*) and (1.5.*c*) hold for all t.

Q.E.D.

Proof of Theorem A. — In the above Proposition, set $\alpha=0$ and $\bar{e}=u/(n-1)$.

Q.E.D.

Remark 7. — In the situation of Theorem A, the change $t\to -t$ of the parameter t preserves the solution. Therefore there is an isometry of $(\overline{M}, \bar{g})$ of order 2 such that all points of M are fixed.

Let g_t be an analytic solution of (1.5) with initial data $g_0=g$ and $g'_0=h$. If the metric g_t+dt^2 on R does not extend to a complete metric, for example, if the sectional curvature of g_t+dt^2 diverges for $t\to t_0$, then we see that (M, g) cannot be immersed in any complete Einstein manifold as a hypersurface with second fundamental form $\alpha=-(1/2)h$. We apply this method to a family $g_t=f(t)^2 g_0$ where g_0 is an Einstein metric and $f(t)$ is a positive function of t such that $f(0)=1$. Let this family g_t be a solution of (1.5). Then:

$$g'_t=2(f'(t)/f(t))\,g_t,$$

$$g''_t=2((f'(t)/f(t))^2+f''(t)/f(t))\,g_t.$$

From now on, we will omit t for simplicity. Since the Ricci tensor is invariant under multiplication by a scalar factor:

$$r=r_0=e_0 g_0=e_0 f^{-2} g,$$

where e_0 is the Ricci curvature of g_0. As a result, (1.5.c) becomes:

$$4n(f'/f)^2-4n^2(f'/f)^2=4(n-1)\bar{e}-4ne_0 f^{-2},$$

$$(2.1) \qquad (f')^2=e_0/(n-1)-(\bar{e}/n)f^2.$$

Further (1.5.a) becomes:

$$ff''=-\bar{e}f^2+e_0-(n-1)(f')^2=-(\bar{e}/n)f^2 \quad \text{[using (2.1)]},$$

$$(2.2) \qquad f''=-(\bar{e}/n)f.$$

Equation (2.2) reduces to (2.1), except in the case where f is constant. We get the following solutions.

(2.3.a) If $\bar{e}>0$, then $e_0>0$ and:

$$f(t)=(\sqrt{e_0/(n-1)}/2\sqrt{\bar{e}/n})\sin(\pm\sqrt{\bar{e}/n}(t+C)).$$

(2.3.b) If $\bar{e}=0$, then $e_0 \geqq 0$ and:

$$f(t)=\pm\sqrt{e_0/(n-1)}\,t+C.$$

(2.3.c) If $\bar{e}<0$, then:

$$f(t)=|(n/4\bar{e})\exp(\pm\sqrt{-\bar{e}/n}(t+C))+(e_0/(n-1))\exp(\mp\sqrt{-\bar{e}/n}(t+C))|.$$

Therefore, if $(\overline{M}, \bar{g})$ is an Einstein manifold and if (M, g_0) is an Einstein manifold which is isometrically immersed into $(\overline{M}, \bar{g})$ as a totally umbilical hypersurface, then $\bar{g}$ is locally isometric with $f(t)^2 g_0+dt^2$, where $f(t)$ is one of the solutions (2.3). In fact, since the equation expressing that a hypersurface is totally umbilical is elliptic, (M, g_0) is analytically immersed into $(\overline{M}, \bar{g})$. Now, we check completeness of the metric $\bar{g}=f(t)^2 g_0+dt^2$.

Remark 8. — If (M, g_0) is a complete Einstein manifold with negative Ricci curvature, then (2.3 c) gives a complete Einstein metric. This metric is not homogeneous by Theorem B, if (M, g_0) does not have constant sectional curvature.

Let $f(t)$ be one of the solutions (2.3) and set $g_t=f(t)^2 g_0$ and $\bar{g}=g_t+dt^2$ on $\overline{M}=M\times I$. Denote by $\overline{K}(V, W)$ [resp. $K_0(X, Y)$] the sectional curvature of $(\overline{M}, \bar{g})$ [resp. (M, g_0)] of the plane spanned by V and W [resp. X and Y . Suppose that X and Y are unit orthogonal vectors on (M, g_0). Then, by the identification $\overline{M}=M\times I$ and the formulae in **1**, we see that:

$$\begin{aligned}(2.4) \quad \overline{K}_t(X, Y)&=\overline{R}(X, Y; X, Y)/(g(X, X)g(Y, Y))\\ &=f^{-4}(R(X, Y; X, Y)+\alpha(X, Y)^2-\alpha(X, X)\alpha(Y, Y))\\ &=f^{-4}(g(R(X, Y)X, Y)-(1/4)g'(X, X)g'(Y, Y))\\ &=f^{-4}(f^2 K_0(X, Y)-f^2(f')^2)=f^{-2}(K_0(X, Y)+(\bar{e}/n)f^2-e_0/(n-1))\\ &=\bar{e}/n+f^{-2}(K_0(X, Y)-e_0/(n-1)),\end{aligned}$$

$$(2.5)\quad \overline{K}_t(X, N)=\overline{R}(X, N; X, N)/g(X, X)$$
$$=f^{-2}((1/4)(g'(X, X))^2-(1/2)g''(X, X))$$
$$=f^{-2}((f'/f)^2 g(X, X)-((f'/f)^2+f''/f)g(X, X))=-f''/f=\bar{e}/n,$$

$$(2.6)\quad \overline{K}_t(X, N+aY)=\overline{R}(X, N+aY; X, N+aY)/(g(X, X)\bar{g}(N+aY, N+aY))$$
$$=f^{-2}(1+a^2f^2)^{-1}(\overline{R}(X, N; X, N)+2a\overline{R}(X, N; X, Y)+a^2\overline{R}(X, Y; X, Y))$$
$$=f^{-2}(1+a^2f^2)^{-1}(f^2\overline{K}(X, N)+a^2f^4\overline{K}(X, Y))$$
$$=(1+a^2f^2)^{-1}(\overline{K}(X, N)+a^2f^2\overline{K}(X, Y)).$$

By these formulae, we see that $\bar{g}$ has constant sectional curvature if and only if g_0 has constant sectional curvature. From now on, we assume that $(\overline{M}, \bar{g})$ extends to a complete Einstein manifold, which we denote by the same symbol $(\overline{M}, \bar{g})$.

LEMMA 9. — *Assume that g_0 does not have constant sectional curvature. Then, (a) $f(t)\neq 0$ for all real number t. (b) If $f(t)$ converges to 0 for $t\to\infty$ or $-\infty$, then the sectional curvature of $(\overline{M}, g)$ is not bounded.*

Proof. — Easy, by (2.4).

Q.E.D.

Denote by G the isometry group of $(\overline{M}, \bar{g})$ and by d the metric on $\overline{M}$ induced by $\bar{g}$.

LEMMA 10. — *Assume that there is a positive number D such that $d(p, G(q))<D$ for all $p, q\in\overline{M}$. If $f(t)$ converges to ∞ for $t\to\infty$ or $-\infty$, then g_0 has constant sectional curvature.*

Proof. — Without loss of generality, we may assume that $f(t)$ converges to ∞ for $t\to\infty$. Let B be the closed ball with center $x\in M$ and radius r in (M, g_0), where r is sufficiently small so that B is compact. By assumption, there exists t_0 such that $f(t)r>D$ for all $t\geqq t_0$. Then for all $t>t_0+D$, $B\times(t_0, \infty)(\subset\overline{M})$ contains the closed ball $\overline{B}_t$ with the center $(x, t)\in\overline{M}$ and the radius D in $(\overline{M}, \bar{g})$. By (2.4), (2.5) and (2.6), the sectional curvature of $(\overline{M}, \bar{g})$ at the point (y, t) converges uniformally in B to $\bar{e}/n$ for $t\to\infty$. Thus the sectional curvature of $(\overline{M}, \bar{g})$ is constant, since:

$$\bigcap_{t>t_0+D} G(\overline{B}_t)=\overline{M}.$$

Q.E.D.

Proof of Theorem B. — Remark that $f'(a)=0$ if and only if i_a ; $(M, g_a)\to(\overline{M}, \bar{g})$ is totally geodesic.

(*a*) $e_0>0$. There is a real number a such that $f(a)=0$. By Lemma 8 (*a*), g_0 and $\bar{g}$ have constant sectional curvature.

(*b*) $e_0=\bar{e}=0$. $f'\equiv 0$. Then $(\overline{M}, \bar{g})$ is the riemannian product $(M, g_0)\times\mathbf{R}$ locally. If $(\overline{M}, \bar{g})$ is simply connected, then $(\overline{M}, \bar{g})$ decomposes globally as $(\tilde{M}, \tilde{g})\times\mathbf{R}$, since $\bar{g}$ is real analytic. Here $(\tilde{M}, \tilde{g})$ is a complete totally geodesic hypersurface of $(\overline{M}, \bar{g})$ which contains M.

(*c*) $e_0=0, \bar{e}<0$. $f(t) \to 0$ for $t \to \infty$ or $-\infty$. By Lemma 8 (*b*), if the sectional curvature of $(\overline{M}, \bar{g})$ is bounded, then g_0 and $\bar{g}$ have constant sectional curvature.

(*d*) $e_0, \bar{e}<0$. There is a real number a such that $f(a)>0$ and $f'(a)=0$. So i_a is totally geodesic. Moreover, $f(t)$ converges to ∞ for $t \to \infty$. If $(\overline{M}, \bar{g})$ satisfies the condition in Lemma 9, then g_0 and $\bar{g}$ have constant sectional curvature.

By Proposition 2, these are the only possible cases.

Q.E.D.

3. Real hypersurfaces of a Kähler-Einstein manifold

In the general situation, we saw in Theorem A that we cannot get much information on (M, g), even if (M, g) is a totally geodesic hypersurface in an Einstein manifold $(\overline{M}, \bar{g})$. But if $(\overline{M}, \bar{g})$ is a Kähler-Einstein manifold, the Gauss-Codazzi equations give more information on (M, g). Let (M, g) be a totally umbilical real hypersurface in a Kähler-Einstein manifold $(\overline{M}, \bar{g})$. By Proposition 2, the second fundamental form α is expressed as $\alpha=ag$ for some real number a. Then, the Codazzi equation and formula (1.1.*a*) become:

$$\bar{R}(X, Y; Z, N)=0, \tag{3.1}$$

$$r=(\bar{e}+(n-1)a^2)g-\beta. \tag{3.2}$$

Denote by J the almost complex structure of $(\overline{M}, \bar{g})$ and set $H=JN$. In equation (3.1), if X is orthogonal to H, then JX is tangent to M, and we see that:

$$\beta(X, Y)=\bar{R}(X, N; Y, N)=-\bar{R}(JX, H; Y, N)=0. \tag{3.3}$$

Then equation (1.1.*c*) implies:

$$\beta(H, H)=\bar{e}. \tag{3.4}$$

PROPOSITION 11. — *Let* $(\overline{M}, \bar{g})$ *be a complete Kähler-Einstein manifold with zero Ricci curvature. Assume that there exists a totally umbilical but not totally geodesic real hypersurface* (M, g) *in* $(\overline{M}, \bar{g})$ (*i.e.*, $a\neq 0$). *Then both* $(\overline{M}, \bar{g})$ *and* (M, g) *have constant sectional curvature.*

Proof. — By equations (3.2), (3.3) and (3.4), g is an Einstein metric with positive Ricci curvature. Thus the proof reduces to Theorem B(*a*).

Q.E.D.

LEMMA 12. — *Let* $(\overline{M}, \bar{g})$ *be a Kähler-Einstein manifold. Assume that there exists a totally geodesic real hypersurface* (M, g) *in* $(\overline{M}, \bar{g})$. *Then there exists a totally geodesic complex hypersurface* $(\tilde{M}, \tilde{g})$ *in* $(\overline{M}, \bar{g})$ *which is contained in* (M, g). *Moreover,* $(\tilde{M}, \tilde{g})$ *is a Kähler-Einstein manifold and* (M, g) *decomposes locally as* $(M, g)=(\tilde{M}, \tilde{g})\times\mathbb{R}$.

Proof. — Since (M, g) is totally geodesic, $\overline{D}_X N=0$ holds for any tangent vector X of M. Then we see that:

$$D_X H=\overline{D}_X H=\overline{D}_X(JN)=J(\overline{D}_X N)=0, \tag{3.5}$$

which implies that there is a hypersurface $(\tilde{M}, \tilde{g})$ in (M, g) and (M, g) decomposes locally as $(M, g)=(\tilde{M}, \tilde{g})\times \mathbf{R}$. Here J preserves the tangent space of $\tilde{M}$. This implies that $\tilde{M}$ is a complex submanifold of $\overline{M}$. Moreover, equations (3.2) and (3.3) imply that $\tilde{g}$ is an Einstein metric.

Q.E.D.

Proof of Theorem C. — Let γ be a geodesic in $(S, \bar{e})$. By Lemma 12, (M, g) may be immersed into $(\tilde{M}, \tilde{g})\times \operatorname{Im}\gamma$. On the other hand, since $\tilde{g}$ is an Einstein metric with Ricci curvature $\bar{e}$, $(\tilde{M}, \tilde{g})\times(S, \bar{e})$ becomes an Einstein manifold and $(\tilde{M}, \tilde{g})\times \operatorname{Im}\gamma$ is totally geodesic in $(\overline{M}, \bar{g})$. Then Proposition 4 implies that $(\tilde{M}, \tilde{g})\times(S, \bar{e})$ is an open set of $(\overline{M}, \bar{g})$. Remark that this identification preserves the complex structure. Since $(\overline{M}, \bar{g})$ is real analytic, this decomposition extends globally. That is, $(\tilde{M}, \tilde{g})$ extends to a complete complex hypersurface of $(\overline{M}, \bar{g})$ and we get a global decomposition.

Q.E.D.

Remark 13. — Even if $(\overline{M}, \bar{g})$ is not complete, the above decomposition holds locally.

Proof of Corollary E. — Assume that there is a compact stable minimal real hypersurface (M, g) in $(\overline{M}, \bar{g})$. Then by Theorem D, (M, g) is totally geodesic. Therefore we can apply Theorem C to the universal riemannian covering of $(\overline{M}, \bar{g})$ and get a global decomposition. This contradicts the assumption.

Q.E.D.

Remark 14. — In Corollary E, if $\overline{M}$ is simply connected, the assumption that M is orientable is not necessary. See Remark 4.

Remark 15. — In particular, there is no compact stable minimal hypersurface in the K3-surfaces $\overline{M}$ with zero Ricci curvature. By Theorem 2.9 in Bourguignon [2], there is no stable closed geodesic in $\overline{M}$. We may say that these results are dual with one another.

Corollary 16. — *Let $(\overline{M}, \bar{g})$ be a compact Kähler-Einstein manifold with zero Ricci curvature of complex dimension* $\leqq 3$. *If $\pi_1(\overline{M})$ is not finite, then* (M, g) *has a local factor* **C**.

Proof. — Since $\pi_1(\overline{M})$ is not finite, $H_n(\overline{M}, \mathbf{Z})$ is not trivial by Poincaré duality. For $\dim_{\mathbf{R}}\overline{M}\leqq 6$, a non-trivial homology class in $H_n(\overline{M}, \mathbf{Z})$ can be represented by stable minimal real hypersurfaces M without singularity (Federer [7], Thm. 5.4.15, Lawson Jr. [9], Remark 3.4). Then by Corollary E, $(\overline{M}, \bar{g})$ decomposes locally with a factor **C**.

Q.E.D.

Remark 17. — We can get Corollary 16 in more general situation by Theorem 3 in Cheeger and Gromoll [3]. But the proof is different.

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