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## A REMARK ON F. KNIGHT'S PAPER

By P. A. MEYER

The aim of this remark is to discuss F. Knight's statement (in the last few lines of the introduction of the preceding paper) that his basic Theorem 1.4 is "almost" true for non-Gaussian processes. We are going to prove it completely, without any assumption on higher moments, by a *very slight* technical modification of Knight's own proof.

Since the reader can refer to the main paper for details and comments, our editing in this note is rather concise.

NOTATION. —  $(\Omega, \mathcal{F}, P)$  is a complete probability space, with a filtration  $(\mathcal{F}_t)$  which satisfies the usual conditions. On this space,  $(X_t)$  is a real valued, progressively measurable process, such that:

$$(1) \quad E \left[ \int_0^\infty e^{-\lambda s} |X_s| ds \right] < \infty \quad \text{for } \lambda > 0.$$

Since (1) is satisfied, we may consider the process  $\left( E \left[ \int_t^\infty e^{-\lambda s} X_s ds \mid \mathcal{F}_t \right] \right)$ , which is a difference of two positive supermartingales (potentials), and therefore has a cadlag <sup>(1)</sup> version. The same is true for the following semimartingale (Knight's  $P_\lambda$  is  $\lambda Y^\lambda$  in our notation):

$$(2) \quad Y_t^\lambda = e^{\lambda t} E \left[ \int_t^\infty e^{-\lambda s} X_s ds \mid \mathcal{F}_t \right] = E \left[ \int_0^\infty e^{-\lambda s} X_{t+s} ds \mid \mathcal{F}_t \right].$$

Formally,  $\lambda Y_t^\lambda$  is an approximation of  $X_t$  for large  $\lambda$ . We now denote by  $Z_t^\lambda$  the local martingale part in the canonical decomposition of  $Y_t^\lambda$ , with the understanding that  $Z_0^\lambda = Y_0^\lambda$ . Let us compute it.

(1) *Cadlag* means *right continuous with left limits* in Brodningag language.

First, consider a process  $(g_t)$ , progressively measurable and such that:

$$\mathbb{E} \left[ \int_0^\infty |g_s| ds \right] < \infty \quad \text{and set } h_t = \mathbb{E} \left[ \int_t^\infty g_s ds \mid \mathcal{F}_t \right] \quad (\text{cadlag, as above}).$$

Then  $k_t = \int_0^t g_s ds + h_t$  is a martingale, and  $j_t = k_0 + \int_0^t e^{\lambda s} dk_s$  (stochastic integral) is a local martingale. A simple integration by parts gives an explicit expression for it:

$$j_t = e^{\lambda t} h_t + \int_0^t e^{\lambda s} (g_s - \lambda h_s) ds,$$

It is very easy to see that  $(j_t)$  belongs to the class (D) on finite intervals, hence is a true martingale. If we now take  $g_t = e^{-\lambda t} X_t$ ,  $e^{\lambda t} h_t$  is the process we called  $Y_t^\lambda$  above, and  $j_t$  turns out to be  $Z_t^\lambda$ :

$$(3) \quad Z_t^\lambda = Y_t^\lambda + \int_0^t (X_s - \lambda Y_s^\lambda) ds.$$

Knight's  $M_\lambda$  is  $\lambda Z^\lambda$  in our notation.

**THE EXTENSION OF KNIGHT'S THEOREM.** — We denote by  $\mathbf{D}$  the Skorohod space of all cadlag mappings from  $\mathbb{R}_+$  to  $\mathbb{R}$ , with its usual Borel structure, and by  $\mathbf{D}_\lambda$  the (Borel) subspace of  $\mathbf{D}$  consisting of all mappings  $y(t)$  such that  $e^{-\lambda t} y(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

We denote by  $\mathbf{A}$  the subspace of  $L_{loc}^1(\mathbb{R}_+)$  consisting of those classes  $x(t)$  such that:

$$\int_0^\infty e^{-\lambda s} |x(s)| ds < \infty \quad \text{for every } \lambda > 0.$$

$\mathbf{A}$  is given the Borel structure inherited from the (Polish space)  $L_{loc}^1$ .

For simplicity of notation we set  $\mathbb{N} = \{1, 2, \dots\}$ , not  $\{0, 1, \dots\}$  as usual. We define a Borel mapping  $\Phi$  from  $\mathbf{B} = \mathbf{A} \times \prod_{\lambda \in \mathbb{N}} \mathbf{D}_\lambda$  to  $\mathbf{D}^\mathbb{N}$ : it associates to the class  $x \in \mathbf{A}$  and the functions  $y_\lambda \in \mathbf{D}_\lambda$  the functions in  $\mathbf{D}$ :

$$(4) \quad z_\lambda(t) = y_\lambda(t) + \int_0^t (x(s) - \lambda y_\lambda(s)) ds, \quad \lambda = 1, 2, \dots$$

The crucial step is the following:

**LEMMA.** — *There exists a Borel mapping  $\Psi$  from  $\mathbf{D}^\mathbb{N}$  to  $\mathbf{B}$  such that  $\Psi \circ \Phi$  is the identity mapping on  $\mathbf{B}$ .*

*Proof.* — We need only show that  $\Phi$  is injective. Indeed, it is well known that  $\mathbf{A}$  and  $\mathbf{D}$  (hence  $\mathbf{D}_\lambda$ ) are Lusin measurable spaces, and so is their product  $\mathbf{B}$ . Then Lusin's theorem implies that  $\Phi(\mathbf{B})$  is Borel in  $\mathbf{D}^\mathbb{N}$ , and  $\Phi$  is a Borel isomorphism from  $\mathbf{B}$  to  $\Phi(\mathbf{B})$ . Then the construction of  $\Psi$  is trivial ( $\Phi^{-1}$  on  $\Phi(\mathbf{B})$ , some constant function on its complement).

To prove injectivity, we need only show that the vanishing of all functions  $z_\lambda$  in (4) implies the vanishing of all  $y_\lambda$ , and the a. s. vanishing of  $x$ . Now if  $z_\lambda = 0$ , (4) can be solved explicitly in  $y_\lambda$ :

$$(5) \quad y_\lambda(t) = -e^{\lambda t} \int_0^t e^{-\lambda s} x(s) ds.$$

We have assumed that this function belongs to  $D_\lambda$ , i. e.  $e^{-\lambda t} y_\lambda(t)$  tends to 0 at infinity. This means that the Laplace transform of  $x$  vanishes at  $\lambda$ . Since  $\lambda$  may take all values  $1, 2, \dots$ ,  $x$  must vanish a. e. Coming back to (5) we see that  $y_\lambda = 0$ .

Returning to the probabilistic situation, we get Knight's result without assumptions on the existence of moments:

**THEOREM.** — *There is a deterministic procedure which, for a. e.  $\omega$ , reconstructs the class of  $X_\cdot(\omega)$  from the paths  $Z^\lambda(\omega)$ ,  $\lambda = 1, 2, \dots$*

*Proof.* — According to (1), we a. s. have:

$$\int_0^\infty e^{-\lambda s} |X_s(\omega)| ds < \infty \quad \text{for all } \lambda > 0.$$

According to (2), we a. s. have for  $\lambda = 1, 2, \dots$ :

$$\lim_{t \rightarrow \infty} e^{-\lambda t} Y_t^\lambda(\omega) = 0,$$

since these processes are differences of two potentials.

According to (3), we a. s. have:

$$(Z^\lambda(\omega))_{\lambda=1, 2, \dots} = \Phi(X_\cdot(\omega), Y^1(\omega), Y^2(\omega), \dots)$$

We now apply  $\Psi$  to the sequence  $(Z^\lambda(\omega))_{\lambda=1, 2, \dots} \in D^{\mathbb{N}}$ , and the main lemma tells us that we a. s. recover the class of  $X_\cdot(\omega)$ , and moreover all mappings  $Y^\lambda(\omega)$ ,  $\lambda = 1, 2, \dots$

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