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ON SPREADING MODELS IN $L^1(E)$

PAR MICHEL TALAGRAND (*)

ABSTRACT. — We construct a Banach space E which has the Schur property (hence l^1 is its only spreading model) but such for each family $(a_{n,k})$, with $a_{n,k} \geqq 1$, $\lim_n a_{n,k} = +\infty$, there is a sequence (f_n) in $L^1(E)$ for which $\| \sum_{k \leqq i \leqq n} \pm f_i \| \leqq a_{n,k}$. In particular, $L^1(E)$ has a spreading model isomorphic to $c_0(\mathbb{N})$.

1. Introduction

Let E be a separable Banach space and (Ω, Σ, μ) a (standard) measure space. We denote by $L^1(E)$ the space of integrable functions $\Omega \rightarrow E$. It is known that if E does not contain $c_0 = c_0(\mathbb{N})$, then $L^1(E)$ does not contain c_0 [2]. The purpose of this work is to show in an opposite direction that even when E is by no way close to c_0, $L^1(E)$ can contain sequences which somehow behave like the unit basis of c_0. Recall that a Banach space has the Schur property if weak null sequences go to zero in norm.

We shall show the following.

THEOREM A. — *There exists a separable Banach space* E *which has the Schur property, such that for each family* $a_{n,k}$ *of real* $a_{n,k} \geqq 1$, *such that:*

$$(1) \qquad \forall k, \qquad \lim_n a_{n,k} = +\infty,$$

there exists a sequence $f_n \in L^1(E)$, *such that:*

$$(2) \qquad \forall \omega, \qquad \| f_n(\omega) \| = 1,$$

(3) $\forall$ *finite set* I, *with* $\operatorname{card} I = n$ *and* $\operatorname{Inf} I \geqq k$ *one has, for* $(b_i) \in \mathbb{R}^I$:

$$\operatorname{Inf} |b_i| \leqq \| \sum_{i \in I} b_i f_i \| \leqq a_{n,k} \sup |b_i|.$$

(*) This paper was written while the author was visiting the Ohio State University.

Since E has the Schur property it follows from Rosenthal's theorem [3] that each sequence (X_n) of E which does not converge in norm has a subsequence equivalent to the basis of l^1. However the sequence (f_n) of $L^1(E)$ has a behavior which is close to the basis of c_0. Since it is possible to choose $(a_{n,k})$ such that for each $n \lim_k (a_{n,k})=1$, in the language of spreading models, $L^1(E)$ has c_0 as a spreading model, while E has l^1 as unique spreading model.

The whole difficulty of the construction is that in E there should be "very few" sequences equivalent to the basis of l^1.

2. Setting of the construction

Let us set $T_n=\{0, 1\}^n$, $T=\bigcup T_n$. For $s\in T$ let $|s|$ be the unique n for which $s\in T_n$. For $s, t\in T$, $|s|=n$, $|t|=m$, $n\leqq m$, $s=(s_1, \ldots, s_n)$, $t=(t_1, \ldots, t_m)$, we write $s<t$ if $\forall i\leqq n$, $s_i=t_i$. With this order, T is the usual dyadic tree. For $t\in T$, $n\leqq |t|$, we write $t|n$ the unique $s\in U_n$ for which $s<t$.

Let us denote by $(e_t)_{t\in T}$ the canonical basis of $\mathbb{R}^{(T)}$. In the next paragraph, we shall construct a family H of $\mathbb{R}^{(T)}$, and we shall define for $x\in\mathbb{R}^{(T)}$:

$$\|x\|=\sup\{|\langle g, x\rangle|, g\in H\}. \tag{4}$$

Let E be the completion of $(\mathbb{R}^{(T)}, \|\,.\,\|)$. It will be true that $\|e_t\|=1$. We denote by e_t^* the element of E^* given by $e_t^*(e_{t'})=1$ if $t=t'$ and zero otherwise.

Let $\Omega=\prod_n T_n$, and let μ be the canonical measure on Ω (i. e. the product measure when each T_n is given the measure which puts weight 2^{-n} at each point).

Let $p_n:\Omega\to T_n$ be the projection of rank n. Let $h_n(\omega)=e_{p_n(\omega)}$. The reader has noticed that the setting of this construction is very similar to the setting of the construction [4] of a space E with the Dunford-Pettis property such that $\mathscr{C}([0, 1], E)$ fails the Dunford-Pettis property. However the idea of the construction of the norm is rather different.

3. Construction of the norming functionals

We start with $H_0=\{e_t^*;\ t\in T\}$. We shall construct inductively subsets H_n of $\mathbb{R}^{(T)}$.

Let X^n be the set of subsets $A=\{t_1, \ldots, t_p\}$ of T with the following property:

$$\forall\, 1\leqq i\leqq p, \qquad |t_i|\geqq n. \tag{5}$$

$$\exists\, s\in T_n, \qquad s<t_i, \quad \forall\, i\leqq p. \tag{6}$$

(7) If $|t_i|=c_i$, for $1\leqq i<j\leqq p$ one has $t_j|c_i=t_{i+1}|c_i$.

The element s will be called the *stem* of A and be denoted by $s(A)$. Let $H_1^n = \{ 1/4 \sum_{t \in A} e_t^*;\ A \in X^n \}$. For $g \in H_1^n$, we call $s(A)$ the stem of g, also denoted by $s(g)$. We set $H_1 = \bigcup_n H_1^n$.

For $g \in \mathbb{R}^{(T)}$, let $V(g) = \sup \{ |t|;\ \langle g, e_t \rangle \neq 0 \}$. Let $n > 0$. Consider a sequence $k(1) = n < k(2) < \ldots < k(p)$ and a sequence $g(i) \in H_1^{k(i)}$ such that:

$$\forall 1 \leqq i \leqq p, \qquad V(g(i)) < k(i+1). \tag{8}$$

$$\exists s \in T_n, \qquad \forall i, \quad s < s(g(i)). \tag{9}$$

$$\forall i < j \leqq p, \qquad s(g(i+1)) \,|\, V(g(i)) = s(g(j)) \,|\, V(g(i)). \tag{10}$$

[The reader should make a picture of the supports of the $g(i)$.] We define H_2^n as the set of sums $1/4 \sum_{i \leqq p} g(i)$ of the above type, and H_2 as $\bigcup_{n \geqq 1} H_2^n$.

The construction continues in the same way. Notice that each $g \in H_n$ is of the type $4^{-n} \sum_{t \in A} e_t^*$. Moreover, as is seen by induction, if $B \subset A$, $g' = 4^{-n} \sum_{t \in B} e_t^*$ still belongs to H_n.

Let H′ be the set of finite sums $\sum_{i \geqq 2} g_i$, where $g_i \in H_i$. Let $H = H_0 \cup H_1 \cup H'$.

4. E has the Schur property

By standard arguments of approximation, it is enough to show that if a sequence $(f_n) \in E$ such that $f_n = \sum_{t \in A_n} x_t^n e_t$ for A_n disjoint sets, $\| f_n \| = 1$ it cannot go to zero weakly.

1*st case*. — The following holds:

$$\forall m, \qquad \lim_n \sup \{ |\langle g, f_n \rangle|;\ g \in H_m \} = 0. \tag{11}$$

For each n, there is $g_n \in H$ with $|\langle g_n, f_n \rangle| \geqq 1/2$. From (11) it follows that $g_n \in H'$ for n large enough. Then we can write $g_n = \sum_{2 \leqq i \leqq k(n)} g_n^i$ where $g_n^i \in H_i$. By taking a subsequence one can assume from (11) that:

$$\Big| \sum_{i \leqq k(n-1)} g_n^i (f_n) \Big| \leqq 1/4.$$

If:

$$g'_n = \sum_{k(n-1) < i \leqq k(n)} g_n^i,$$

one has $|g'_n(f_n)| \geqq 1/4$. Let $\overline{g_n^i}$ obtained from g_n^i by restricting its support to A_n. Then $\overline{g_n^i} \in H_n$. Let:

$$g''_n = \sum_{k(n-1) < i \leqq k(n)} \overline{g_n^i}.$$

Then $|g''_n(f_n)| \geqq 1/4$. Moreover, $g''_n(f_p)=0$ for $p \neq n$. Let $h_n = \sum_{p \leqq n} g''_p$. Then $h_n \in H'$. Indeed, $h_n = \sum_{i < k(n)} h^i$ where $h^i = \overline{g_p^i}$ for the unique p such that $k(p-1) < i \leqq k(p)$. We have $|h_n(f_p)| > 1/4$ for $p < n$. Hence if h is a weak* cluster point of (h_n), we have $|h(f_p)| \geqq 1/4 \, \forall p$, which finishes the proof in this case.

2nd case. — There is $m, \alpha > 0$ and a sequence k_n such that $\sup \{ |\langle g, f_{k_n} \rangle| ; g \in H_m \} > \alpha \, \forall n$. One can suppose $k_n = n$. One can also suppose that m is the smallest integer for which the above is true, i. e.:

$$\lim_{n \to \infty} \sup \{ |\langle g, f_n \rangle| ; g \in H_{m-1} \} = 0. \tag{12}$$

For convenience of notation suppose now on that $m \geqq 1$. (The same argument works for $m=0$.)

Let $g_n \in H_m$ with $|\langle g_n, f_n \rangle| > \alpha$. One can suppose that g_n is supported by A_n. It follows from the definition of H_m that for each k one can write $g_n = g_n^1 + \ldots + g_n^k + g'_n$ where $g_n^i \in H_{m-1}$ for $i \leqq n$, and $g'_n \in H_m^k$. It follows, by taking a subsequence, that one can assume $g_n \in H_m^n$ and $|\langle g_n, f_n \rangle| \geqq \alpha/2$. Another extraction of subsequence will give $g_n \in H_m^{k(n)}$ where $k(n) > V(g_{n-1})$. Let $s_n = s(g_n) \in T_{k(n)}$. By taking a subsequence, one can assume that for each p, the sequence $s_n | p$ is eventually constant. A further subsequence will satisfy $s_n | V(g_p) = s_{p+1} | V(g_p)$ for $n \geqq p+1$.

It follows from the definition of H_{m+1} that for each n, $h_n = 1/4 \sum_{p \leqq n} g_p \in H_{m+1}$. Moreover, for $p \leqq n$ we have $|h_n(f_p)| > \alpha/8$. Let h be a weak* cluster point of (h_n). Then $|h(f_p)| \geqq \alpha/8$ for each p, which finishes the proof.

5. Construction of (f_n)

In fact, (f_n) will be a subsequence of h_n.

LEMMA. — *Let (u_i) be a sequence of independent random variables uniformly distributed in $\{1, \ldots, g\}$. Let $P(q, n) = \text{Prob}(\exists i, j \leqq n, u_i = u_j)$. Then $\lim_{q \to \infty} P(q, n) = 0$. Moreover, $P(q, n)$ is increasing in n and decreasing in q.*

Proof. — $P(q, n) \leqq q^{-2}(n(n-1))/2$.

Let $(a_{n,k})$ be the sequence of theorem A. One can suppose that $a_{n,k} \leqq a_{n+1,k}$ and $a_{n,k} \geqq a_{n,k+1}$ for each n, k. Let $n(k)$ be the smallest integer such that $a_{n(k),k} \geqq k+1$. From the lemma, there exists an increasing sequence $q(k)$ such that for each $k \geqq 1$ one has the following conditions :

$$n(k) \, P(2^{-q(k)}, n(k)) \leqq \frac{1}{2}. \tag{13}$$

(14) For each integer n such that $a_{n,k} \leqq 2$, $n \, P(2^{-q(k)}, n) \leqq a_{n,k} - 1$.

We shall prove that the sequence $f_n = h_{q(n)}$ satisfies the theorem. Let I be a finite set of integers, with $k = \operatorname{Inf} I$ and $\operatorname{card} I = n$. Let l the greatest integer such that $l+1 \leqq a_{n,k}$. (It is possible that $l=0$.) Let $m = k + l + 1$.

We have:

$$a_{n,m} \leqq a_{n,k} < l+2 \leqq m \leqq a_{n(m),m} \quad \text{so} \quad n \leqq n(m).$$

Hence:

$$n\,P(2^{-q(m)}, n) \leqq \frac{1}{2}. \tag{15}$$

Let us define $a_i(\omega)$ by $f_i(\omega) = e_{a_i(\omega)}$. Let:

$$Z = \{\omega \in \Omega;\ \exists\, i, j \in I,\ i, j \geqq m,\ i \neq j,\ a_i(\omega) \,|\, q(m) = a_j(\omega) \,|\, q(m)\}.$$

Since the maps $\omega \to a_i(\omega)$ are independent and $a_i(\omega) \,|\, q(m)$ takes for value each element of $T_{q(m)}$ with equal probability, one has $\mu(Z) \leqq P(2^{-q(m)}, n)$. For $\omega \in Z$, we have the trivial estimate $\| \sum_{i \in I} f_i(\omega) \| \leqq n$.

We show by induction over p that for $\omega \notin Z$ and $g \in H_p$, we have:

$$|\langle g, \sum_{i \in I} f_i(\omega) \rangle| \leqq 2^{-p}(l+1). \tag{16}$$

The result is obvious for $p=0$. Assume it has been proved for p. Let $g \in H_{p+1}$. Then we have a decomposition $g = 1/4 \sum_{i \leqq r \leqq n} g(r)$ which satisfy (8) to (10). Let j be the largest integer $j \leqq n$ for which $V(g(j)) < m$.

Let $g' = 1/4 \sum_{i \leqq r \leqq j} g(r)$. Then $g' = 4^{-p-1} \sum_{t \in A} e_t^*$ where $\sup\{|t|;\ t \in A\} < m$. Since there are at most l indexes i for which $|a_i(\omega)| < m$ we have $|\langle g', \sum_{i \in I} f_i(\omega) \rangle| \leqq 4^{-p-1} l$.

If $j=p$, the proof is finished. Otherwise $|\langle g(j+1), \sum_{i \in I} f_i(\omega) \rangle| \leqq 2^{-p}(l+1)$ by induction hypothesis. If $j+1=p$, the proof is finished. Otherwise let $g'' = \sum_{r > j+1} g(r)$. It follows from condition (10) that there is $s \in T_m$ such that for each $t \in T$ one has $s < t$. But since there is at most one $i \in I$ for which $s < a_i(\omega)$, we have $|\langle g'', \sum_{i \in I} f_i \rangle| \leqq 4^{-p-1}$. Adding these three estimates gives (16). It follows that for $g \in H$ one has:

$$|\langle g, \sum_{i \in I} f_i(\omega) \rangle| \leqq \sup\left(1, \frac{l+1}{2}\right)$$

and hence $\|\sum_{i\in I} f_i(\omega)\| \leqq \sup(1, (l+1)/2)$. So we have:

$$\|\sum_{i\in I} f_i\|_1 \leqq \int_Z \|\sum_{i\in I} f_i(\omega)\| d\mu(\omega) + \int_{\Omega \setminus Z} \|\sum_{i\in I} f_i(\omega)\| d\mu(\omega).$$

$$\leqq n\,\mu(Z) + \sup\left(1, \frac{l+1}{2}\right),$$

$$\leqq n\,P(2^{-q(m)}, n) + \sup\left(1, \frac{l+1}{2}\right).$$

If $l=0$, we have $a_{n,k} \leqq 2$, so $n\,P(2^{-q(m)}, n) \leqq a_{n,k}-1$ from (14) and since $q(m) \geqq q(k)$, so the right hand side is $\leqq a_{n,k}$. If $l \geqq 1$, we have $n\,P(2^{-q(m)}, n) \leqq (1/2)$ from (14), so the right hand side is less than $l/2+1 \leqq l+1 \leqq a_{n,k}$ which concludes the proof of the theorem.

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