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BOUNDDED DOMAINS WHICH ARE ISOSPECTRAL
BUT NOT CONGRUENT

BY HAJIME URAKAWA

1. Introduction

The purpose of this paper is to give examples of bounded domains of \( \mathbb{R}^n \) of dimension not less than four which are isospectral but not congruent.

Let \( \Omega \) be a bounded domain in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) with the appropriately regular boundary \( \partial \Omega \). For the Laplacian \( \Delta_0 = -\sum \frac{\partial^2}{\partial x_i^2} \) on \( \mathbb{R}^n \), let us consider the following problems:

**Dirichlet Problem**
\[
\begin{align*}
\Delta_0 f &= \lambda f \quad \text{in } \Omega, \\
 f &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

**Neumann Problem**
\[
\begin{align*}
\Delta_0 f &= \lambda f \quad \text{in } \Omega, \\
\frac{\partial f}{\partial v} &= 0 \quad \text{a.e. } \partial \Omega, \text{ i.e., where the exterior normal } v \text{ of } \partial \Omega \text{ is defined.}
\end{align*}
\]

It is well known that each problem has a discrete spectrum which consists of the eigenvalues with finite multiplicities. We denote by \( \text{Spec}_D(\Omega) \) (resp. \( \text{Spec}_N(\Omega) \)) the spectrum of the Dirichlet problem (resp. the Neumann Problem) for the domain \( \Omega \) in \( \mathbb{R}^n \).

One of the important problems of the spectra is to find how the spectra \( \text{Spec}_D(\Omega) \) or \( \text{Spec}_N(\Omega) \) reflect the shape of \( \Omega \). In his paper [K], M. Kac gave the following interesting expression of this problem: thinking of \( \Omega \) as a drum and its eigenvalues as its fundamental tones, *is it possible, just by listening with a perfect ear, to hear the shape of \( \Omega \)?* (See also [M.S.]).

Many mathematicians, e.g., Weyl [W], Carleman [C], Kac [K], McKean-Singer [M.S.] and others challenged it, so that one can hear the several geometric quantities of \( \Omega \), that is, the
Problem (cf. [K]).

For two bounded domains $\Omega_1$, $\Omega_2$ in $\mathbb{R}^n (n \geq 2)$, assume that $\text{Spec}_D(\Omega_1) = \text{Spec}_D(\Omega_2)$ or $\text{Spec}_N(\Omega_1) = \text{Spec}_N(\Omega_2)$. Are the domains $\Omega_1$, $\Omega_2$ congruent in $\mathbb{R}^n$? Here two domains $\Omega_1$, $\Omega_2$ are congruent in $\mathbb{R}^n$ if there exists an isometry $\Phi$ of $\mathbb{R}^n$ such that $\Phi(\Omega_1) = \Omega_2$.

It is just the problem proposed by Kac (cf. [K], see also [Yau], problem No. 67). A partial answer is known: in case of $\Omega_1$ = a disc, due to the celebrated inequality of Faber-Krahn [F], [Kr] (resp. that of Weinberger [Wr]) related to the first eigenvalue of the Dirichlet problem (resp. the Neumann problem), $\text{Spec}_D(\Omega_1) = \text{Spec}_D(\Omega_2)$ (resp. $\text{Spec}_N(\Omega_1) = \text{Spec}_N(\Omega_2)$) implies that $\Omega_2$ is the disc with the same radius as $\Omega_1$.

In this paper, we give an eventual answer of the problem of Kac:

Theorem 4.4. — There exist two domains $\Omega_1$, $\Omega_2$ in $\mathbb{R}^n (n \geq 4)$ such that

$$\text{Spec}_D(\Omega_1) = \text{Spec}_D(\Omega_2) \quad \text{and} \quad \text{Spec}_N(\Omega_1) = \text{Spec}_N(\Omega_2),$$

but $\Omega_1$ and $\Omega_2$ are not congruent in $\mathbb{R}^n$.

In case of dimension two or three, the problem is still open. By the way note that one can formulate an analogous problem for compact Riemannian manifolds without boundary and the answer is negative by virtue of examples of Milnor [M], Ikeda [I] and Vigneras [V].

The proof of Theorem 4.4 is very simple. Our examples can be found among the truncated cones $D_\varepsilon (0 < \varepsilon < 1)$ given by $D_\varepsilon = \{ r \omega; \varepsilon < r < 1, \omega \in C_1 \}$ where $C_1$ are the domains in the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. The outline of the proof is as follows:

First, for a fixed $\varepsilon (0 < \varepsilon < 1)$, we show by the separation of the variables, that $\text{Spec}_D(D_\varepsilon)$ (resp. $\text{Spec}_N(D_\varepsilon)$) is completely determined by the number $\varepsilon$ and the spectrum $\text{Spec}_D(C_1)$ (resp. $\text{Spec}_N(C_1)$) of the Dirichlet problem (resp. the Neumann problem) of the spherical domain $C_1$ for the Laplacian on the standard unit sphere $S^{n-1}$ (cf. § 4). Then we have only to answer the following problem:

(A) Find two domains $C_1$, $\bar{C}_1$ in $S^{n-1}$ which satisfy $\text{Spec}_D(C_1) = \text{Spec}_D(\bar{C}_1)$ and $\text{Spec}_N(C_1) = \text{Spec}_N(\bar{C}_1)$, but are not congruent in $S^{n-1}$.

Recently, Bérard-Besson [B.B.] determined the spectra $\text{Spec}_D(C_1)$, $\text{Spec}_N(C_1)$ of the spherical domains $C_1$ which are the intersections of $S^{n-1}$ with the chambers of the Weyl groups $W$ (i.e., the finite reflection groups). They showed that the spectra $\text{Spec}_D(C_1)$, $\text{Spec}_N(C_1)$ are completely determined by the set of the exponents of $W$. Hence due to their results, the problem (A) for these domains $C_1$ can be modified into the following:

(B) Find two finite reflection groups $W$, $\hat{W}$ acting on the same Euclidean space $\mathbb{R}^n$ which satisfy the conditions: (i) the sets of the exponents of $W$, $\hat{W}$ coincide each other and (ii) the intersections $C_1$, $\bar{C}_1$ of their chambers with $S^{n-1}$ are not congruent in $S^{n-1}$.

Notice that the condition (ii) is equivalent to that the Coxeter graphs of $W$, $\hat{W}$ are not isomorphic (cf. § 3). Thus we have only to consider the following:

(C) Does the set of the exponents of the finite reflection group $W$ acting on $\mathbb{R}^n$ determine the Coxeter graph of $W$ uniquely?
In case of $n \geq 4$, the answer of the problem (C) is NO, i.e., there exist examples of finite reflection groups with the same set of the exponents and the different Coxeter graphs (cf. § 3). Thus we obtain Theorem 4.4 and the following:

**Theorem 3.8.** — There exist two domains $C_1, \bar{C}_1$ in the unit sphere $S^{n-1}$ in $\mathbb{R}^n(n \geq 4)$ such that

$$\text{Spec}_D(C_1) = \text{Spec}_D(\bar{C}_1) \quad \text{and} \quad \text{Spec}_N(C_1) = \text{Spec}_N(\bar{C}_1),$$

but $C_1, \bar{C}_1$ are not congruent in $S^{n-1}$.

**Remark.** — The boundaries of our examples are not smooth, but polygons. The boundary value problems in non-smooth domains have been treated by Agmon [A], Grisvard [Gd], Brownell [B], Kac [K], p. 19 and others. But the original version of the problem of Kac was proposed for domains of smooth boundaries. In this sense, the problem of Kac is still open for every dimension $n \geq 2$.

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### 2. Preliminaries

In this section, we will review reflection groups following Bourbaki [B.N.].

Let $(E, \langle , \rangle)$ be a finite dimensional real vector space with an inner product $( , )$. Put $n = \text{dim}(E)$. Let $\mathfrak{h}$ be a finite set consisting of hyperplanes of $E$. In this paper, we deal only finite reflection groups, so we always assume that each hyperplane belonging to $\mathfrak{h}$ passes through the origin $0$ of $E$. Let $O(E)$ be the orthogonal group of $E$ with respect to the inner product $( , )$. For $H \in \mathfrak{h}$, let $s_H \in O(E)$ be the reflection relative to $H$, i.e.,

$$s_H(x) = x - \frac{2(x, a)}{(a, a)} a, \quad x \in E,$$

where $a$ is a vector orthogonal to the hyperplane $H$. The subgroup $W$ of $O(E)$ generated by $\{ s_H; H \in \mathfrak{h} \}$ is called a reflection group on $E$ (cf. [B.N], p. 72) if it satisfies the conditions (D1), (D2):

(D1) If $w \in W$ and $H \in \mathfrak{h}$, then $w(H) \in \mathfrak{h}$.

(D2) $W$ is finite, so $W$ acts properly discontinuously on $E$.

A connected component $C$ of $E \setminus \cup \{ H; H \in \mathfrak{h} \}$ is called a chamber of $W$ in $E$ and a hyperplane $H$ of $\mathfrak{h}$ is called a wall of the chamber $C$ if the intersection of the closure $\overline{C}$ of $C$ with $H$ includes a non-empty open subset of $H$. Then it is known that (1) $W$ acts simply transitively on the set of all chambers, (2) the set of all hyperplanes $H$ such that $s_H \in W$ coincides with $\mathfrak{h}$ and (3) for every chamber $C$, its closure $\overline{C}$ is a fundamental domain of $W$ in $E$ (cf. [B.N], p. 74, 75).

Let $W_i(0 \leq i \leq s)$ be reflection groups on the Euclidean spaces $(E_i, ( , ))$, $\mathfrak{h}_i$ the sets of their hyperplanes in $E_i(1 \leq i \leq s)$ and $W_0 = \{ \text{id} \}$. Let $E = E_0 \times E_1 \times \ldots \times E_s$ be their direct
product of which inner product $(\ , \ )$ is given by $(x, y) = \sum_{i=0}^{s} x_i y_i$ for $x = (x_0, \ldots, x_s)$, $y = (y_0, \ldots, y_s) \in E$. The direct product $W = W_0 \times \ldots \times W_s$ acts on $E$ by $w(x) = (w_0(x_0), \ldots, w_s(x_s))$ for $w = (id, w_1, \ldots, w_s) \in W$. Then $W$ is a reflection group on $(E, (\ , \ ))$ generated by reflections relative to the hyperplanes all of which are of the form:

\[(2.1) \quad H = E_0 \times E_1 \times \ldots \times E_{i-1} \times H_i \times E_{i+1} \times \ldots \times E_s,\]

where $H_i$ belong to $h_i$, $i = 1, \ldots, s$. Each chamber of $(W, E)$ is of the form:

\[(2.2) \quad C = E_0 \times C_1 \times \ldots \times C_s,\]

where $C_i$ are chambers of $(W_i, E_i)$, $i = 1, \ldots, s$. Each reflection group $W$ on the Euclidean space $(E, (\ , \ ))$ is decomposed as the direct product of reflection groups $W_i$, $i = 0, 1, \ldots, s$, $W_0 = \{id\}$, in such a way that the Euclidean space $(E, (\ , \ ))$ is decomposed as the direct product of the Euclidean spaces $(E_i, (\ , \ ))$, $i = 0, 1, \ldots, s$ and $W_i$ act irreducibly on $E_i$ as subgroups of $O(E_i)$, $i = 1, \ldots, s$ (cf. [B.N.], p. 82). The subgroup $W' = W_1 \times \ldots \times W_s$ of $W$ is called the essential part of $W$. Put $E' = E_1 \times \ldots \times E_s$ and $l = \text{dim}(E')$. For an arbitrary fixed chamber $C'$ of $W'$ in $E'$, let $m_0$ be the set of all walls of $C'$. For $H_\in\mathbb{m}$, let $e_H$ be the unit vector in $E'$ which is orthogonal to $H$ and belongs to the one of two connected components of $E' \setminus H$ containing $C'$. Then $\{e_H; \ H \in \mathbb{m}\}$ is a basis of $E'$ (cf. [B.N.], p. 85). So we may put $m = \{H_i\}_{i=1}^{l}$. Let $\{\omega_i\}_{i=1}^{l}$ be the dual basis of $\{e_{H_i}\}_{i=1}^{l}$, i.e., $(\omega_i, e_{H_j}) = \delta_{ij}$. Then the chamber $C'$ of $W'$ in $E'$ is an open simplex cone in $E'$ with the vertex $o$ given by

\[(2.3) \quad C' = \left\{ \sum_{i=1}^{l} x_i \omega_i \in E'; \ x_i > 0 (i = 1, \ldots, l) \right\} \quad (\text{cf. [B.N.], p. 85}).\]

For a chamber $C'$ of $W'$ and the set $\{H_i\}_{i=1}^{l}$ of all the walls of $C'$, an element $c = s_{H_1} \ldots s_{H_l}$ of $W'$ is called a Coxeter transformation of $W'$. Each Coxeter transformation of $W'$ has the same order $h = h(W')$, which is called the Coxeter number of $W'$ and the same characteristic polynomial $P(T) = \text{det}(T\text{id} - c)$ which can be written of the form:

\[
P(T) = \prod_{j=1}^{l} (T - \exp(2\pi \sqrt{-1} m_j/h)) \quad (\text{cf. [B.N.], 116}).\]

Here $m_j (j = 1, \ldots, l)$ are integers which can be arranged by $0 \leq m_1 \leq m_2 \leq \ldots \leq m_l < h$. These $l$ non-negative integers $m_j$ are called the exponents of $W'$ (cf. [B.N.], p. 118).

Then the number of all the hyperplanes of $W$ (or $W'$) is given by

\[(2.4) \quad \#H = \frac{1}{2} l \sum_{i=1}^{s} l_i h_i = \sum_{i=1}^{l} m_i,\]

where $l_i = \text{dim}(E_i)$ and $h_i$ is the Coxeter number of $W_i$, $i = 1, \ldots, s$. In fact, since a chamber $C'$ of $W'$ is given by $C_1 \times \ldots \times C_s$ where $C_i$ is a chamber of $W_i$, a Coxeter transformation $c$ of $W'$ relative to $C'$ is given as a product $c = c_1 \ldots c_s$ of the ones of the irreducible reflection
groups \( W_i \) relative to \( C_i \). Since the number of all the hyperplanes of \( W_i \) is \( 2^{-1} l_i h_i \) which coincides with the sum of all the exponents of \( W_i \) (cf. [B.N.], p. 119,118), we have (2.4).

Moreover the order of the group \( W \) (or \( W' \)) coincides with \( (m_1 + 1) \ldots (m_\ell + 1) \) (cf. [B.N.], p. 122).

For a chamber \( C' \) of the essential part \( W' \) of a reflection group \( W' \), let \( \{ H_i \}_{i=1}^\ell \) be the set of all the walls of \( C' \). Let \( \{ e_{H_i} \}_{i=1}^\ell \) be the unit vectors in \( E' \) defined as above. Let \( m_{ij}, i, j=1, \ldots, \ell \), be the order of the element of \( s_{H_i} s_{H_j} \) in \( W' \). Then we have:

**Lemma 2.1.** The positive integers \( m_{ij} \) satisfy the following conditions:

1. \( (e_{H_i}, e_{H_j}) = -\cos(\pi/m_{ij}) \),
2. \( m_{ij} \geq 2 \) if \( i \neq j \), i.e., \( (e_{H_i}, e_{H_j}) \leq 0 \),
3. \( m_{ij} = m_{ji}, \ i, j=1, \ldots, \ell \), and \( m_{ii} = 1, \ i=1, \ldots, \ell \).

**Proof.** See [B.N.], p. 77.

Due to this lemma, for a reflection group \( W \), we give a graph \( \Gamma \) consisting of \( \ell \) vertices \( \{ 1, \ldots, \ell \} \) and the numbers \( m_{ij} \). Two vertices \( i, j \) of \( \Gamma \) are joined by an edge if \( m_{ij} \geq 3 \) and the edge is labelled with the number \( m_{ij} \) if \( m_{ij} > 3 \). Such a graph is called a Coxeter graph (cf. [B.N.], p. 20). Two vertices \( a, b \) of the graph \( \Gamma \) are connected if there exist vertices \( \{ x_j \}_{j=0}^\ell \) of \( \Gamma \) such that \( a = x_0, b = x_\ell \), and each \( x_j \) is joined to \( x_{j+1} \) by an edge. A maximal set of connected vertices and edges of \( \Gamma \) is called a connected component of \( \Gamma \). For a reflection group \( W \), let \( W' = W_1 \times \ldots \times W_\ell \) be the decomposition of the essential part \( W' \) of \( W \). Then the Coxeter graph \( \Gamma \) corresponding to \( W' \) consists of \( s \) connected components \( \{ \Gamma_i \}_{i=1}^s \) such that each graph \( \Gamma_i \) is the Coxeter graph of the irreducible subgroup \( W_i \) of \( W' \) (cf. [B.N.], p. 22). Note that two reflection groups are isomorphic if and only if their Coxeter graphs coincide. Furthermore the classification of irreducible reflection groups is as follows:

**Lemma 2.2.** The irreducible reflection group is the one of which Coxeter graph is some of the following table.

Here \( h, \# h \) and "order" in the table are the Coxeter number, the number of all the hyperplanes in \( h \) and the order of the corresponding reflection group, respectively.

**Proof.** See [B.N.], p. 193, 200-221 and 231, exercices 11, 12).

A reflection group corresponding the graph \( A_1 \sim G_2 \) is called a crystallographic group which is given as a Weyl group of a root system.

### 3. Case of spherical domains

3.1. Let \((E, (\ , \ ))\) be the \( n \)-dimensional Euclidean space. Let \((x_1, \ldots, x_n)\) be the coordinate of \( E \) with respect to a fixed orthonormal basis \( \{ e_i \}_{i=1}^n \) of \( E \). We identify \( E \) with...
Coxeter graph & Exponents & \( h \) & \# h & Order \\
\( A_1(\ell \geq 1) \) & \( 1, 2, \ldots, \ell \) & \( l+1 \) & \( (l+1)/2 \) & \( (l+1)! \) \\
\( B_1(\ell \geq 2) \) & \( 1, 3, 5, \ldots, 2l-3 \) & \( \frac{2l}{2} \) & \( l^2 \) & \( 2^l l! \) \\
\( D_1(\ell \geq 4) \) & \( 1, 3, 5, \ldots, 2l-3 \) & \( \frac{2l-2}{2} \) & \( (l-1) \) & \( 2^{l-1} l! \) \\
\( E_6 \) & \( 1, 4, 5, 7, 8, 11 \) & 12 & 36 & \( 2^7 \cdot 3^4 \cdot 5 \) \\
\( E_7 \) & \( 1, 5, 7, 9, 11, 13, 17 \) & 18 & 63 & \( 2^{10} \cdot 3^4 \cdot 5^2 \cdot 7 \) \\
\( E_8 \) & \( 1, 7, 11, 13, 17, \ldots, 30 \) & 120 & \( 2^{14} \cdot 3^3 \cdot 5^2 \cdot 7 \) \\
\( F_4 \) & \( 1, 5, 7, 11 \) & 12 & 24 & \( 2^7 \cdot 3^2 \) \\
\( G_2 \) & \( 1, 5 \) & 6 & 6 & 12 \\
\( H_3 \) & \( 1, 5, 9 \) & 10 & 15 & 120 \\
\( H_4 \) & \( 1, 11, 19, 29 \) & 30 & 60 & \( 2^6 \cdot 3^2 \cdot 5^2 \) \\
\( I_2(p) (p = 5 \text{ or } p \geq 7) \) & \( 1, p-1 \) & \( p \) & \( p \) & \( 2p \) \\

\( \mathbb{R}^n \) by the mapping \( E \ni x = \sum_{i=1}^{n} x_i e_i \mapsto (x_1, \ldots, x_n) \in \mathbb{R}^n \). For \( x = \sum_{i=1}^{n} x_i e_i \in \mathbb{E} \), put \( |x| = (x, x)^{1/2} = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2} \). Let \( S_{n-1} = \{ x \in \mathbb{E}; |x| = 1 \} \), the unit sphere in \( \mathbb{E} \). For \( x \in \mathbb{E} - (0) \), let \( (r, \omega) \) be the polar coordinate of \( x \) defined by

\[
|x| = (x, x)^{1/2} = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}.
\]

Then the Laplacian \( \Delta_0 = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \) of the Euclidean space \( (\mathbb{E}, (\ , \ )) \) is expressed relative to the polar coordinate \( (r, \omega) \) as:

\[
(3.1) \quad \Delta_0 = -\frac{\partial^2}{\partial r^2} - \frac{(n-1)}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta,
\]

where the operator \( \Delta \) is the Laplacian of the standard unit sphere \( (S_{n-1}, g_0) \) whose metric \( g_0 \) is induced from the inner product \( (\ , \ ) \) of \( \mathbb{E} \).

Now let \( W \) be a finite reflection group of \( (\mathbb{E}, (\ , \ )) \) defined by a finite set \( \mathfrak{h} \) of hyperplanes of \( \mathbb{E} \) passing through the origin \( \mathfrak{o} \). Let \( C \) be a chamber of \( W \) in \( \mathbb{E} \). Then, \( C \) is given as \( E_0 \times C' \) where \( W = \{ \text{id} \} \times W', E = E_0 \times E', W' \) is the essential part of \( W \) and \( C' \) is a chamber of \( E' \), which is an open simplex cone in \( E' \).

**Definition 3.1.** Let \( C_1 \) be the intersection of the chamber \( C \) with the unit sphere \( S_{n-1} \), which is an open simplex of \( S_{n-1} \). For \( 0 < \varepsilon < 1 \), let \( D_\varepsilon \) be the domain "truncated cone" in \( \mathbb{E} \) given by

\[
(3.2) \quad D_\varepsilon = \{ r, \omega; \ v < r < 1, \ \omega \in C_1 \}.
\]

Then we have:

**Lemma 3.2.** (1) The boundary \( \partial C_1 \) of \( C_1 \) in \( S_{n-1} \) is given by

\[
\partial C_1 = \partial C \cap S_{n-1} = \cup \{ H \cap S_{n-1}; H \in \mathfrak{h} \},
\]
where $\partial C$ is the boundary of $C$ in $E$ and $m$ is the set of all the walls of the chamber $C$ of $W$ in $E$.

(2) The boundary $\partial D_\epsilon$ of $D_\epsilon$ in $E$ is given by

$$\partial D_\epsilon = C_1 \cup \varepsilon C_1 \cup \{ r\omega ; \omega \in \partial C_1, \varepsilon < r < 1 \},$$

where $C_1$ is the closure of $C_1$ in $S^{n-1}$.

(3) The closure $C_1$ is the fundamental domain in $S^{n-1}$ relative to the isometry actions of $W$ in $(S^{n-1}, g_0)$.

Proof. — (1) and (2) follow from the fact that $C = E_0 \times C'$ and $C'$ is an open simplex cone in $E'$. (3) follows from that $C$ is the fundamental domain of $W$ in $E$.

Let us consider the following boundary value problems for the domains $C_1$ and $D_\epsilon$.

Case 1. — The spherical domain $C_1$ of $S^{n-1}$:

(S.D.P.) \[
\begin{cases}
\Delta f = \lambda f & \text{in } C_1, \\
f = 0 & \text{on } \partial C_1.
\end{cases}
\]

(S.N.P.) \[
\begin{cases}
\Delta f = \lambda f & \text{in } C_1, \\
\frac{\partial f}{\partial v} = 0 & \text{a.e. } \partial C_1, \text{where the exterior normal } v \text{ of } \partial C_1 \text{ is defined.}
\end{cases}
\]

Case 2. — The Euclidean domain $D_\epsilon(0<\varepsilon<1)$ of $E$:

(E.D.P.) \[
\begin{cases}
\Delta_0 f = \lambda f & \text{in } D_\epsilon, \\
f = 0 & \text{on } \partial D_\epsilon.
\end{cases}
\]

(E.N.P.) \[
\begin{cases}
\Delta_0 f = \lambda f & \text{in } D_\epsilon, \\
\frac{\partial f}{\partial v} = 0 & \text{a.e. } \partial D_\epsilon, \text{where the exterior normal } v \text{ of } \partial D_\epsilon \text{ is defined.}
\end{cases}
\]

In this section, we treat with Case 1. Case 2 will be dealt in section 4.

3.2. In this subsection, we review the works of Bérard-Besson [B.B.] who determined the spectrum $\text{Spec}_0(C_1)$ (resp. $\text{Spec}_N(C_1)$) of (S.D.P.) (resp. (S.N.P.)). Their results are valid in case of the reflection groups (cf. [B2]).

First for the above domain $C_1$ of $S^{n-1}$ corresponding to the reflection group $W$, we define the inner product $(\ , \ )$ on $C^\infty(C_1)$ by

$$\langle f_1, f_2 \rangle = \int_{C_1} f_1(x) f_2(x) d\omega(x), \quad f_1, f_2 \in C^\infty(C_1),$$

where $d\omega$ is the volume element of the standard unit sphere $(S^{n-1}, g_0)$. Let $L^2(C_1)$ be the completion of $C^\infty(C_1)$ with respect to the inner product $(\ , \ )$.

Now consider a $C^\infty$ function $f$ on $S^{n-1}$ satisfying the conditions

(3.3) \[ \Delta f = \lambda f \text{ in } S^{n-1} \]
and
\[(3.4)\]
\[w \cdot f = \varepsilon(w) f, \quad w \in W,\]
where \((w \cdot f)(x) = f(w^{-1}(x))\) for \(w \in W\) and \(x \in S^{n-1}\) and \(\varepsilon(w) = \det(w)\) is given by
\[(3.5)\]
\[\varepsilon(w) = 1 \quad \text{for every} \quad w \in W,\]
or
\[(3.6)\]
\[\varepsilon(w) = \det(w) \quad \text{for every} \quad w \in W.\]

Then the restriction of \(f\) to \(C_1\) satisfies (S.D.P.) (resp. (S.N.P.)) if \(\varepsilon\) satisfies (3.6) resp. (3.5)). Furthermore, the set of all restrictions of \(C^\infty\) eigenfunctions of \(\Delta\) on \(S^{n-1}\) with the condition (3.4) is dense in \(L^2(C_1)\) (cf. [B.B.], p. 239). Thus, to determine \(\text{Spec}_D(C_1)\) and \(\text{Spec}_N(C_1)\), we have only to consider the set of all \(C^\infty\) eigenfunctions of \(\Delta\) on \(S^{n-1}\) satisfying the condition (3.4). Of course, every solution \(f_1\) of (S.D.P.) or (S.N.P.) can be extended to a function \(\tilde{f}\) on \(S^{n-1}\) by
\[f_1(x) = \tilde{f}_1(x), \quad x \in C_1\]
and
\[w \cdot f(x) = \varepsilon(w) \tilde{f}_1(x), \quad x \in S^{n-1}, \quad w \in W.\]

Then it is well defined on \(S^{n-1}\) due to Lemma 3.2, moreover, it can be proved by the same manner as Lemma 8 in [B1] that \(\tilde{f}\) is \(C^\infty\) on \(S^{n-1}\).

Now we set:

\[H_k(E) = \{ P \in C^\infty_0(E) : P(w^{-1}(x)) = \det(w) P(x) \text{ for all } w \in W \text{ and } x \in E \},\]

\[H_k^s(E) = \{ P \in H_k(E) : P(w(x)) = P(x) \text{ for all } w \in W \text{ and } x \in E \},\]

\[h_k^s(E) = \dim(H_k^s(E)) \text{ and } h_k^t(E) = \dim(H_k(E)).\]

Then the inclusion \(i : S^{n-1} \to E\) induces a linear mapping \(i^* : C^\infty(E) \to C^\infty(S^{n-1})\) by \(P \to P \circ i\). The mapping \(i^*\) is injective and its image of the space \(\sum_{k=0}^{\infty} H_k(E)\) is dense in \(C^\infty(S^{n-1})\). Furthermore, the image of \(H_k(E)\) by \(i^*\) coincides with the eigenspace of \(\Delta\) on \(S^{n-1}\) with the eigenvalue \(k(k+n-2), k = 0, 1, 2, \ldots\)

Therefore the spectrum \(\text{Spec}_D(C_1)\) (resp. \(\text{Spec}_N(C_1)\)) of the Dirichlet problem (resp. the Neumann problem) of the domain \(C_1\) in \(S^{n-1}\) is determined as follows:

1. The set of all the eigenvalues of the Dirichlet problem (S.D.P.) and the Neumann problem (S.N.P.) is included in the set \(\{ k(k+n-2) : k = 0, 1, 2, \ldots \}\).
2. If \(h_k^s(E) \neq 0\) (resp. \(h_k^t(E) \neq 0\)), \(k(k+n-2)\) is really the eigenvalues of (S.D.P.) (resp. (S.N.P.)) with multiplicity \(h_k^s(E)\) (resp. \(h_k^t(E)\)).

Thus to determine \(\text{Spec}_D(C_1)\) and \(\text{Spec}_N(C_1)\), we have only to compute \(h_k^s(E)\) and \(h_k^t(E)\) \((k = 0, 1, 2, \ldots)\). For this purpose, consider the Poincaré series:

\[(3.7)\]
\[F^s(T) = \sum_{k=0}^{\infty} h_k^s(E) T^k,\]
(3.8) \[ F^i(T) = \sum_{k=0}^{\infty} h_i^k(E) T^k, \]

where $T$ is an indeterminate.

Bérard-Besson [B.B.] computed the series (3.7), (3.8) making use of the Poincaré series of the subring of the polynomial ring consisting of invariant polynomials under the action of the reflection group $W$ as follows:

**Proposition 3.5 (the Neumann problem (S.N.P.)).** Let $W$ be a reflection group of $(E, (\cdot, \cdot))$ (dim $E = n$) defined by a finite set $\mathfrak{h}$ of hyperplanes of $E$ passing through the origin $o$. Let $W = \{ \text{id} \} \times W'$, $E = E_0 \times E'$ be the decomposition of $(W, E)$ such that $(W', E')$ is the essential part of $(W, E)$. Let $C_1$ be the intersection of a chamber $C$ of $(W, E)$ with the unit sphere $S^{n-1}$ in $(E, (\cdot, \cdot))$. Then the series (3.7) which determines the spectrum $\text{Spec}_N(C_1)$ of the Neumann problem (S.N.P.) is given as follows:

\[ F^i(T) = \frac{1 - T^2}{\prod_{j=1}^{n} (1 - T^{m_j+1})}, \]

where $\{ m_j \}_{j=1}^{n}$ is the set consisting of 0, ..., 0 (dim $E_0 = l_0$) and the exponents of the reflection group $W'$.

**Proof.** See [B.B.], p. 241, Propositions 2 and 6.

**Proposition 3.6 (the Dirichlet problem (S.D.P.)).** Under the same assumptions of Proposition 3.5, the Poincaré series $F^a(T)$ which determines the spectrum $\text{Spec}_D(C_1)$ is given by

\[ F^a(T) = T^d \cdot F^i(T), \]

where $d$ is the number $\# \mathfrak{h}$ of all the elements in $\mathfrak{h}$, which is given by (2.4).

**Proof.** See [B.B.], p. 242, Proposition 4.

3.3. Due to Propositions 3.5, 3.6, we have:

**Theorem 3.7.** Let $W$ (resp. $\tilde{W}$) be a finite reflection group defined by a finite set $\mathfrak{h}$ (resp. $\tilde{\mathfrak{h}}$) of hyperplanes of the Euclidean space $(E, (\cdot, \cdot))$, dim $E = n$, passing through the origin $o$. Let $W = \{ \text{id} \} \times W'$, $E = E_0 \times E'$ (resp. $\tilde{W} = \{ \text{id} \} \times \tilde{W}'$, $\tilde{E} = \tilde{E}_0 \times \tilde{E}'$) be the decomposition of $(W, E)$ resp. $(\tilde{W}, \tilde{E})$ such that $(W', E')$ (resp. $(\tilde{W}', \tilde{E}')$) is the essential part of $(W, E)$ resp. $(\tilde{W}, \tilde{E})$. Let $C = E_0 \times C'$ (resp. $\tilde{C} = \tilde{E}_0 \times \tilde{C}'$) be a chamber of $(W, E)$ (resp. $(\tilde{W}, \tilde{E})$), where $C'$ (resp. $\tilde{C}'$) is a chamber of the essential part $(W', E')$ (resp. $(\tilde{W}', \tilde{E}')$). Put $C_1 = C \cap S^{n-1}$ (resp. $\tilde{C}_1 = \tilde{C} \cap S^{n-1}$) where $S^{n-1}$ is the unit sphere of the Euclidean space $(E, (\cdot, \cdot))$. Then we have:

1. If the sets of the exponents of $W'$ and $\tilde{W}'$ coincide each other and dim $E_0 = \text{dim} (\tilde{E}_0)$, then

\[ \text{Spec}_D(C_1) = \text{Spec}_D(\tilde{C}_1) \quad \text{and} \quad \text{Spec}_N(C_1) = \text{Spec}_N(\tilde{C}_1). \]
Let $\dim(E_0) = \dim(\overline{E}_0)$. Then the following conditions are equivalent:

(i) The domains $\overline{C}_1$ and $\overline{\mathcal{C}}_1$ are congruent in the unit sphere $(S^{n-1}, g_0)$, i.e., there exist an isometry $\Psi$ of $(S^{n-1}, g_0)$ such that $\Psi(\overline{C}_1) = \overline{\mathcal{C}}_1$.

(ii) The chambers $\overline{C}$ and $\overline{\mathcal{C}}$ are congruent in the Euclidean space $(E, (\ , \ ))$.

(iii) The Coxeter graphs of $\overline{W}'$ and $\overline{\mathcal{W}}'$ coincide.

Proof. — Propositions 3.5, 3.6 and (2.4) imply the assertion (1). (2) The chamber $\overline{C}'$ (resp. $\overline{\mathcal{C}}'$) is an open simplex cone in $E'$ (resp. $\overline{E}'$). Combining this with the definitions of $\overline{C}_1$ and $\overline{\mathcal{C}}_1$, we have the equivalence between (i) and (ii). The equivalence between (ii) and (iii) follows from (2.3), Lemma 2.1 and the definition of the Coxeter graph.

O.E.D.

Notice that the set of the exponents does not determine the reflection group uniquely. There exist many examples of pairs of the reflection groups of which have the same set of the exponents but the different Coxeter graphs as in the table below.

Moreover, for such a pair of reflection groups $(W', E')$, $(\overline{W}', \overline{E}')$ and an arbitrary dimensional Euclidean space $(E_0, (\ , \ ))$, define the direct products $W = \{id\} \times W'$, $\overline{W} = \{id\} \times \overline{W}'$ and $E = E_0 \times E'$. Then for these reflection groups $(W, E)$ and $(\overline{W}, \overline{E})$, the spectra of $(S.D.P.)$ and $(S.N.P.)$ for the intersections $\overline{C}_1, \overline{\mathcal{C}}_1$ of their chambers with the unit sphere coincide each other, but $\overline{C}_1$ and $\overline{\mathcal{C}}_1$ are not congruent in the unit sphere by Theorem 3.7. Therefore we have:

**Theorem 3.8.** There exist two domains $\overline{C}_1$ and $\overline{\mathcal{C}}_1$ in the unit sphere $(S^{n-1}, g_0) (n \geq 4)$ such that:

$$\text{Spec}_D(\overline{C}_1) = \text{Spec}_D(\overline{\mathcal{C}}_1) \quad \text{and} \quad \text{Spec}_N(\overline{C}_1) = \text{Spec}_N(\overline{\mathcal{C}}_1).$$

but $\overline{C}_1$ is not congruent to $\overline{\mathcal{C}}_1$ in the unit sphere $(S^{n-1}, g_0)$.

**Remark 3.9.** There exist many examples other than the above table. For example,

<table>
<thead>
<tr>
<th>Exponents</th>
<th>$# h$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_i \times I_2(l) (l \geq 4)$</td>
<td>$1, 3, 5, \ldots, 2l-3, 2l-1, l-1$</td>
<td>$l^2+l$</td>
</tr>
<tr>
<td>$D_1 \times I_2(2l)$</td>
<td>$1, 3, 5, \ldots, 2l-3, l+1, 1, 2l-1$</td>
<td>$(l+1)^2+2l$</td>
</tr>
<tr>
<td>$E_6 \times A_1 \times A_1$</td>
<td>$1, 4, 5, 7, 8, 11, 1, 1$</td>
<td>$36+2$</td>
</tr>
<tr>
<td>$F_4 \times I_2(5) \times I_2(9)$</td>
<td>$1, 5, 7, 11, 1, 4, 1, 8$</td>
<td>$24+5+9$</td>
</tr>
</tbody>
</table>

**Examples 3.10.** The simplest cases in the above table are:

(1) $A_3 \times A_1$ and $I_2(3) \times I_2(4)$,

(2) $B_3 \times A_1$ and $I_2(4) \times I_2(6)$,

where $I_2(3) = A_2$, $I_2(4) = B_2$ and $I_2(6) = G_2$. The chambers of these reflection groups are given as follows:
BOUNDED DOMAINS WHICH ARE ISOSPECTRAL

<table>
<thead>
<tr>
<th>Pairs of Coxeter graphs</th>
<th>Exponents</th>
<th>#(b)</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_1 \times A_1 \times \ldots \times A_1 (I \geq 3))</td>
<td>(l, 2, 1, 1, \ldots, 1)</td>
<td>(l(l+1)/2 + (l-2))</td>
<td>((l+1)! \cdot 2^{l-2})</td>
</tr>
<tr>
<td>(I_2(3) \times I_2(4) \times \ldots \times I_2(l+1))</td>
<td>(1, 2, 1, 3, \ldots, 1, l)</td>
<td>(3 + 4 + \ldots + (l+1))</td>
<td>(\prod_{i=1}^{l-1} 2^{i+2})</td>
</tr>
<tr>
<td>(B_1 \times A_1 \times \ldots \times A_1 (I \geq 3))</td>
<td>(1, 3, 5, \ldots, 2l-1, 1, \ldots, 1)</td>
<td>(l^2 + (l-2))</td>
<td>(2^{(l+1)/2} \cdot 2^{l-2})</td>
</tr>
<tr>
<td>(I_2(4) \times I_2(6) \times \ldots \times I_2(2l))</td>
<td>(1, 3, 1, 5, \ldots, 1, 2l-1)</td>
<td>(4 + 6 + \ldots + 2l)</td>
<td>(\prod_{i=1}^{2l} 2^{2l})</td>
</tr>
<tr>
<td>(D_1 \times A_1 \times \ldots \times A_1 (I \geq 4))</td>
<td>(1, 3, 5, \ldots, 2l-3, l-1,\ldots, 1)</td>
<td>((l-1) + (l+2))</td>
<td>(2^{l-1} \cdot 2^{l-1})</td>
</tr>
<tr>
<td>(I_2(4) \times I_2(6) \times \ldots \times I_2(2l-2) \times I_2(l))</td>
<td>(1, 3, 5, \ldots, 1, 2l-3, l)</td>
<td>(4 + 6 + \ldots + (2l-2) + l)</td>
<td>(\prod_{i=2}^{l-1} 2(2l) \times 2l)</td>
</tr>
<tr>
<td>(E_6 \times A_1 \times \ldots \times A_1)</td>
<td>(1, 4, 5, 7, 8, 11, 1)</td>
<td>(36 + 6)</td>
<td>(2^7 \cdot 3^4 \cdot 5 \cdot 2^4)</td>
</tr>
<tr>
<td>(E_7 \times A_1 \times \ldots \times A_1)</td>
<td>(1, 5, 7, 9, 11, 13, 17)</td>
<td>(63 + 5)</td>
<td>(2^{19} \cdot 3^4 \cdot 5 \cdot 7 \cdot 2^5)</td>
</tr>
<tr>
<td>(I_2(5) \times I_2(6) \times I_2(8) \times I_2(9) \times I_2(12))</td>
<td>(1, 4, 1, 5, 1, 7, 1, 8, 1, 11)</td>
<td>(5 + 6 + 8 + 9 + 12)</td>
<td>(2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 12)</td>
</tr>
<tr>
<td>(I_2(6) \times I_2(8) \times I_2(10))</td>
<td>(1, 5, 1, 7, 1, 9, 1, 11)</td>
<td>(6 + 8 + 10 + 12 + 14 + 18)</td>
<td>(2^9 \cdot 6 \cdot 8 \cdot 10 \cdot 12)</td>
</tr>
<tr>
<td>(E_6 \times A_1 \times \ldots \times A_1)</td>
<td>(1, 7, 11, 13, 17, 19, 23, 29)</td>
<td>(120 + 6)</td>
<td>(2^{14} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 2^6)</td>
</tr>
<tr>
<td>(I_2(8) \times I_2(12) \times I_2(14) \times I_2(18))</td>
<td>(1, 7, 1, 11, 1, 13, 1, 17, 29, 1)</td>
<td>(8 + 12 + 14 + 18)</td>
<td>(2^7 \cdot 8 \cdot 12 \cdot 14 \cdot 18)</td>
</tr>
<tr>
<td>(F_4 \times A_1 \times A_1)</td>
<td>(1, 5, 7, 11, 1, 1)</td>
<td>(24 + 2)</td>
<td>(2^7 \cdot 3^2 \cdot 2^7)</td>
</tr>
<tr>
<td>(I_2(6) \times I_2(8) \times I_2(12))</td>
<td>(1, 5, 1, 7, 1, 11)</td>
<td>(6 + 8 + 12)</td>
<td>(2^5 \cdot 6 \cdot 8 \cdot 12)</td>
</tr>
<tr>
<td>(H_3 \times A_1)</td>
<td>(1, 5, 9, 1)</td>
<td>(15 + 1)</td>
<td>(120 \cdot 2)</td>
</tr>
<tr>
<td>(G_2 \times I_2(10))</td>
<td>(1, 5, 1, 9)</td>
<td>(6 + 10)</td>
<td>(2^5 \cdot 6 \cdot 10)</td>
</tr>
<tr>
<td>(H_4 \times A_1 \times A_1)</td>
<td>(1, 11, 19, 29, 1)</td>
<td>(60 + 2)</td>
<td>(2^6 \cdot 3^2 \cdot 5^2 \cdot 2^2)</td>
</tr>
<tr>
<td>(I_2(12) \times I_2(20) \times I_2(30))</td>
<td>(1, 11, 1, 19, 1, 29)</td>
<td>(12 + 20 + 30)</td>
<td>(2^3 \cdot 12 \cdot 20 \cdot 30)</td>
</tr>
</tbody>
</table>

(1) A chamber \(C_{11}\) of \(A_3 \times A_1\) is given by \(\left\{ \sum_{i=1}^{4} x_i \omega_i; x_i > 0, i = 1, \ldots, 4 \right\}\) as a cone in the 4-dimensional Euclidean space \((\mathbb{R}^4, (\ , ))\), where the vector \(\omega_{4}\) is orthogonal to each \(\omega_{i}\), \(i = 1, 2, 3\) which are given such as in the Figure 1. That is, let \(e_1, e_2, e_3\) be the orthonormal basis of the 3-dimensional subspace of \((\mathbb{R}^4, (\ , ))\) orthogonal to \(\omega_{4}\). Then \(\omega_{1} = e_3, \omega_{2} = e_1 - e_2 + e_3\) and \(\omega_{3} = e_1 + e_2 + e_3\).

A chamber \(\bar{C}_{11}\) of \(I_2(3) \times I_2(4)\) is given by \(\left\{ \sum_{i=1}^{4} y_i \tilde{\omega}_i; y_i > 0, i = 1, \ldots, 4 \right\}\) as a cone in the 4-dimensional Euclidean space \((\mathbb{R}^4, (\ , ))\), where both vectors \(\tilde{\omega}_1\) and \(\tilde{\omega}_2\) are orthogonal to both vectors \(\tilde{\omega}_3\) and \(\tilde{\omega}_4\) and the angle between \(\tilde{\omega}_1\) and \(\tilde{\omega}_2\) (resp. \(\tilde{\omega}_3\) and \(\tilde{\omega}_4\)) is \(\pi/3\).
On the other hand, since the angle between $c_0$ and $c_2$ is arc-tan$(2^{1/2})$, it is impossible that $C_{(1)}$ and $\tilde{C}_{(1)}$ are congruent in the 4-dimensional Euclidean space.

(2) A chamber $C_{(2)}$ of $B_3 \times A_1$ is given by \[ \{ \sum_{i=1}^{4} x_i \eta_i; x_i > 0, \, i = 1, \ldots, 4 \} \], where $\eta_1 = \omega_1 = e_3$, $\eta_2 = \omega_2 = e_1 - e_2 + e_3$, $\eta_3 = e_1 + e_3$, and $\eta_4 = \omega_4$ in the example (1). A chamber of $\tilde{C}_{(2)}$ of $I_2(4) \times I_2(6)$ is given by \[ \{ \sum_{i=1}^{4} y_i \tilde{\eta}_i; \, y_i > 0, \, i = 1, \ldots, 4 \} \], where both vectors $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are orthogonal to both vectors $\tilde{\eta}_3$ and $\tilde{\eta}_4$, and the angle between $\tilde{\eta}_1$ and $\tilde{\eta}_2$ (resp. $\tilde{\eta}_3$ and $\tilde{\eta}_4$) is $\pi/4$ (resp. $\pi/6$).

4. Case of Euclidean domains

In this section, we consider the boundary value problems (E.D.P.), (E.N.P.) (Case 2) for the domains $D_{\epsilon}(0<\epsilon<1)$ (3.2) of the Euclidean space $(E, (\ , \ ))$ of dimension $n$ as in 3.1. We preserve the situations in 3.1. Recall that, for $0<\epsilon<1$, $D_{\epsilon} = \{ r\omega; \, \epsilon < r < 1, \, \omega \in C_1 \}$, where $C_1 = C \cap S^{n-1}$, $C$ is a chamber of a finite reflection group $W$ in $E$.

Firstly, note that the volume element $dx = dx_1 \ldots dx_n$ can be expressed on $E(\omega)$ by the polar coordinate $(r, \omega)$ as

\[ dx = r^{n-1} \, dr \, d\omega. \]
where $d\omega$ is the volume element of the standard unit sphere $(S^{n-1}, g_0)$. Let $L^2(D_e, dx)$ be the space of all square integrable functions on $D_e$ with respect to the measure $dx$, and $L^2((e, 1) \times C_1, dr \, d\omega)$ the space of all square integrable functions on the product space $(e, 1) \times C_1$ of the open interval $(e, 1)$ and $C_1$ with respect to the product measure $dr \, d\omega$. Since $0 < e^{n-1} < r^{n-1} < 1$ on the interval $(e, 1)$, $L^2(D_e, dx)$ can be identified with $L^2((e, 1) \times C_1, dr \, d\omega)$ by the mapping $D_e \ni \omega \mapsto (r, \omega) \in (e, 1) \times C_1$.

Now let $\{\lambda_1 \leq \lambda_2 \leq \ldots\}$ be the set of all the eigenvalues (counted repeatedly as many as their multiplicities) of the Dirichlet problem (S.D.P.) (resp. the Neumann problem (S.N.P.)) of the Laplacian $\Delta$ of $(S^{n-1}, g_0)$ for the domain $C_1$ in $S^{n-1}$. Let $\{\psi_j\}_{j=1}^\infty$ be a complete basis of $L^2(C_1, d\omega)$ such that

$$(4.1) \quad \Delta \psi_j = \lambda_j \psi_j \quad \text{in } C_1,$$

and

$$(4.2) \quad \psi_j = 0 \quad \text{on } \partial C_1 \quad \text{resp. } \frac{\partial \psi_j}{\partial v} = 0 \quad \text{a.e. } \partial C_1, \text{i.e., where the exterior normal } v \text{ of } \partial C_1 \text{ is defined}.$$ 

For each eigenvalue $\lambda$ of $\Delta$, recalling (3.1), define a differential operator $L_\lambda$ on the open interval $(e, 1)$ by

$$(4.3) \quad L_\lambda = -\frac{d^2}{dr^2} - \frac{n-1}{r} \frac{d}{dr} + \frac{\lambda}{r^2}.$$

Let $L^2_2(e, 1), L^2_2(e, 1)$ be the spaces of all square integrable functions on the interval $(e, 1)$ with respect to the measure $dr, r^{n-1} dr$, respectively. Note that a $C^\infty$ function $\Phi$ on $(e, 1)$ is an eigenfunction of $L_\lambda$ with an eigenvalue $\mu$:

$$L_\lambda \Phi = \mu \Phi,$$

if and only if $\Phi$ satisfies the following equation of Sturm-Liouville type on $(e, 1)$:

$$(4.4) \quad \frac{d}{dr} \left(r^{n-1} \frac{d\Phi}{dr}\right) - \lambda r^{n-3} \Phi + \mu r^{n-1} \Phi = 0.$$ 

**Lemma 4.1.** Let us consider the boundary value problem of (4.4) with the boundary condition:

$$(4.5) \quad \Phi(e) = \Phi(1) = 0 \quad \text{resp. } \frac{d}{dr} \Phi(e) = \frac{d}{dr} \Phi(1) = 0.$$

Let $\{\mu_j^2\}_{j=1}^\infty$ be the set of all eigenvalues of the boundary value problem (4.4) and (4.5), and let $\Phi_j^2(j = 1, 2, \ldots)$ be the eigenfunction with the eigenvalue $\mu_j^2$. Then $\{\Phi_j^2\}_{j=1}^\infty$ is a complete basis of $L^2_2(e, 1)$. 

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Proof. — See [P], p. 508 or [Y], p. 109, Theorem 1.

Now, for the eigenvalues \( \lambda_i (i = 1, 2, \ldots) \) of the boundary value problem (S.D.P.) (resp. (S.N.P.)) for the domain \( C_1 \), consider \( C^\infty \) functions \( \Phi_j^i \) \( (j = 1, 2, \ldots) \) on the interval \( (\varepsilon, 1) \). For a \( C^\infty \) function \( \Phi \), (resp. \( \Psi \)) on the interval \( (\varepsilon, 1) \) (resp. \( C_1 \)), we define a \( C^\infty \) function \( \Phi \otimes \Psi \) on \( D_\varepsilon \) (or \( (\varepsilon, 1) \times C_1 \)) by:

\[
\Phi \otimes \Psi (r \omega) = \Phi (r) \Psi (\omega), \quad r \omega \in D_\varepsilon \text{ or } (\varepsilon, 1) \times C_1.
\]

Then, by (3.1), the \( C^\infty \) functions \( \Phi_j^i \otimes \Psi_i \) \( (i, j = 1, 2, \ldots) \) on \( D_\varepsilon \) satisfy the equation

\[
(4.6) \quad \Delta_0 (\Phi_j^i \otimes \Psi_i) = - \left( \frac{d^2 \Phi_j^i}{dr^2} + \frac{n-1}{r} \frac{d \Phi_j^i}{dr} \right) \otimes \Psi_i + \frac{1}{r^2} \Phi_j^i \otimes \Delta \Psi_i
\]

\[
= (\lambda_i \Phi_j^i) \otimes \Psi_i = \mu_j^i \Phi_j^i \otimes \Psi_i \quad \text{in } D_\varepsilon,
\]

and the boundary condition

\[
(4.7) \quad \Phi_j^i \otimes \Psi_i = 0 \quad \text{on } \partial D_\varepsilon,
\]

(resp. \( \partial / \partial n (\Phi_j^i \otimes \Psi_i) = 0 \) a.e. \( \partial D_\varepsilon \), where the exterior normal of \( \partial D_\varepsilon \) is defined), since \( \Phi_j^i \) satisfies (4.1) and (4.2) and \( \Psi_i \) satisfies (4.4) and (4.5). In fact, \( \partial / \partial n (\Phi_j^i \otimes \Psi_i) \) \( (r \omega) \) coincides with

\[
- \left( \frac{d}{dr} (\Phi_j^i (\varepsilon) \Psi_i (\omega)), \frac{d}{dr} (\Phi_j^i (1) \Psi_i (\omega)) \right), \quad \text{or } \Phi_j^i (r) \frac{\partial}{\partial n} \Psi_i (\omega)
\]

a.e. \( \partial D_\varepsilon \). Here \( (\partial / \partial n) \Psi_i \) is the derivation of \( \Psi_i \) with respect to the exterior normal of \( \partial C_1 \) (cf. Lemma 3.2(2)).

Furthermore we have the following lemma.

**Lemma 4.2.** — \( \{ \Phi_j^i \otimes \Psi_i; i, j = 1, 2, \ldots \} \) is a complete basis of \( L^2 (D_\varepsilon, dx) \).

**Proof.** — It can be proved by the similar manner as Theorem 2.1 in [E]. Consider the following boundary value problem on the interval \( (\varepsilon, 1) \):

\[
\begin{cases}
- \frac{d^2}{dr^2} u = \lambda u & \text{on } (\varepsilon, 1), \\
u(\varepsilon) = u(1) = 0 & \text{resp. } \frac{d}{dr} u(\varepsilon) = \frac{d}{dr} u(1) = 0.
\end{cases}
\]

Let \( \{ u_i \}_{i=1}^\infty \) be a complete basis of \( L^2 (\varepsilon, 1) \) such that \( u_i \) is the eigenfunction of the above problem with the eigenvalue \( \lambda_i \) \( (i = 1, 2, \ldots) \). Let \( \| \cdot \|_{L^2 (D_\varepsilon, dx)} \), \( \| \cdot \|_{L^2 (\varepsilon, 1)} \), \( \| \cdot \|_{L^2 (S^{n-1}, d\sigma)} \) be the \( L^2 \)-norms of \( L^2 (D_\varepsilon, dx) \), \( L^2 (\varepsilon, 1) \), \( L^2 (\varepsilon, 1) \), respectively. Since \( \{ \Phi_j^i \}_{j=1} \) for each \( \lambda_i (i = 1, 2, \ldots) \) is a complete basis of \( L^2 (\varepsilon, 1) \), for each \( \lambda_i (i = 1, 2, \ldots) \) and \( \lambda_l (l = 1, 2, \ldots) \), there exist \( a_{l,k}^i \in \mathbb{R} \) \( (k = 1, 2, \ldots) \) such that

\[
\lim_{p \to \infty} \left\| u_i - \sum_{k=1}^p a_{l,k}^i \Phi_k^i \|_{L^2 (\varepsilon, 1)} = 0.
\]
On the other hand, we have
\[
\left\| u_i \otimes \Psi_i - \sum_{k=1}^{p} \alpha_{i,k}^{p} \Phi_k^{i} \otimes \Psi_i \right\|_{L^2(D, dx)} = \left\| u_i - \sum_{k=1}^{p} \alpha_{i,k}^{p} \Phi_k^{i} \right\|_{L^2(D, dx)} \cdot \| \Psi_i \|_{L^1(S^{n-1}, d\omega)}.
\]
Thus we obtain
\[
\lim_{p \to \infty} \left\| u_i \otimes \Psi_i - \sum_{k=1}^{p} \alpha_{i,k}^{p} \Phi_k^{i} \otimes \Psi_i \right\|_{L^2(D, dx)} = 0.
\]
On the other hand, \(\{ u_i \otimes \Psi_i \}_{i=1}^{\infty} \) is a complete basis of \(L^2((\epsilon, 1) \times C_1, dr d\omega)\), due to the Stone-Weierstrass theorem (cf. [B.G.M.], p. 144). As \(L^2(D, dx)\) can be identified with \(L^2((\epsilon, 1) \times C_1, dr d\omega)\), \(\{ \Phi_k^{i} \otimes \Psi_i \}_{i=1}^{\infty} \) is a complete basis of \(L^2(D, dx)\).

Q.E.D.

Therefore the spectrum \(\text{Spec}_D(D_\epsilon)\) (resp. \(\text{Spec}_N(D_\epsilon)\)) of the Dirichlet problem (E.D.P.) (resp. the Neumann problem (E.N.P.)) for the domain \(D_\epsilon\) in \(E\) is given by
\[
\{ \mu_j^{\epsilon}; i=1, 2, \ldots \},
\]
where \(\{ \lambda_j \}_{j=1}^{\infty} \) is the spectrum \(\text{Spec}_D(C_1)\) (resp. \(\text{Spec}_N(C_1)\)) for the domain \(C_1\) in the unit sphere \(S^{n-1}\). Since \(\mu_j^{\epsilon}\) depend on \(\lambda_j\) but not on \(\Psi_i\), we obtain the following theorem.

**Theorem 4.3.** — For two reflection groups \(W, \tilde{W}\) on the same Euclidean space \((E, (., .))\), let \(C, \tilde{C}\) be their chambers and \(C_1 = C \cap S^{n-1}, \tilde{C}_1 = \tilde{C} \cap S^{n-1}\), where \(S^{n-1}\) is the unit sphere in \((E, (., .))\). For each \(0 < \epsilon < 1\), define the domains \(D_\epsilon = \{ r\omega; \epsilon < r < 1, \omega \in C_1 \}, \tilde{D}_\epsilon = \{ r\omega; \epsilon < r < 1, \omega \in \tilde{C}_1 \}\) respectively. Let \(\text{Spec}_D(D_\epsilon), \text{Spec}_N(D_\epsilon)\) (resp. \(\text{Spec}_N(D_\epsilon), \text{Spec}_N(\tilde{D}_\epsilon)\)) be the spectra of the Dirichlet problems (E.D.P.) (resp. the Neumann problems (E.N.P.)) for the domains \(D_\epsilon, \tilde{D}_\epsilon\) in \(E\). Let \(\text{Spec}_D(C_1), \text{Spec}_N(C_1)\) (resp. \(\text{Spec}_N(C_1), \text{Spec}_N(\tilde{C}_1)\)) be the spectra of the Dirichlet problems (S.D.P.) (resp. the Neumann problems (S.N.P.)) for the domains \(C_1, \tilde{C}_1\) in \(S^{n-1}\). Then we have:

If \(\text{Spec}_D(C_1) = \text{Spec}_D(\tilde{C}_1)\) (resp. \(\text{Spec}_N(C_1) = \text{Spec}_N(\tilde{C}_1)\)), then \(\text{Spec}_D(D_\epsilon) = \text{Spec}_D(\tilde{D}_\epsilon)\) (resp. \(\text{Spec}_N(D_\epsilon) = \text{Spec}_N(\tilde{D}_\epsilon)\)) for each \(0 < \epsilon < 1\).

We note that, if \(C_1\) is not congruent to \(\tilde{C}_1\) in the unit sphere \(S^{n-1}\), then \(D_\epsilon\) is not congruent to \(\tilde{D}_\epsilon\) in the Euclidean space \((E, (., .))\) for each \(0 < \epsilon < 1\). Therefore by Theorems 3.8, 4.3, we have:

**Theorem 4.4.** — There exist domains \(D_\epsilon, \tilde{D}_\epsilon(0 < \epsilon < 1)\) in the \(n\)-dimensional Euclidean space \(\mathbb{R}^n(n \geq 4)\) such that
\[
\text{Spec}_D(D_\epsilon) = \text{Spec}_D(\tilde{D}_\epsilon) \quad \text{and} \quad \text{Spec}_N(D_\epsilon) = \text{Spec}_N(\tilde{D}_\epsilon),
\]
but these domains \(D_\epsilon, \tilde{D}_\epsilon(0 < \epsilon < 1)\) are not congruent each other in the Euclidean space \(\mathbb{R}^n\).
REFERENCES


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