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## ABELIAN SURFACES AND KOWALEWSKI'S TOP

By A. LESFARI <sup>(1)</sup>

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**ABSTRACT.** — This paper presents a new and systematic method to integrate the problem of Kowalewski's rigid body motion, and leads to a detailed geometric description of the invariant surfaces (tori) on which the motion evolves.

### Introduction

This paper deals with a geometric and systematic approach to the integration of Kowalewski's rigid body motion. It is well known that this motion is completely integrable and Kowalewski [16] integrates the problem in terms of hyperelliptic quadratures after a complicated and mysterious change of variables. The classical approach to solving integrable systems was based on solving the Hamilton-Jacobi equation by separation of variables, after an appropriate change of coordinates; for every problem finding this transformation required a great deal of ingenuity. Up to now Kowalewski's method has been neither understood, nor improved nor extended to other cases except for some modest amelioration contributed by Kötter [15] and Kolossoff [14]. This paper presents a new and systematic method to integrate the problem, and leads to a detailed geometric description of the invariant surfaces (tori) on which the motion evolves.

As is well known, a Hamiltonian system

$$\dot{z} = J \frac{\partial H}{\partial z}, \quad J = J(z) \text{ antisymmetric, } z \in \mathbb{R}^{2n}$$

is called completely integrable (in the  $\mathcal{C}^\infty$  sense), if it has  $n$  constants of the motion  $H_1, \dots, H_n$  in involution with linearly independent gradients. By the Arnold-Liouville theorem, the compact and connected invariant manifolds

$$\bigcap_{j=1}^n \{H_j = c_j, z \in \mathbb{R}^{2n}\}, \quad c_j \in \mathbb{R}$$

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are diffeomorphic to a real torus and there is a transformation to so-called action-angle variables, mapping the flow into a straight line motion on that torus. In most examples, the tori (of the Arnold-Liouville theorem) are real parts of complex algebraic tori (called Abelian varieties): they come equipped with an algebraic addition law. Adler and van Moerbeke ([3], [4]) have called such systems algebraically completely integrable. They have developed methods, at first, to recognize such integrable systems among families of Hamiltonian systems and, at second, to integrate such problem in terms of Abelian integrals; this approach is inspired by the work of Kowalewski. For example in [4], the criterion is used to detect the algebraic completely integrable geodesic flows on  $SO(4)$  for a left invariant diagonal metric: the only one leading to an integrable flow is Manakov's metric. This problem was integrated using coadjoint orbits on Kac-Moody Lie algebras [2]. Mumford [4] then recognized the nature of its invariant tori and Haine [10] used the Laurent solutions to the differential equations to realize the invariant tori as Prym varieties on which the flow linearizes. Recently, Adler and van Moerbeke ([5], [6], [7]) have classified the algebraically completely integrable geodesic flows on  $SO(4)$  for a left invariant metric and developed a general and effective method to integrate such systems.

This paper deals with the Kowalewski case in the dynamic of the rigid body, which will now be explained. With Arnold [8], the differential equations of motion of a rigid body about a fixed point are given by the customary Euler-Poisson equations

$$(Int 1) \quad \begin{cases} \dot{M} = [M, \Lambda M] + \mu g [\Gamma, L] \\ \dot{\Gamma} = [\Gamma, \Lambda M] \end{cases}$$

where

$$M = (M_{jk})_{1 \leq j, k \leq 3} \equiv \sum m_j e_j \equiv \begin{bmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix} \in so(3)$$

$$\Lambda M = (\Lambda_{jk} M_{jk})_{1 \leq j, k \leq 3} = \sum I_j^{-1} m_j e_j \in so(3)$$

$$\Gamma = (\Gamma_{jk})_{1 \leq j, k \leq 3} \equiv \sum \gamma_j e_j \equiv \begin{bmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{bmatrix} \in so(3)$$

and

$$L = \begin{bmatrix} 0 & -l_3 & l_2 \\ l_3 & 0 & -l_1 \\ -l_2 & l_1 & 0 \end{bmatrix}.$$

$M$ ,  $\Gamma$ ,  $L$ ,  $I = \text{diag}(I_1, I_2, I_3)$ ,  $\mu$  and  $g$  denote respectively the angular momentum, the directional cosine of the  $z$ -axis (fixed in space), the center of gravity, the principal moment of inertia of the body, the mass of the body and the acceleration of gravity, all expressed in the body coordinates. In the absence of gravity i. e.  $L=0$ , we have the Euler free

rigid body motion around a fixed point. The Lagrange top corresponds to the case  $I_1=I_2, l_1=l_2=0$  i. e. the motion of a body around a fixed point symmetric around one principal axis of inertia and the center of gravity is on the axis of symmetry. The so-called Kowalewski top corresponds to the case  $I_1=I_2=2I_3, l_3=0$  i. e. the center of gravity belongs to the equatorial plane, passing through the fixed point. Moreover, we may choose  $l_2=0, \mu g l_1=1$  and  $I_3=1$ . After the substitution  $t \rightarrow 2t$  the system (Int 1) becomes

$$(Int\ 2) \quad \left\{ \begin{array}{l} \dot{m}_1 = m_2 m_3 \\ \dot{m}_2 = -m_1 m_3 + 2\gamma_3 \\ \dot{m}_3 = -2\gamma_2 \\ \dot{\gamma}_1 = 2m_3 \gamma_2 - m_2 \gamma_3 \\ \dot{\gamma}_2 = m_1 \gamma_3 - 2m_3 \gamma_1 \\ \dot{\gamma}_3 = m_2 \gamma_1 - m_1 \gamma_2 \end{array} \right.$$

and possesses the four invariants

$$(Int\ 3) \quad \left\{ \begin{array}{l} H_1 = \frac{1}{2}(m_1^2 + m_2^2) + m_3^2 + 2\gamma_1 = C_1 \\ H_2 = m_1 \gamma_1 + m_2 \gamma_2 + m_3 \gamma_3 = C_2 \\ H_3 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = C_3 = 1 \\ H_4 = \left[ \left( \frac{m_1 + im_2}{2} \right)^2 - (\gamma_1 + i\gamma_2) \right] \left[ \left( \frac{m_1 - im_2}{2} \right)^2 - (\gamma_1 - i\gamma_2) \right] = C_4, \end{array} \right.$$

where we may choose  $C_3=1$  without loss of generality. Let  $A$  be the complex affine variety defined by the intersection of the constants of the motion

$$(Int\ 4) \quad A = \bigcap_{j=1}^4 \{H_j = c_j\} \subseteq \mathbb{C}^6.$$

The first section explains how the affine variety  $A$  and vector-fields behave after the quotient by some natural involution on  $A$  and how these vector-fields become well defined when we take Kowalewski's variables. We show that these variables are naturally related to the so-called Euler's differential equations and can be seen as the addition-formula for the Weierstrassian elliptic function. In the second section, which is the main part of the paper, we show that the Kowalewski top is algebraically completely integrable in the Adler-van Moerbeke sense discussed above. The basic tool for doing this, is to consider

the five parameter family of Laurent solutions

$$M(t) = \frac{M^0}{t} + M^1 + M^2 t + \dots$$

$$\Gamma(t) = \frac{\Gamma^0}{t^2} + \frac{\Gamma^1}{t} + \Gamma^2 + \Gamma^3 t + \dots$$

where the five free parameters  $\alpha, \beta, \lambda, \theta, \mu$  appear linearly, whenever they appear for the first time. In fact these expansions contain a lot of information, which can be used to construct the abelian surfaces on which the flow linearizes. For instance, substituting these expansions into the constants of motion, leads to 4 polynomial relations between  $\alpha, \beta, \lambda, \theta, \mu$ , hence defining a reducible algebraic curve  $\mathcal{D}$  of genus 9, with two components of genus 3, each of which is a double ramified cover of an elliptic curve. Then, we complete  $A$  into an abelian surface by adjoining the curve  $\mathcal{D}$ ; the abelian surface obtained this way can be embedded into  $\mathbb{C}P^7$  via the sections of the line bundle going with  $\mathcal{D}$  and its period matrix has the type

$$\begin{pmatrix} 1 & 0 & a & c \\ 0 & 2 & c & b \end{pmatrix} \text{ with } \text{Im} \begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0.$$

It can also be realized as the dual of the Prym variety of the double cover of the elliptic curve mentioned above.

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This paper is dedicated to Professor A. Sadel.

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### *References.*

*Note.* Some results were obtained recently by Horozov-van Moerbeke [Abelian surfaces of polarization (1, 2) and Kowalewski's top, *Comm. Pure Applied Math.*, 1987], Adler-van Moerbeke [About Lax pair for the Kowalewski's top (to appear)] and Haine-Horozov in a forthcoming paper.

## 1. About Kowalewski's procedure

Let

$$f: A \rightarrow A: (m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3) \mapsto (x_1, x_2, m_3, y_1, y_2, \gamma_3)$$

be a birational map on the affine variety A (Int 4) where  $x_1, x_2, y_1$  and  $y_2$  are defined as

$$(1.1) \quad \begin{cases} 2x_1 = m_1 + im_2 \\ 2x_2 = m_1 - im_2 \\ y_1 = x_1^2 - (\gamma_1 + i\gamma_2) \\ y_2 = x_2^2 - (\gamma_1 - i\gamma_2). \end{cases}$$

In term of these new variables, equations (Int. 2) with  $t \rightarrow it$  and (Int. 3) take the following form where  $C_1 \equiv 6h_1, C_2 \equiv 2h_2$  and  $C_4 \equiv k^2$

$$(1.2) \quad \begin{cases} \dot{x}_1 = m_3 x_1 - \gamma_3 \\ \dot{x}_2 = -m_3 x_2 + \gamma_3 \\ \dot{m}_3 = -x_1^2 + y_1 + x_2^2 - y_2 \\ \dot{y}_1 = 2m_3 y_1 \\ \dot{y}_2 = -2m_3 y_2 \\ \dot{\gamma}_3 = x_1(x_2^2 - y_2) - x_2(x_1^2 - y_1) \end{cases}$$

and

$$(1.3) \quad A: \begin{cases} y_1 y_2 = k^2 \\ m_3^2 = 6h_1 + y_1 + y_2 - (x_1 + x_2)^2 \\ m_3 \gamma_3 = 2h_2 + x_1 y_2 + x_2 y_1 - x_1 x_2 (x_1 + x_2) \\ \gamma_3^2 = 1 - k^2 + x_1^2 y_2 + x_2^2 y_1 - x_1^2 x_2^2. \end{cases}$$

Note that

$$(1.4) \quad \tau: (x_1, x_2, m_3, y_1, y_2, \gamma_3) \mapsto (x_1, x_2, -m_3, y_1, y_2, -\gamma_3)$$

is an automorphism of A, of order two. The quotient  $\mathbb{B} \equiv A/\tau$  by the involution  $\tau$ , is a *Kummer surface* defined by

$$(1.5) \quad \mathbb{B}: \begin{cases} y_1 y_2 = k^2 \\ y_1 R(x_2) + y_2 R(x_1) + R_1(x_1, x_2) + k^2 (x_1 - x_2)^2 = 0 \end{cases}$$

with

$$(1.6) \quad \left\{ \begin{array}{l} R(X) \equiv -x^4 + 6h_1x^2 - 4h_2x + 1 - k^2 \\ R_1(x_1, x_2) \equiv -6h_1x_1^2x_2^2 + 4h_2x_1x_2(x_1 + x_2) \\ \qquad \qquad \qquad - (1 - k^2)(x_1 + x_2)^2 + 6h_1(1 - k^2) - 4h_2^2. \end{array} \right.$$

The variety A is a double cover of the surface  $\mathbb{B}$  branched over the fixed points of the involution  $\tau$ . To find them, we substitute  $m_3 = \gamma_3 = 0$  in the system (1.3), to wit

$$(1.7) \quad \left\{ \begin{array}{l} (a) \quad y_1y_2 = k^2 \\ (b) \quad y_1 + y_2 = (x_1 + x_2)^2 - 6h_1 \\ (c) \quad x_2y_1 + x_1y_2 = x_1x_2(x_1 + x_2) - 2h_2 \\ (d) \quad x_2^2y_1 + x_1^2y_2 = x_1^2x_2^2 + k^2 - 1. \end{array} \right.$$

Away from the  $x_1^2 = x_2^2$ , we may solve (b) and (d) in  $y_1$  and  $y_2$  and substitute into the remaining equations; one then finds two curves in  $x_1$  and  $x_2$

$$(1.8) \quad \left\{ \begin{array}{l} R(x_1, x_2) \equiv -x_1^2x_2^2 + 6h_1x_1x_2 - 2h_2(x_1 + x_2) + 1 - k^2 = 0 \\ S(x_1, x_2) \equiv (x_1^4 + 2x_1^3x_2 - 6h_1x_1^2 + 1 - k^2) \\ \qquad \qquad \qquad \times (x_2^4 + 2x_1x_2^3 - 6h_1x_2^2 + 1 - k^2) + k^2(x_1^2 - x_2^2)^2 = 0 \end{array} \right.$$

which intersect at the zeroes of the resultant of R, S:

$$(1.9) \quad \text{Res}(R, S)_{x_2} = x_1^2(x_1^4 + 6h_1x_1^2 + k^2 - 1)^2 P_8(x_1),$$

where  $P_8(x_1)$  is a monic polynomial of degree 8. Since the root  $x_1$  must be excluded (it indeed implies that the leading terms of R and S vanish), the possible intersections of the curve R and S will be

(i) at the roots of  $x_1^4 + 6h_1x_1^2 + k^2 - 1 = 0$ : this is unacceptable, because then one checks that the common roots of R and S would have the property that  $x_1^2 = x_2^2$ , which was excluded.

(ii) at the roots of  $P_8(x_1) = 0$ ; there, for generic  $k$  and  $h$ ,  $x_1^2 \neq x_2^2$ .

Finally, we must analyze the case  $x_1^2 = x_2^2$  for which one checks that (1.7) has no common roots. Consequently the involution  $\tau$  has 8 fixed points on the affine variety A. Clearly the vector field (1.2) vanishes at the fixed points of  $\tau$ .

Now, equations (1.5) imply that

$$y_1 = \frac{-1}{2R(x_2)} [R_1(x_1, x_2) + k^2(x_1 - x_2)^2 + w]$$

$$y_2 = \frac{-1}{2R(x_1)} [R_1(x_1, x_2) + k^2(x_1 - x_2)^2 - w]$$

with a radical  $w$  such that

$$w^2 = [\mathbf{R}_1(x_1, x_2) + k^2(x_1 - x_2)^2]^2 - 4k^2 \mathbf{R}(x_1) \mathbf{R}(x_2) \equiv \Psi(x_1, x_2).$$

This shows that the surface  $\mathbb{B}$  is a double cover of the plane  $x_1, x_2$  ramified along the curve

$$(1.10) \quad \mathcal{C}: \Psi(x_1, x_2) = 0.$$

This equation is reducible and can be written as the product  $\Psi_1(x_1, x_2) \cdot \Psi_2(x_1, x_2)$  of two symmetric polynomials (in  $x_1, x_2$ ) of degree two in each one of the variables  $x_1, x_2$ , i. e.,

$$\Psi_1(x_1, x_2) = A(x_1)x_2^2 + 2B(x_1)x_2 - C(x_1) = A(x_2)x_1^2 + 2B(x_2)x_1 - C(x_2)$$

where  $A(x)$ ,  $B(x)$  and  $C(x)$  are three polynomials of degree two in  $x$ :

$$A(x) \equiv -2(k + 3h_1)x^2 + 4h_2x - 1$$

$$B(x) \equiv 2h_2x^2 + (2k(k + 3h_1) - 1)x - 2h_2k$$

$$C(x) \equiv x^2 + 4h_2kx + 2(k^2 - 1)(k + 3h_1) + 4h_2^2.$$

The curve  $\mathcal{C}_1$  given by the symmetric equation

$$(1.11) \quad \mathcal{C}_1: \Psi_1(x_1, x_2) = 0$$

is elliptic:

$$(1.12) \quad x_2 = \frac{-B(x_1) \pm \sqrt{2(k + 3h_1) - 4h_2^2} \sqrt{\mathbf{R}(x_1)}}{A(x_1)}$$

$$x_1 = \frac{-B(x_2) \pm \sqrt{2(k + 3h_1) - 4h_2^2} \sqrt{\mathbf{R}(x_2)}}{A(x_2)}$$

where  $\mathbf{R}(x)$  is given by (1.6). Let  $\mathcal{C}_2$  be the curve defined by (1.11) but after switching the sign of  $k$ . The curve  $\mathcal{C}_1$  and  $\mathcal{C}_2$  intersect in 8 distinct points which happen to be the fixed points of the involution  $\tau$ , because

$$\text{Res}(\Psi_1, \Psi_2)_{x_2} = 16k^2 P_8(x_1)$$

where  $P_8(x_1)$  is given by (1.9).

Now, differentiating the symmetric equation  $\Psi_1(x_1, x_2) = 0$  [or  $\Psi_2(x_1, x_2) = 0$ ] with regard to  $t$ , one finds

$$\frac{\partial \Psi_1}{\partial x_1} \dot{x}_1 + \frac{\Psi_2}{\partial x_2} \dot{x}_2 = 0$$

where

$$\frac{\partial \Psi_1}{\partial x_1} = 2(A(x_2)x_1 + B(x_2)) = \pm 2\sqrt{2(k+3h_1)-4h_2^2}\sqrt{R(x_2)} \quad \text{by (1.12).}$$

Hence

$$(1.13) \quad \frac{\dot{x}_1}{\sqrt{R(x_1)}} \pm \frac{\dot{x}_2}{\sqrt{R(x_2)}} = 0.$$

Since  $R(x_1)$  and  $R(x_2)$  are two polynomials of the fourth degree in  $x_1$  and  $x_2$  respectively and having the same coefficients, then (1.13) is the so-called Euler's equation. The reader is referred to Halpen [12] and Weil [23] for this theory which we summarize here after.

Let  $F(x) = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4$  be a polynomial of the fourth degree. The general integral of Euler's equation

$$\frac{\dot{x}}{\sqrt{F(x)}} \pm \frac{\dot{y}}{\sqrt{F(y)}} = 0$$

can be written in two different ways:

$$(i) \quad F_1(x, y) + 2sF(x, y) - s^2(x-y)^2 = 0$$

where

$$F(x, y) = a_0x^2y^2 + 2a_1xy(x+y) + 3a_2(x^2+y^2) + 2a_3(x+y) + a_4$$

and

$$F_1(x, y) \equiv \frac{F(x)F(y) - F^2(x, y)}{(x-y)^2}.$$

(ii) or in an irrational form

$$\frac{F(x, y) \mp \sqrt{F(x)}\sqrt{F(y)}}{(x-y)^2} = s$$

which can be seen as the addition-formula for the Weierstrassian elliptic function

$$2\mathcal{P}(u+v) = \frac{(\mathcal{P}(u) + \mathcal{P}(v))(2\mathcal{P}(u)\mathcal{P}(v) - (1/2)g_2) - g_3 - \mathcal{P}'(u)\mathcal{P}'(v)}{(\mathcal{P}(u) - \mathcal{P}(v))^2}$$

$$\mathcal{P}'(u) \equiv \frac{d\mathcal{P}}{du} = \pm \sqrt{4\mathcal{P}^3 - g_2\mathcal{P} - g_3}$$

with  $\mathcal{P}(u) = x$ ,  $\mathcal{P}(v) = y$ ,  $\mathcal{P}'^2(u) = F(x)$ ,  $\mathcal{P}'^2(v) = F(y)$ ,  $F(x) = 4x^3 - g_2x - g_3$ ,  $2\mathcal{P}(u+v) = s$  and  $g_2, g_3$  constants.

Let us now apply these facts to Kowalewski's problem with  $a_0 = -1$ ,  $a_1 = 0$ ,  $a_2 = h_1$ ,  $a_3 = -h_2$ ,  $a_4 = 1 - k^2$ ,  $F(x) = R(x)$ ,  $F(x_1, x_2) = R(x_1, x_2) + 3h_1(x_1 - x_2)^2$  and  $s = k + 3h_1$ . So the polynomial (1.11) which can also be regarded as a solution of (1.13), can also be written as:

$$R_1(x_1, x_2) + 2kR(x_1, x_2) - k^2(x_1 - x_2)^2 = 0$$

where  $R_1(x_1, x_2)$  is given by (1.6) and has the form

$$R_1(x_1, x_2) = \frac{R(x_1)R(x_2) - R^2(x_1, x_2)}{(x_1 - x_2)^2}.$$

Remember that  $R(x_1, x_2)$  is given by (1.8). The solution of (1.13) can also be expressed as:

$$(1.14) \quad \frac{R(x_1, x_2) \mp \sqrt{R(x_1)}\sqrt{R(x_2)}}{(x_1 - x_2)^2} + 3h_1 = s.$$

Let us carry out the calculations, assuming the polynomial  $R(x)$  reduced to the form  $4x^3 - g_2x - g_3$  and call  $s_1$  (resp.  $s_2$ ) the relation (1.14) with the sign  $-$  (resp.  $+$ ). Now, outside the branch locus of  $\mathbb{B}$  (1.5) over  $\mathbb{C}^2$ , the equation (1.13) is not identically zero and may be written in the form

$$(1.15) \quad \begin{cases} \frac{\dot{x}_1}{\sqrt{R(x_1)}} + \frac{\dot{x}_2}{\sqrt{R(x_2)}} = \frac{\dot{s}_1}{\sqrt{4s_1^3 - g_2s_1 - g_3}} \neq 0 \\ \frac{\dot{x}_1}{\sqrt{R(x_1)}} - \frac{\dot{x}_2}{\sqrt{R(x_2)}} = \frac{\dot{s}_2}{\sqrt{4s_2^3 - g_2s_2 - g_3}} \neq 0 \end{cases}$$

where  $g_2 = k^2 - 1 + 3h_1^2$  and  $g_3 = h_1(k^2 - 1 - h_1^2) + h_2^2$ . After some algebraic manipulation we deduce from (1.4)

$$\begin{aligned} (m_3x_1 - \gamma_3)^2 &= R(x_1) + (x_1 - x_2)^2 y_1 \\ (m_3x_2 - \gamma_3)^2 &= R(x_2) + (x_1 - x_2)^2 y_2 \\ (m_3x_1 - \gamma_3)(m_3x_2 - \gamma_3) &= R(x_1, x_2) \end{aligned}$$

and from (1.3)

$$\begin{aligned} \dot{x}_1^2 &= R(x_1) + (x_1 - x_2)^2 y_1 \\ \dot{x}_2^2 &= R(x_2) + (x_1 - x_2)^2 y_2. \end{aligned}$$

This together with (1.5) and (1.15) implies that

$$\frac{\dot{s}_1^2}{4s_1^3 - g_2s_1 - g_3} = \left( \frac{\dot{x}_1}{\sqrt{R(x_1)}} + \frac{\dot{x}_2}{\sqrt{R(x_2)}} \right)^2$$

$$= \frac{(x_1 - x_2)^4}{R(x_1)R(x_2)} \left[ \left( \frac{R(x_1, x_2) - \sqrt{R(x_1)}\sqrt{R(x_2)}}{(x_1 - x_2)^2} \right)^2 - k^2 \right] = 4 \frac{(s_1 - 3h_1)^2 - k^2}{(s_1 - s_2)^2}.$$

In the same way, we find

$$\frac{\dot{s}_2^2}{4s_2^3 - g_2s_2 - g_3} = \left( \frac{\dot{x}_1}{\sqrt{R(x_1)}} - \frac{\dot{x}_2}{\sqrt{R(x_2)}} \right)^2 = 4 \frac{(s_2 - 3h_1)^2 - k^2}{(s_2 - s_1)^2}.$$

Consequently, the system (1.1) can be written as follows

$$(1.16) \quad \begin{cases} \frac{ds_1}{\sqrt{P_5(s_1)}} + \frac{ds_2}{\sqrt{P_5(s_2)}} = 0 \\ \frac{s_1 ds_1}{\sqrt{P_5(s_1)}} + \frac{s_2 ds_2}{\sqrt{P_5(s_2)}} = i dt \end{cases}$$

where  $P_5(s) \equiv ((s - 3h_1)^2 - k^2)(4s^3 - g_2s - g_3)$ . As known, such integrals are called *hyperelliptic integrals* and the problem can be integrated in terms of genus two hyperelliptic functions of time. Finally, we have the

**THEOREM 1.1.** — (i) *The complex affine variety A (Int 4) is a double ramified cover on the Kummer surface B (1.5), with eight branch points (= zeroes of the polynome  $P_8(x_1)$  (1.9)).* (ii) *The surface B is a double cover of plane  $(x_1, x_2)$  ramified along two elliptic curves intersecting each other at the 8 points above.* (iii) [16] *The system of differential equations (Int 2) is reduced to the system (1.16) which can be integrated in terms of genus 2 hyperelliptic functions.*

### 2. A geometric approach to study Kowalewski's top

The system (Int 2) can be written as a Hamiltonian vector field

$$(2.1) \quad \dot{z} = J \frac{\partial H_1}{\partial z}; \quad z = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}, \quad J = \begin{pmatrix} M & \Gamma \\ \Gamma & 0 \end{pmatrix} \quad \text{with } M, \Gamma \in \text{so}(3).$$

The second Hamiltonian vector field

$$(2.2) \quad \dot{z} = J \frac{\partial H_4}{\partial z}$$

is quartic and is written explicitly as

$$\dot{m}_1 = A m_2 m_3 + B(-m_1 m_3 + 2\gamma_3)$$

$$\begin{aligned}
\dot{m}_2 &= B m_2 m_3 - A (-m_1 m_3 + 2\gamma_3) \\
\dot{m}_3 &= 2A (-m_1 m_2 + \gamma_2) + B(m_1^2 - m_2^2 - 2\gamma_1) \\
\dot{\gamma}_1 &= (A m_2 - B m_1) \gamma_3 \\
\dot{\gamma}_2 &= (A m_1 + B m_2) \gamma_3 \\
\dot{\gamma}_3 &= (B m_1 - A m_2) \gamma_1 - (B m_2 + A m_1) \gamma_2
\end{aligned}$$

where  $A \equiv m_1^2 - m_2^2 - 4\gamma_1$  and  $B \equiv 2(m_1 m_2 - 2\gamma_2)$ . These vector fields are in *involution*, i. e.

$$\{H_1, H_4\} = \left\langle \frac{\partial H_1}{\partial z}, J \frac{\partial H_4}{\partial z} \right\rangle = 0$$

and the remaining ones are *Casimir functions*, i. e.

$$J \frac{\partial H_2}{\partial z} = J \frac{\partial H_3}{\partial z} = 0.$$

To illustrate the method (announced in the introduction) in a simple example, let us first examine the *Euler rigid body motion* around a fixed point. The system (Int 1) reduces in this case to the following equation

$$(a) \quad \dot{M} = [M, \Lambda M]$$

which is explicitly given by

$$(b) \quad \begin{cases} \dot{m}_1 = (\lambda_3 - \lambda_2) m_2 m_3 \\ \dot{m}_2 = (\lambda_1 - \lambda_3) m_1 m_3 \\ \dot{m}_3 = (\lambda_2 - \lambda_1) m_1 m_2 \end{cases}$$

where  $\lambda_j = 1/I_j$  ( $1 \leq j \leq 3$ ) and has the two first integrals

$$(c) \quad \begin{cases} H_1 = m_1^2 + m_2^2 + m_3^2 = c_1 \\ H_2 = \lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2 = c_2 \end{cases}$$

where  $c_1$  and  $c_2$  are constants. The system (b) with Hamiltonian  $(1/2)H_1$  is completely integrable on the phase space which is the sphere  $H_1 = c_1$ . For appropriate values of the constants, this sphere intersects the ellipsoid  $H_2 = c_2$  in two circles and any representative point of (b) moves on a circle of the sphere (uniform motion). It is well known that the system (b) can be integrated in terms of elliptic functions. From system (b) together with conditions (c) we obtain the following expression

$$(d) \quad \int_{m_3(t_0)}^{m_3(t)} \frac{dm}{[(a_1^2 - m^2)(a_2^2 - m^2)]^{1/2}} = a_3(t - t_0)$$

where

$$a_3^2 = (\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1), \quad a_1^2 = \frac{\lambda_2 c_1 - c_2}{\lambda_2 - \lambda_3}$$

and

$$a_2^2 = \frac{c_2 - \lambda_1 c_1}{\lambda_3 - \lambda_1}, \quad \lambda_1 \neq \lambda_2 \neq \lambda_3.$$

If we replace  $t \in \mathbb{R}$  by  $t \in \mathbb{C}$  then the function  $m_3(t)$  is an elliptic function on a complex torus and must have a Laurent expansion around an arbitrary complex valued constant  $t_0$ . Therefore since  $a_1$ ,  $a_2$  and  $a_3$  only depend on  $c_1$  and  $c_2$ , these two free parameters must enter somewhere in the expansion of  $m_3$  around the blow up point. Indeed, it is easy to see that the Laurent series solution of the system (a) has the form

$$M(t) = \sum_{k=0}^{\infty} M^k (t - t_0)^{k-1}.$$

Substituting this series into (a), one finds at the 0-th step a non-linear equation

$$(e) \quad M^0 + [M^0, \Lambda M^0] = 0,$$

and at the  $k$ -th step, a linear equation

$$(\mathcal{L} - k I)(M^k) = \text{terms containing } M^j \text{ for } 1 \leq j < k,$$

with  $\mathcal{L}$  being the Jacobian of (e). The matrix  $(\mathcal{L} - k I)$  is always invertible, unless  $k=2$  and then its rank equals 1. Consequently the coefficient  $M^2$  contains two free parameters, which account for  $c_1$  and  $c_2$ . In fact, there is a much richer structure involved in this example. Namely the circle of the sphere ( $H_1 = c_1$ ) extends to the complex torus mentioned above, the flow is mapped by the integral (d) into a straight line motion on that torus and the functions  $M(t)$  is meromorphic. The complex intersection

$$\bigcap_{i=1}^2 \{H_i(M) = c_i\} \subseteq \mathbb{C}^3$$

is the affine part of an elliptic curve  $\subseteq \mathbb{C}\mathbb{P}^3$  which is the above torus. This torus has an algebraic addition law connecting  $p(t_1 + t_2)$  to  $p(t_1)$  and  $p(t_2)$  where  $p(t) \equiv (m_1(t), m_2(t), m_3(t))$  is a solution of equations (b).

This example shows that the classical way of solving Euler's equations in terms of elliptic functions can be understood as a characterization of this complex torus. So the main question is how to complete the affine variety (defined by the intersection of the constants of the motion) and the differential equations on it into a non-singular compact complex algebraic variety? In the above example, we have completed the affine variety by the points of blow up, which are captured automatically by projectivizing the equations of this variety in  $\mathbb{C}\mathbb{P}^3$ . But this procedure can never work in general; indeed, an abelian

variety of dimension bigger or equal than two is never a complete intersection. Therefore if the affine part of an abelian surface is defined by 4 equations in  $\mathbb{C}^6$ , then the obvious embedding into  $\mathbb{C}\mathbb{P}^6$ , by making the equations projective, must have one or more singularities at infinity. In fact, we shall show that the Laurent expansions can be used to manufacture the tori, without ever going through the delicate procedure of blowing up and down.

A. ASYMPTOTIC EXPANSIONS. — Let  $M$  and  $\Gamma$  have the following asymptotic expansions

$$(2.3) \quad M = \sum_{k=0}^{\infty} M^k t^{k-1} \quad \text{and} \quad \Gamma = \sum_{k=0}^{\infty} \Gamma^k t^{k-2}.$$

Substituting (2.3) in the differential equations (Int 1), at the 0-th step, the coefficients of  $t^{-2}$  (for  $M$ ) and  $t^{-3}$  (for  $\Gamma$ ), yield a non-linear system

$$(2.4) \quad \begin{cases} M^0 + [M^0, \Lambda M^0] + [\Gamma^0, L] = 0 \\ 2\Gamma^0 + [\Gamma^0, \Lambda M^0] = 0 \end{cases}$$

and at the  $k$ -th step ( $k \geq 1$ ), the coefficients of  $t^{k-2}$  (for  $M$ ) and  $t^{k-3}$  (for  $\Gamma$ ) lead to a system of linear equations in  $M^k$  and  $\Gamma^k$

$$(2.5) \quad (\mathcal{L} - kI) \begin{pmatrix} M^k \\ \Gamma^k \end{pmatrix} = 0 \quad \text{for } k=1$$

$$= \begin{bmatrix} -\sum_{j=1}^{k-1} [M^j, \Lambda M^{k-j}] \\ -\sum_{j=1}^{k-1} [\Gamma^j, \Lambda M^{k-j}] \end{bmatrix} \quad \text{for } k > 1.$$

where  $\mathcal{L}$  denotes the linear operator

$$\mathcal{L} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} [M^0, \Lambda X] + [X, \Lambda M^0] + [Y, L] + X \\ [\Gamma^0, \Lambda X] + [Y, \Lambda M^0] + 2Y \end{pmatrix}.$$

In the basis  $(e_j)_{1 \leq j \leq 6}$ ,  $\mathcal{L}$  is given by the matrix

$$\mathcal{L} = \begin{bmatrix} 1 & m_3^0 & m_2^0 & 0 & 0 & 0 \\ -m_3^0 & 1 & -m_1^0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & -\gamma_3^0 & 2\gamma_2^0 & 2 & 2m_3^0 & -m_2^0 \\ \gamma_3^0 & 0 & -2\gamma_1^0 & -2m_3^0 & 2 & m_1^0 \\ -\gamma_2^0 & \gamma_1^0 & 0 & m_2^0 & -m_1^0 & 2 \end{bmatrix}$$

which is the Jacobian of (2.4).

LEMMA 2.1. — *The non linear system (2.4) defines two lines and two points.*

*Proof.* — The system (2.4) has the following explicit form

$$(2.6) \quad \begin{cases} (a) & m_1^0 + m_2^0 m_3^0 = 0 \\ (b) & m_2^0 - m_1^0 m_3^0 + 2\gamma_3^0 = 0 \\ (c) & m_3^0 - 2\gamma_2^0 = 0 \\ (d) & 2\gamma_1^0 + 2m_3^0 \gamma_2^0 - m_2^0 \gamma_3^0 = \\ (e) & 2\gamma_2^0 + m_1^0 \gamma_2^0 - 2m_3^0 \gamma_1^0 = \\ (f) & 2\gamma_3^0 + m_2^0 \gamma_1^0 - m_1^0 \gamma_2^0 = 0. \end{cases}$$

Equations (a), (b) and (c) imply that

$$(g) \quad m_1^0 + 2m_2^0 \gamma_3^0 = 0$$

$$(h) \quad 2\gamma_3^0 + (1 + (2\gamma_2^0)^2) m_2^0 = 0.$$

From equations (c), (d) and (h), it follows that

$$(j) \quad 4\gamma_1^0 + 2(2\gamma_2^0)^2 + (1 + (2\gamma_2^0)^2) (m_2^0)^2 = 0$$

and by (c), (e), (f), (g), (h) and (j), we obtain

$$m_2^0 (1 - \gamma_1^0 + 2(\gamma_2^0)^2) = 0$$

$$\gamma_2^0 (1 - 4\gamma_1^0 - (2\gamma_2^0)^2) = 0.$$

We now distinguish several cases:

*Case I.* — If  $m_2^0 = \gamma_2^0 = 0$  then the solution of (2.6) is identically zero.

*Case II.* — If  $\gamma_2^0 = 0$  and  $\gamma_1^0 = 1$  it follows that the set of solutions of (2.6) is

$$(2.7) \quad (m_1^0, m_2^0, m_3^0, \gamma_1^0, \gamma_2^0, \gamma_3^0) = (0, 2\varepsilon, 0, 1, 0, -\varepsilon), \quad \varepsilon \equiv \pm i.$$

*Case III.* — If  $1 - \gamma_1^0 + 2(\gamma_2^0)^2 = 1 - 4\gamma_1^0 - (2\gamma_2^0)^2 = 0$ , this immediately implies  $\gamma_1^0 = 1/2$  and  $\gamma_2^0 = \pm i/2$ . From equations (c), (g) and (h), it follows that  $m_3^0 = \pm i$ ,  $m_1^0 \pm im_2^0 = 0$  and  $\gamma_3^0 = 0$ . Consequently, we find the two lines

$$(2.8) \quad (m_1^0, m_2^0, m_3^0, \gamma_1^0, \gamma_2^0, \gamma_3^0) = (\alpha, \varepsilon\alpha, \varepsilon, 1/2, \varepsilon/2, 0), \quad \varepsilon \equiv \pm i$$

where  $\alpha$  is a free parameter. This concludes the proof of lemma 2.1.

LEMMA 2.2. — *The system (2.5) has in*

- (i) *case II (2.7), 2 degrees of freedom for  $k=2$  and 1 degree of freedom for  $k=3$  and 4.*
- (ii) *case III (2.8), 1 degree of freedom for  $k=1, 2, 3$  and 4.*

The *proof* of this lemma, is a linear algebra problem which is a straight forward computation. Since we are interested in 5-parameter Laurent solution, we only consider (ii). The eigenvectors  $V_1, V_2, V_3$  and  $V_4$  corresponding to the eigenvalues  $k=1, 2, 3$

and 4 of the matrix

$$\mathcal{L} = \begin{bmatrix} 1 & \varepsilon & \varepsilon\alpha & 0 & 0 & 0 \\ -\varepsilon & 1 & -\alpha & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & \varepsilon & 2 & 2\varepsilon & -\varepsilon\alpha \\ 0 & 0 & -1 & -2\varepsilon & 2 & \alpha \\ -\varepsilon/2 & 1/2 & 0 & \varepsilon\alpha & -\alpha & 2 \end{bmatrix}$$

are

$$V_1 = (\varepsilon(\alpha^2 - 2), -\alpha^2, \alpha, 0, 0, 1)$$

$$V_2 = (\alpha, \varepsilon\alpha, -2\varepsilon, 1, \varepsilon, 0)$$

$$V_3 = (-\varepsilon(\alpha^2 - 4), \alpha^2 + 8, -3\alpha, -3\varepsilon\alpha, 3\alpha, 6)$$

$$V_4 = (\alpha(\alpha^2 - 8), \varepsilon\alpha(\alpha^2 + 12), -4\varepsilon(\alpha^2 - 3), 6(\alpha^2 + 2), 6\varepsilon(\alpha^2 - 3), 20\varepsilon\alpha).$$

We denote by  $\beta$ ,  $\lambda$ ,  $\theta$  and  $\mu$ , the free parameters obtained respectively at  $k=1, 2, 3$  and 4. If we put

$$X^j \equiv (m_1^j, m_2^j, m_3^j, \gamma_1^j, \gamma_2^j, \gamma_3^j), \quad 1 \leq j \leq 4$$

the solutions of (2.5) are given by

$$X^1 = \beta V_1$$

$$X^2 = -\beta^2 A_1 + 3\lambda V_2$$

$$X^3 = -\frac{\alpha\beta^3}{4} A_2 + \frac{\beta\lambda}{2} A_3 + \theta V_3$$

$$X^4 = \frac{\beta^4}{80} A_4 - \frac{\beta^2\lambda}{40} A_5 + \frac{\lambda^2}{10} A_6 + \frac{\beta\theta}{20} A_7 - \mu V_4$$

where

$$A_1 \equiv (\alpha(\alpha^2 + 2)/2, \varepsilon\alpha(\alpha^2 - 2)/2, 0, 1, 0, \varepsilon\alpha)$$

$$A_2 \equiv (\varepsilon\alpha(\alpha^2 - 2), -\alpha(\alpha^2 + 6), \alpha^2 + 6, \varepsilon(\alpha^2 - 2), -\alpha^2 - 6, 0)$$

$$A_3 \equiv (\varepsilon(7\alpha^2 + 8), -7\alpha^2 + 16, 3\alpha, 3\varepsilon\alpha, -3\alpha, 0)$$

$$A_4 \equiv (\alpha(13\alpha^4 + 74\alpha^2 + 40), \varepsilon\alpha(13\alpha^4 - 26\alpha^2 - 80), -4\varepsilon(3\alpha^4 - 11\alpha^2 + 10),$$

$$2(9\alpha^4 + 32\alpha^2 - 20), 2\varepsilon(9\alpha^4 - 33\alpha^2 + 30), 0)$$

$$A_5 \equiv (\alpha(85\alpha^2 - 208), \varepsilon\alpha(85\alpha^2 + 212), -4\varepsilon(5\alpha^2 + 37), -6(5\alpha^2 + 2), 2\varepsilon(15\alpha^2 + 111), 0)$$

$$A_6 \equiv (63\alpha, 63\varepsilon\alpha, -72\varepsilon, 108, 108\varepsilon, 0)$$

$$A_7 \equiv (\alpha(31\alpha^2 + 80), \varepsilon(31\alpha^2 - 100), -4\varepsilon(11\alpha^2 - 5), 6(11\alpha^2 - 10), 6\varepsilon(11\alpha^2 - 5), 0).$$

To conclude the generic solution blows up after a finite time according to a Laurent series within a 5 parameters family of Laurent solutions. By the majorant method, any formal Laurent series solution of a system of differential equations with quadratic right hand side automatically converges. Now it is easily checked that

$$(\mathcal{L} + \mathbf{I}) \begin{pmatrix} \mathbf{M}^0 \\ \Gamma^0 \end{pmatrix} = 0$$

and it follows from Lemma 2.2 that

$$\det(\mathcal{L} - k \mathbf{I}) = (k+1)k(k-1)(k-2)(k-3)(k-4).$$

Consequently, we have

**THEOREM 2.1.** — *The system of differential equations (Int 2) possesses Laurent series solution*

$$(2.9) \quad m_j = \sum_{k=0}^{\infty} m_j^k t^{k-1} \quad \text{and} \quad \gamma_j = \sum_{k=0}^{\infty} \gamma_j^k t^{k-2}, \quad 1 \leq j \leq 3$$

which depend on 5 free parameters (=dim (phase space) - 1), with leading terms given by (2.8).

**B. DIVISORS OF POLES.** — We now search for the set of Laurent solution which remain confined to a fixed affine invariant surface, related to specific values of  $c_1, c_2, c_4$ , i. e.

$$\begin{aligned} \mathcal{D}_\varepsilon &= \left\{ \begin{array}{l} \text{The Laurent solutions } m_j(t), \gamma_j(t), 1 \leq j \leq 3 \\ \text{such that } H_k(m_j(t), \gamma_j(t)) = c_k, 1 \leq k \leq 4 \end{array} \right\} \\ &= 4 \text{ polynomial relations between } \alpha, \beta, \lambda, \theta \text{ and } \mu; \\ &\quad c_1 = (\alpha^2 - 4)\beta^2 + 18\lambda \\ &\quad -\varepsilon c_2 = (\alpha^2 + 2)\beta^3 - 6\beta\lambda + 12\theta \\ &\quad 8 = (5\alpha^2 - 2)\beta^4 - 6\beta^2\lambda + 84\beta\theta - 240\mu \\ &\quad 8c_4 = (\alpha^2 - 1)((11\alpha^2 + 10)\beta^4 + 54\beta^2\lambda + 108\beta\theta + 240\mu) \\ &= \text{algebraic curve} \\ &= \left\{ \begin{array}{l} (\alpha, \beta, \varepsilon) \text{ such that } P(\alpha, \beta, \varepsilon) \equiv (\alpha^2 - 1)(\alpha^2 - 1)\beta^4 - P(\beta) + c_4 = 0 \\ P(\beta) \equiv c_1\beta^2 - 2\varepsilon c_2\beta - 1 \\ \varepsilon \equiv \pm i \end{array} \right\}. \end{aligned}$$

The map

$$(2.10) \quad \sigma_\varepsilon : \mathcal{D}_\varepsilon \rightarrow \mathcal{D}_\varepsilon : (\alpha, \beta, \varepsilon) \mapsto (-\alpha, \beta, \varepsilon)$$

is an involution on  $\mathcal{D}_\varepsilon$ . The quotient  $\mathcal{D}_\varepsilon^0 = \mathcal{D}_\varepsilon / \sigma_\varepsilon$  by the involution  $\sigma_\varepsilon$  is an elliptic curve defined by

$$(2.11) \quad \mathcal{D}_\varepsilon^0 : u^2 = P^2(\beta) - 4c_4\beta^4.$$

The curve  $\mathcal{D}_\varepsilon$  is a 2-sheeted ramified covering of  $\mathcal{D}_\varepsilon^0$

$$(2.12) \quad \varphi_\varepsilon : \mathcal{D}_\varepsilon \rightarrow \mathcal{D}_\varepsilon^0 : (\alpha, u, \beta, \varepsilon) \mapsto (u, \beta, \varepsilon)$$

$$(2.13) \quad \mathcal{D}_\varepsilon : \begin{cases} \alpha^2 = \frac{2\beta^4 + P(\beta) + u}{2\beta^4} \\ u^2 = P^2(\beta) - 4c_4\beta^4. \end{cases}$$

Let us now look more closely at certain points of interest on the non-singular version of the curve  $\mathcal{D}_\varepsilon$ . For  $\beta$  sufficiently small,

$$\alpha^2 = \frac{2\beta^4 + P(\beta) + \sqrt{P^2(\beta) - 4c_4\beta^4}}{2\beta^4} = 1 - c_4 + O(\beta)$$

and

$$\alpha^2 = \frac{2\beta^4 + P(\beta) - \sqrt{P^2(\beta) - 4c_4\beta^4}}{2\beta^4} = \frac{1}{\beta^4}(-1 + O(\beta)).$$

At  $\beta = \infty$ , the curve  $\mathcal{D}_\varepsilon$  behaves as follows

$$2(\alpha^2 - 1)\beta^2 = c_1 \pm \sqrt{c_1^2 - 4c_4} + O(\beta).$$

Now, the curve  $\mathcal{D}_\varepsilon$  has 4 points at infinity  $p_j (1 \leq j \leq 4)$  and 4 branch points  $q_j \equiv (\alpha = 0, u = -2\beta^4 - P(\beta), \beta^4 + P(\beta) + c_4 = 0) (1 \leq j \leq 4)$  on the elliptic curve  $\mathcal{D}_\varepsilon^0$ . The divisor structure of  $\alpha, \beta$  on  $\mathcal{D}_\varepsilon$  is

$$\begin{aligned} (\alpha) &= \sum_{1 \leq j \leq 4} q_j - \sum_{1 \leq j \leq 4} p_j \\ (\beta) &= 4 \text{ zeroes} - \sum_{1 \leq j \leq 4} p_j. \end{aligned}$$

Let  $g(\mathcal{D}_\varepsilon^0) = \text{genus of } \mathcal{D}_\varepsilon^0, g(\mathcal{D}_\varepsilon) = \text{genus of } \mathcal{D}_\varepsilon, n = \# \text{ of sheets and } v = \# \text{ of branch points. Then by the Riemann-Hurwitz's formula}$

$$g(\mathcal{D}_\varepsilon) = n(g(\mathcal{D}_\varepsilon^0) - 1) + 1 + \frac{v}{2} = 3.$$

The map

$$(\alpha, u, \beta, \varepsilon) \mapsto (\alpha, u, -\beta, -\varepsilon)$$

is an isomorphism between  $\mathcal{D}_{\varepsilon=i}$  and  $\mathcal{D}_{\varepsilon=-i}$  and so we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{D}_{\varepsilon=i} & \xrightarrow{\sim} & \mathcal{D}_{\varepsilon=-i} \\ \varphi_{\varepsilon=i} \downarrow & & \downarrow \varphi_{\varepsilon=-i} \\ \mathcal{D}_{\varepsilon=i}^0 & \xrightarrow{\sim} & \mathcal{D}_{\varepsilon=-i}^0 \end{array}$$

Thus, we have proved

**THEOREM 2.2.** — *The divisors of poles  $\mathcal{D}_{\varepsilon=\pm i}$  (2.13) of the functions  $m_j, \gamma_j (1 \leq j \leq 3)$  are two isomorphic irreducible Riemann surfaces of genus 3. They are 2-sheeted ramified coverings of two elliptic curves  $\mathcal{D}_{\varepsilon=\pm i}^0$  (2.11).*

*Remark.* — From the Poincaré residue formula, we know that the 3 holomorphic differentials on  $\mathcal{D}_\varepsilon$  are of the form

$$\frac{g(\alpha, \beta, \varepsilon) d\beta}{(\partial P / \partial \alpha)(\alpha, \beta, \varepsilon)} = \frac{g(\alpha, \beta, \varepsilon) d\beta}{\alpha u}$$

where  $g(\alpha, \beta, \varepsilon)$  is a polynomial of at most degree five in  $\alpha$  and  $\beta$ . It is easy to verify that

$$(2.14) \quad \omega_0 = \frac{d\beta}{u}, \quad \omega_1 = \frac{(\alpha^2 - 1)\beta^2 d\beta}{\alpha u}, \quad \omega_2 = \frac{d\beta}{\alpha u}$$

form effectively a basis of holomorphic differentials on  $\mathcal{D}_\varepsilon$ . Observe that  $\sigma_\varepsilon^* \omega_0 = \omega_0$  and  $\sigma_\varepsilon^* \omega_j = -\omega_j (j=1, 2)$  for the involution  $\sigma_\varepsilon$  (2.10).

**C. ABELIAN SURFACE.** — Let  $T$  be a smooth surface compactifying  $A$  (Int 4) and let

$$\mathcal{D} \equiv \mathcal{D}_{\varepsilon=i} + \mathcal{D}_{\varepsilon=-i} \subseteq T$$

be a divisor (to be shown later). Consider a basis  $1, f_1, \dots, f_N$  of the vector space

$$\mathcal{L}(\mathcal{D}) = \{f, f \text{ meromorphic on } T, (f) \geq -\mathcal{D}\}$$

and the holomorphic map

$$T \rightarrow \mathbb{C}\mathbb{P}^N : p \mapsto (1, f_1(p), \dots, f_N(p)),$$

considered projectively, because if at  $p$  some  $f_j(p) = \infty$ , we divide by  $f_j$  having the highest order pole near  $p$ , which makes every element finite. This defines a map of  $T$  into  $\mathbb{C}\mathbb{P}^N$ . The *Kodaira embedding theorem* tells us that if the line bundle associated with the divisor is positive, then for some  $k \geq 0, k \in \mathbb{Z}$ , the functions of  $\mathcal{L}(k\mathcal{D})$  embed smoothly  $T$  into  $\mathbb{C}\mathbb{P}^N$  and then by Chow's theorem,  $T$  can be realized as an algebraic variety, i. e.

$$T = \bigcap_j \{P_j(z) = 0, z \in \mathbb{C}\mathbb{P}^N\}$$

where  $P_j(z)$  are homogeneous polynomials. In fact we shall show that in our case,  $k=1$  suffices i. e. the divisor  $\mathcal{D}$  provides a smooth embedding into  $\mathbb{C}\mathbb{P}^7$ , via the meromorphic section of  $\mathcal{L}(\mathcal{D})$ . The Riemann-Roch theorem and the adjunction formula (on abelian surfaces) together imply that

$$\dim \mathcal{L}(\mathcal{D}) = N + 1 = g(\mathcal{D}) - 1.$$

Based on this motivation, we wish to find a set of polynomial functions  $\{f_0=1, f_1, \dots, f_N\}$  having a simple pole along  $\mathcal{D}$  such that the embedding of  $\mathcal{D}$  with those functions into  $\mathbb{C}\mathbb{P}^N$  yields a curve of genus  $N+2$ . Let

$$\mathcal{L}^r = \left\{ \begin{array}{l} \text{polynomials } f \text{ of degree } \leq r \text{ behaving like} \\ f^{(r)} \\ t + f^1 + O(t) \text{ mod. the constants of motion} \end{array} \right\}$$

and let  $\{f_0=1, f_1, \dots, f_{N_r}\}$  be a basis of  $\mathcal{L}^r$ . The map

$$\mathcal{D} \rightarrow \mathbb{C}\mathbb{P}^{N_r}: p \mapsto \lim_{t \rightarrow 0} t(f_0(p), \dots, f_{N_r}(p)) = (0, f_1^0(p), \dots, f_{N_r}^0(p))$$

maps the curve  $\mathcal{D}$  into  $\tilde{\mathcal{D}}^r \subseteq \mathbb{C}\mathbb{P}^{N_r}$ . We look for  $r$  such that

$$g(\tilde{\mathcal{D}}^r) = N_r + 2, \quad \tilde{\mathcal{D}}^r \subseteq \mathbb{C}\mathbb{P}^{N_r}.$$

LEMMA 2.3:

$$\begin{aligned} \mathcal{L}^0 &= \{f_0=1\} \\ \mathcal{L}^1 &= \mathcal{L}^0 \oplus \{f_1=m_1, f_2=m_2, f_3=m_3\} \\ f_1 &= \frac{\alpha}{t} + \varepsilon(\alpha^2 - 2)\beta + O(t) \\ f_2 &= \frac{\varepsilon\alpha}{t} - \alpha^2\beta + O(t) \\ f_3 &= \frac{\varepsilon}{t} + \alpha\beta + O(t) \\ \mathcal{L}^2 &= \mathcal{L}^1 \oplus \left\{ f_4=\gamma_3, f_5 = -\frac{1}{4}(f_1^2 + f_2^2) \right\} \\ f_4 &= \frac{\beta}{t} - \varepsilon\alpha\beta^2 + O(t) \\ f_5 &= \frac{\varepsilon\alpha\beta}{t} + \beta^2 + O(t) \\ \mathcal{L}^3 &= \mathcal{L}^2 \oplus \{f_6=f_3f_5 + f_1f_4\} \\ f_6 &= \frac{\varepsilon(\alpha^2 - 1)\beta^2}{t} - \alpha(\varepsilon c_2 - c_1\beta + (\alpha^2 - 1)\beta^3) + O(t) \\ \mathcal{L}^4 &= \mathcal{L}^3 \oplus \{f_7=(f_2\gamma_1 - f_1\gamma_2)f_3 + 2f_4\gamma_2\} \\ f_7 &= \frac{\varepsilon}{t}(-\varepsilon c_2 + c_1\beta - 2(\alpha^2 - 1)\beta^3) - \alpha(2 + 3\varepsilon c_2\beta - c_1\beta^2 + 2(\alpha^2 - 1)\beta^4) + O(t). \end{aligned}$$

The *proof* of the previous lemma can easily be done by inspection of the expansions (2.9).

PROPOSITION 2.1. —  $\mathcal{L}^4$  provides an embedding of  $\tilde{\mathcal{D}}^4$  into projective space such that

$$(\alpha, u, \beta) \in \mathcal{D} \mapsto \lim t(f_0, f_1, \dots, f_7) = (0, f_1^0, \dots, f_7^0) \in \mathbb{C}\mathbb{P}^7$$

and the genus of  $\tilde{\mathcal{D}}^4$  is

$$g(\tilde{\mathcal{D}}^4) = 9.$$

*Proof.* — It turns out that neither  $\mathcal{L}^1$ , nor  $\mathcal{L}^2$ , nor  $\mathcal{L}^3$  yields a curve of the right genus; in fact

$$g(\tilde{\mathcal{D}}^r) \neq \dim \mathcal{L}^r + 1, \quad r = 1, 2, 3$$

For instance, the embedding into  $\mathbb{C}\mathbb{P}^3$  via  $\mathcal{L}^1$  does not separate the sheets, so we proceed to  $\mathcal{L}^2$  and we show that

$$g(\tilde{\mathcal{D}}^2 \text{ as embedded into } \mathbb{C}\mathbb{P}^5) - 2 > 5$$

which contradicts the fact that  $N + 1 = g(\tilde{\mathcal{D}}) - 1$ , so we look at  $\mathcal{L}^3$  and we find that

$$g(\tilde{\mathcal{D}}^3 \text{ as embedded into } \mathbb{C}\mathbb{P}^6) - 2 > 6$$

and the contradiction persists. We proceed now to  $\mathcal{L}^4$  and if we denote by

$$\begin{pmatrix} f_0 \\ \vdots \\ f_7 \end{pmatrix} \equiv F_k = F_k^0 + F_k^1 t + \dots, \quad 0 \leq k \leq 7$$

and

$$p_j \equiv (\alpha = \pm 1, u = \pm \beta^2 \sqrt{c_1^2 - 4c_4}, \beta = \infty) \quad (1 \leq j \leq 4)$$

the 4 points at infinity of  $\mathcal{D}_\varepsilon$ , we obtain

$$\begin{aligned} F_k^0(p_j) &= \left( 0, \frac{\alpha}{\beta}, \frac{\varepsilon\alpha}{\beta}, \frac{\varepsilon}{\beta}, 1, \varepsilon\alpha, \varepsilon(\alpha^2 - 1)\beta, -\frac{c_2}{\beta} + \frac{\varepsilon(1-u)}{\beta^2} \right)(p_j) \\ &= (0, 0, 0, 0, 1, \pm\varepsilon, 0, \mp\varepsilon\sqrt{c_1^2 - 4c_4}) = 4 \text{ distinct points} \end{aligned}$$

and using the transformation

$$\mathcal{D}_{\varepsilon=i} \rightarrow \mathcal{D}_{\varepsilon=-i} : (\alpha, u, \beta) \mapsto (-\alpha, -u, \beta),$$

we have that

$$F_k^0(p_j)|_{\mathcal{D}_{\varepsilon=i}} = F_k^0(p_j)|_{\mathcal{D}_{\varepsilon=-i}},$$

implying that the 4 points  $p_j$  on one curve are identified pairwise with the 4 corresponding points on the other curve. Let  $s \equiv 1/\beta$  be a local parameter for  $p_j$ , we have

$$\frac{\partial F_k^0}{\partial s}(p_j) = (0, \pm 1, \pm\varepsilon, \varepsilon, 0, \pm\varepsilon, 0, -c_2)$$

which shows that

$$\left. \frac{\partial F_k^0}{\partial s}(p_j) \right|_{\mathcal{D}_{\varepsilon=i}} \neq \left. \frac{\partial F_k^0}{\partial s}(p_j) \right|_{\mathcal{D}_{\varepsilon=-i}}$$

and consequently the curve  $\mathcal{D}_{\varepsilon=i}$  intersects transversely the curve  $\mathcal{D}_{\varepsilon=-i}$  in 4 points at infinity (fig. 1).

$$b_1, \dots, b_4 \equiv (\alpha = \infty, \beta = 0)$$

$$p_1, \dots, p_4 \equiv (\alpha = \pm 1, u = \pm \beta^2 \sqrt{c_1^2 - 4c_4}, \beta = \infty)$$

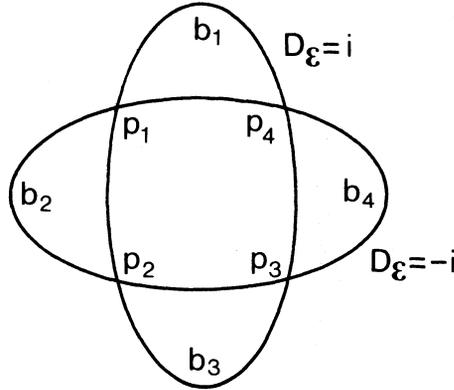


Fig. 1

In a neighbourhood of the points  $b_j \equiv (\alpha = \infty, \beta = 0)$  ( $1 \leq j \leq 4$ ) it is more convenient to divide the functions  $f_0, \dots, f_7$  by  $f_1$  and one can see that if we put

$$\left( \frac{f_0}{f_1}, \dots, \frac{f_7}{f_1} \right) \equiv \mathcal{F}_k = \mathcal{F}_k^0 + \mathcal{F}_k^1 t + \dots, \quad 0 \leq k \leq 7$$

that

$$\mathcal{F}_k^0(b_j) = (0, 1, \varepsilon, 0, 0, 0, \mp 1, 0) = 4 \text{ distinct points}$$

and consequently

$$g(\tilde{\mathcal{D}}^4 \text{ as embedded into } \mathbb{C}\mathbb{P}^7) - 2 = 7$$

i. e.  $g(\tilde{\mathcal{D}}^4) = 9$ . This completes the proof of proposition 2. 1.

Let  $\mathcal{L} = \mathcal{L}^4$  and  $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}^4 \subseteq \mathbb{C}\mathbb{P}^7$ . Next we wish to construct a surface strip around  $\tilde{\mathcal{D}}$  which will support the commuting vector fields.

(i) At all points where  $\alpha, \beta \neq 0, \infty$ , the Laurent solutions are nicely convergent (by the majorant method). Therefore, at most points of  $\tilde{\mathcal{D}}$  there is a transversal fiber to the curve. Hence this defines a smooth surface strip around  $\tilde{\mathcal{D}}$  except at the bad points.

(ii) Now we need to construct a surface strip around  $\mathcal{D}$  at the bad points as well. For doing that, we must introduce the concept of projective normality. Ultimately, we wish to prove that in the various charts

$$(2.15) \quad \left(\frac{f_j}{f_k}\right)' = \text{polynomial} \left(\frac{f_j}{f_k}\right), \quad 0 \leq j \leq 7, \quad k \text{ fixed.}$$

This enables one to show that  $f_j/f_k$  is a bona fide Taylor series starting from every point in a neighbourhood of the point in question  $\subseteq \mathbb{C}\mathbb{P}^7$ .

PROPOSITION 2.2. — *The orbits of the vector field (2.1) going through the curve  $\mathcal{D}$  form a smooth surface  $\Sigma$  near  $\mathcal{D}$  such that*

$$\Sigma \setminus \mathcal{D} \subseteq A.$$

Moreover, the variety

$$T = A \cup \Sigma$$

is smooth, compact and connected.

*Proof.* — Let  $I = \{t \in \mathbb{C}, -\delta < t < \delta\}$  be an interval, let

$$(t, p) \mapsto \varphi(t, p) = \{(M(t, p), \Gamma(t, p)), t \in I, p \in \mathcal{D}\},$$

be the orbit of the vector field (2.1) going through the point  $p \in \mathcal{D}$ , let  $\Sigma_p \subset \mathbb{C}\mathbb{P}^7$  be the surface element formed by the divisor  $\mathcal{D}$  and the orbits going through  $p$ , and set  $\Sigma \equiv \bigcup_{p \in \mathcal{D}} \Sigma_p$ . Consider the curve  $\mathcal{D}' = \pi \cap \Sigma$  where  $\pi \subset \mathbb{C}\mathbb{P}^7$  is a hyperplane transversal to

the direction of the flow. If  $\mathcal{D}'$  is smooth, then using the implicit function theorem the surface  $\Sigma$  is smooth. But if  $\mathcal{D}'$  is singular at 0, then  $\Sigma$  would be singular along the trajectory ( $t$ -axis) which goes immediately into the affine part  $A$ . Hence,  $A$  would be singular which is a contradiction because  $A$  is the fibre of a morphism from  $\mathbb{C}^6$  to  $\mathbb{C}^4$  and so smooth for almost all the four constants of the motion  $c_j$ . Next, let  $\bar{A}$  be the projective closure of  $A$  into  $\mathbb{C}\mathbb{P}^6$ , let  $z = (z_0, z_1 = m_1/z_0, \dots, \gamma_3/z_0) \in \mathbb{C}\mathbb{P}^6$ , let  $\mathcal{C} = \bar{A} \cap (z_0 = 0)$  be the locus at infinity. (In fact, we have  $\mathcal{C} = d_1 \cup d_2 \cup S'$  where  $d_1, d_2$  are straight lines and  $S'$  a circle with  $d_1 \cap d_2 = \emptyset$ ,  $d_j \cap S' = 1$  point,  $j = 1, 2$ ) and let  $f = (f_0, f_1, \dots, f_7) \in \mathcal{L}(\mathcal{D})$ . Consider the map

$$\bar{A} \subseteq \mathbb{C}\mathbb{P}^6 \rightarrow \mathbb{C}\mathbb{P}^7 : z \mapsto f(z),$$

and let  $T = f(\bar{A})$ . In a neighbourhood  $\mathcal{V}(p) \subseteq \mathbb{C}\mathbb{P}^7$  of  $p$ , we have

$$\begin{aligned} \Sigma_p &= T \\ \Sigma_p \setminus \mathcal{D} &\subseteq A. \end{aligned}$$

Otherwise there would be an element of surface  $\Sigma'_p \subseteq T$  such that

$$\Sigma_p \cap \Sigma'_p = t\text{-axis}$$

$$\text{orbit } \varphi(t, p) = t\text{-axis} \setminus p \subseteq A,$$

and hence  $A$  would be singular along the  $t$ -axis which is impossible. Since the variety  $\bar{A} \cap (z_0 \neq 0)$  is irreducible and since the generic hyperplane section  $\pi_{\text{gen.}}$  of  $\bar{A}$  is also irreducible, all hyperplane sections are connected and hence  $\mathcal{C}$  is also connected. Now, consider the graph  $(f) \subseteq \mathbb{C}\mathbb{P}^6 \times \mathbb{C}\mathbb{P}^7$  of the map  $f$ , which is irreducible together with  $\bar{A}$ . It follows from the irreducibility of  $\mathcal{C}$  that a generic hyperplane section graph  $(f) \cap (\pi_{\text{gen.}} \times \mathbb{C}\mathbb{P}^7)$  is irreducible, hence the special hyperplane section

$$\pi_{\text{sp.}} \equiv \text{graph}(f) \cap ((z_0 = 0) \times \mathbb{C}\mathbb{P}^7)$$

is connected and therefore the projection map

$$p'_{\mathbb{C}\mathbb{P}^7}(\pi_{\text{sp.}}) = f(\mathcal{C}) \equiv \mathcal{D}$$

is connected. Hence, the variety

$$A \cup \Sigma = T$$

is compact, connected and embeds smoothly into  $\mathbb{C}\mathbb{P}^7$  via  $f$ . This concludes the proof of proposition 2.2.

In fact, we shall prove a somewhat stronger statement than (2.15), namely that (2.15) is namely that (2.15) is satisfied with quadratic polynomials. By inspection one sees that the functions  $f_0, \dots, f_7$  do not satisfy that property. For example

$$\left(\frac{f_0}{f_4}\right)' = \frac{f_1 \gamma_2 - f_2 \gamma_1}{(f_4)^2} \neq \text{polynomial} \left(\frac{f_j}{f_4}\right), \quad 0 \leq j \leq 7.$$

Hence we must take functions with higher order poles. Let us consider for instance  $\mathcal{L}(2\mathcal{D}_{\varepsilon=i} + \mathcal{D}_{\varepsilon=-i})$ .

**PROPOSITION 2.3.** — *We have  $\dim \mathcal{L}(2\mathcal{D}_{\varepsilon=i} + \mathcal{D}_{\varepsilon=-i}) = 18$  and  $2\mathcal{D}_{\varepsilon=i} + \mathcal{D}_{\varepsilon=-i}$  is projectively normal and embeddable into  $\mathbb{C}\mathbb{P}^{17}$ .*

*Proof.* — For ease of manipulation, we are going to use  $x_1, x_2, y_1, y_2$  given by (1.1) instead of the functions  $m_1, m_2, \gamma_1, \gamma_2$ . Let  $g_0 = 1, g_1, \dots, g_{17}$  be a basis of  $\mathcal{L}(2\mathcal{D}_{\varepsilon=i} + \mathcal{D}_{\varepsilon=-i})$  where  $g_1 = x_1, g_2 = x_2, g_3 = f_3, g_4 = f_4, g_5 = f_5, g_6 = f_6, g_7 = f_7, g_8 = g_2^2, g_9 = g_2 g_3, g_{10} = g_2 g_4, g_{11} = y_2, g_{12} = g_1 g_{11}, g_{13} = g_1^2 g_{11}, g_{14} = g_1^3 g_{11}, g_{15} = g_2 g_5, g_{16} = g_2 g_6$  and  $g_{17} = g_2 g_7$ . These meromorphic functions have the properties

$$\begin{aligned} (g_1) &= -\mathcal{D}_{\varepsilon=-i} \\ (g_2) &= -\mathcal{D}_{\varepsilon=i} \\ (g_k) &= -\mathcal{D}_{\varepsilon=i} - \mathcal{D}_{\varepsilon=-i} \quad 3 \leq k \leq 7 \\ (g_8) &= -2\mathcal{D}_{\varepsilon=i} \\ (g_k) &= -2\mathcal{D}_{\varepsilon=i} - \mathcal{D}_{\varepsilon=-i} \quad k = 9, 10, 15, 16, 17 \\ (g_{11}) &= -2\mathcal{D}_{\varepsilon=i} + 2\mathcal{D}_{\varepsilon=-i} \end{aligned}$$

$$(g_{12}) = -2 \mathcal{D}_{\varepsilon=i} + \mathcal{D}_{\varepsilon=-i}$$

$$(g_k) = -2 \mathcal{D}_{\varepsilon=i}, \quad k = 13, 14.$$

Hence, the map

$$2 \mathcal{D}_{\varepsilon=i} + \mathcal{D}_{\varepsilon=-i} \rightarrow \mathbb{C}P^{17} : p \rightsquigarrow \lim_{t \rightarrow 0} t(g_0(p), \dots, g_{17}(p)) = (0, g_1^0(p), \dots, g_{17}^0(p))$$

maps the curve  $2 \mathcal{D}_{\varepsilon=i} + \mathcal{D}_{\varepsilon=-i}$  into  $\mathbb{C}P^{17}$ . In fact, in a neighbourhood of the points at infinity  $p_j = (\alpha = \pm 1, u = \pm \beta^2 \sqrt{c_1^2 - 4c_4}, \beta = \infty)$  ( $1 \leq j \leq 4$ ) it is simpler to divide the functions  $g_0, \dots, g_{17}$  by  $g_{10}$  which makes  $(g_k/g_{10})(p_j)$  ( $0 \leq k \leq 17$ ) finite. Whereas in the neighbourhood of the points  $b_j = (\alpha = \infty, \beta = 0)$ , it is more convenient to divide by  $g_8$  which makes  $(g_k/g_8)(p_j)$  ( $0 \leq k \leq 17$ ) finite. Next, using the vector field (2.1), we show that in a neighbourhood of the points  $p_j$ , and modulo linear combination of the constants of motion

$$\left( \frac{g_k}{g_{10}} \right)' = \frac{\dot{g}_k g_{10} - g_k \dot{g}_{10}}{(g_{10})^2}, \quad 0 \leq k \leq 17$$

$$= \frac{\text{quadratic polynomial in } (g_0, \dots, g_{17})}{(g_{10})^2}$$

$$= \text{quadratic polynomial} \left( \frac{g_0}{g_{10}}, \dots, \frac{g_{17}}{g_{10}} \right).$$

Also, in a neighbourhood of the points  $b_j$ , we have

$$\left( \frac{g_k}{g_8} \right)' = \text{quadratic polynomial} \left( \frac{g_0}{g_8}, \dots, \frac{g_{17}}{g_8} \right).$$

This finishes the proof of proposition 2.3.

It follows from the previous proposition that at the bad points  $p_j, b_j$  the series  $(g_k/g_{10})(p_j), (g_k/g_8)(b_j)$  ( $0 \leq k \leq 17$ ) converges as a consequence of Picard's theorem applied to the system of ordinary differential equations  $(g_k/g_l)'$  ( $l=8$  or  $10$ ).

**PROPOSITION 2.4.** — *The two commuting vector fields (2.1) and (2.2) extend holomorphically and remain independent on T.*

*Proof.* — Let  $\varphi^{t_1}$  and  $\varphi^{t_2}$  be the flows generated respectively by vector fields (2.1) and (2.2) and consider a point  $p \in T \setminus A = \mathcal{D}$ . For  $\delta$  sufficiently small.

$$\varphi^{\tau_1}(p), \quad \forall \tau_1, \quad -\delta < \tau_1 < \delta$$

is well defined and  $\varphi^{\tau_1}(p) \in A$ . Then we may define  $\varphi^{t_2}$  on T by

$$\varphi^{t_2}(q) = \varphi^{-\tau_1} \varphi^{t_2} \varphi^{\tau_1}(q), \quad q \in \mathcal{V}(p) = \varphi^{-\tau_1}(\mathcal{V}(\varphi^{\tau_1}(p))),$$

where  $\mathcal{V}(p)$  is a neighbourhood of  $p$ . By commutativity one can see that  $\varphi^{t_2}$  is independent of  $t_1$ ,

$$\varphi^{-t_1 - \delta_1} \varphi^{t_2} \varphi^{t_1 + \delta_1}(q) = \varphi^{-t_1} \varphi^{-\varphi_1} \varphi^{t_2} \varphi^{t_1} \varphi^{\delta_1} = \varphi^{-t_1} \varphi^{t_2} \varphi^{t_1}(q)$$

We affirm that  $\varphi^{t_2}(q)$  is holomorphic away from  $\mathcal{D}$ . This because  $\varphi^{t_2} \varphi^{t_1}(q)$  is holomorphic away from  $\mathcal{D}$  and that  $\varphi$  is holomorphic in  $\mathcal{V}(p)$  and maps bi-holomorphically  $\mathcal{V}(p)$  onto  $\mathcal{V}(\varphi^1(p))$ . This completes the proof of proposition 2.4.

Since the flows  $\varphi^{t_1}$  and  $\varphi^{t_2}$  are holomorphic and independent on  $\mathcal{D}$ , we can show along the same lines as in the Arnold-Liouville theorem that  $T$  is a torus. And that will be done, by considering the holomorphic map

$$\Psi: \mathbb{C}^2 \rightarrow T: (t_1, t_2) \mapsto \Psi(t_1, t_2) = \varphi^{t_1} \varphi^{t_2}(p)$$

for a base point  $p \in A$ . Then

$$L = \{(t_1, t_2) \in \mathbb{C}^2 : \Psi(t_1, t_2) = p\}$$

is a lattice of  $\mathbb{C}^2$ , hence

$$\Psi: \mathbb{C}^2/L \rightarrow T$$

is a biholomorphic diffeomorphism. Therefore  $T \cong \mathbb{C}^2/L$  is conformal to a complex torus  $\mathbb{C}^2/L$  and an abelian surface as a consequence of Chow. Finally, we have the

**THEOREM 2.3.** — *T is an abelian surface on which the Hamiltonian flows (2.1) and (2.2) are straight lines motions.*

**PROPOSITION 2.5.** — *There are on T two holomorphic differentials  $dt_1$  and  $dt_2$  such that*

$$dt_1|_{\mathcal{D}_\varepsilon} = \omega_1 \quad \text{and} \quad dt_2|_{\mathcal{D}_\varepsilon} = \omega_2$$

where  $\omega_1$  and  $\omega_2$  and the two holomorphic differentials (2.14) on  $\mathcal{D}_\varepsilon$ .

*Proof.* — Let  $p \in \mathcal{D}_\varepsilon \cap \{\alpha u \neq 0\}$  where  $u$  is given by (2.11). Around the point  $p$ , we consider two coordinates on  $T$ ,

$$\tau = \frac{1}{m_3} = -\varepsilon t + O(t^2)$$

$$x = \begin{cases} x_1 = -i\beta + O(t) \text{ along } \mathcal{D}_{\varepsilon=i} \\ x_2 = i\beta + O(t) \text{ along } \mathcal{D}_{\varepsilon=-i} \end{cases}$$

We denote by  $\partial/\partial t_1$  (resp.  $\partial/\partial t_2$ ) the derivate according to the vector field (2.1) [resp. (2.2)]. Obviously we then have

$$dt_1 = \frac{1}{\Delta(\tau, x)} \left( \frac{\partial x}{\partial t_2} d\tau - \frac{\partial \tau}{\partial t_2} dx \right)$$

$$dt_2 = \frac{1}{\Delta(\tau, x)} \left( -\frac{\partial x}{\partial t_1} dt + \frac{\partial \tau}{\partial t_1} dx \right)$$

with

$$\Delta(\tau, x) \equiv \frac{\partial \tau}{\partial t_1} \cdot \frac{\partial x}{\partial t_2} - \frac{\partial \tau}{\partial t_2} \cdot \frac{\partial x}{\partial t_1}.$$

By direct computation using the asymptotic expansions, we find that

$$\begin{aligned} \frac{\partial \tau}{\partial t_1} &= -\varepsilon + O(t), & \frac{\partial \tau}{\partial t_2} &= -4\varepsilon(\alpha^2 - 1)\beta^2 + O(t) \\ \frac{\partial x}{\partial t_1} &= -2\alpha\beta^2 + O(t), & \frac{\partial x}{\partial t_2} &= 8\alpha((\alpha^2 - 1)\beta^4 - P(\beta)) + O(t) \end{aligned}$$

where  $P(\beta) \equiv c_1\beta^2 - 2\varepsilon c_2\beta - 1$ . From which one can deduce the two differentials  $dt_1$  and  $dt_2$ . The restrictions of  $dt_1$  and  $dt_2$  to the curve  $\mathcal{D}_\varepsilon$  are given by

$$\left. \begin{aligned} dt_1|_{\mathcal{D}_\varepsilon} &= k_1 \frac{(\alpha^2 - 1)\beta^2 d\beta}{\alpha u} \\ dt_2|_{\mathcal{D}_\varepsilon} &= k_2 \frac{d\beta}{\alpha u} \end{aligned} \right\} k_1, k_2 \in \mathbb{C}$$

and are the two holomorphic differentials  $\omega_1, \omega_2$  (2.14) on  $\mathcal{D}_\varepsilon$ . This completes the proof of proposition 2.5.

**PROPOSITION 2.6.** — *The vector field (2.1) [resp. (2.2)] is regular along  $\mathcal{D}$ , transversal to  $\mathcal{D}$  at every point  $\beta \neq 0$  (resp.  $\beta \neq \infty$ ) and (doubly) tangent at  $\beta = 0$  (resp.  $\beta = \infty$ ).*

*Proof.* — Using the same notation as in proposition 2.1, one can see that

$$\exists F_k^0, F_l^1: \det \begin{bmatrix} \frac{\partial}{\partial s} F_k^0(p_j) & \frac{\partial}{\partial s} F_l^0(p_j) \\ F_k^1(p_j) & F_l^1(p_j) \end{bmatrix} \neq 0,$$

and consequently the vector field (2.1) is transversal to  $\mathcal{D}$  at the 4 points  $p_j$  of  $\mathcal{D}_{\varepsilon=i} \cap \mathcal{D}_{\varepsilon=-i}$ . From proposition 2.5, the function

$$\frac{\omega_1}{\omega_2} = \frac{k_1}{k_2} (\alpha^2 - 1) \beta^2 \sim \frac{1}{\beta^2}$$

is meromorphic along a neighbourhood of  $b_j = (\alpha = \infty, \beta = 0)$  ( $1 \leq j \leq 4$ ) and provides the tangent to the curve  $\mathcal{D}$  in the coordinates  $t_1$  and  $t_2$ . The function  $\omega_1/\omega_2$  vanishes, whenever the vector field (2.2) is tangent to  $\mathcal{D}$  and has a pole whenever (2.1) is tangent to  $\mathcal{D}$ . Hence the zeroes  $b_j$  of  $\omega_2$  provide the 4 points of tangency of the vector field

(2.1) to  $\mathcal{D}$ . We find that

$$\forall \mathcal{F}_x^0, \mathcal{F}_i^1, \det \begin{bmatrix} \frac{\partial}{\partial \beta} \mathcal{F}_x^0(b_j) & \frac{\partial}{\partial \beta} \mathcal{F}_i^0(b_j) \\ \mathcal{F}_x^1(b_j) & \mathcal{F}_i^1(b_j) \end{bmatrix} = 0,$$

and consequently (2.1) is (doubly) tangent to  $\mathcal{D}$  at 4 points  $b_j$ , which concludes the proof of proposition 2.6.

PROPOSITION 2.7. — *The differentials  $\omega_1, \omega_2, f_j^0 \omega_2 (1 \leq j \leq 7)$  form a basis for the space  $\Omega(\mathcal{D})$  of holomorphic differentials on  $\mathcal{D}$ .*

*Proof.* — The adjunction formula gives us a map, the Poincaré residue map, between meromorphic 2-forms on  $T$  with a pole along  $\mathcal{D}$  and holomorphic 1-forms on  $\mathcal{D}$ . Applied to the 2-form  $\omega = f_j dt_1 \wedge dt_2$  with  $f_j \in \mathcal{L}(\mathcal{D})$ ,

$$\begin{aligned} \omega &= \frac{dt_1 \wedge dt_2}{(1/f_j)} \rightarrow \text{Res } \omega \Big|_{\mathcal{D}} = - \frac{dt_1}{(\partial/\partial t_2)(1/f_j)} \Big|_{\mathcal{D}} \\ &= \frac{dt_2}{(\partial/\partial t_1)(1/f_j)} \Big|_{\mathcal{D}} = \frac{dt_2}{(\partial/\partial t_1)(t_1/f_j^0) + O(t_1^2)} \Big|_{\mathcal{D}} \\ &= f_j^0 dt_2 \Big|_{\mathcal{D}} = f_j^0 \omega_2. \end{aligned}$$

Hence

$$\Omega(\mathcal{D}) = \{ \omega_1, \omega_2 \} \oplus \{ f_j^0 \omega_2, 1 \leq j \leq 7 \}$$

and this finishes the proof of proposition 2.7.

*Remark.* — It is interesting to observe that the embedding of  $\mathcal{D}$  into  $\mathbb{C}P^7$  is the canonical embedding,

$$\mathcal{D} \rightarrow \mathbb{C}P^7 : p \mapsto (\omega_2, f_1^0 \omega_2, \dots, f_7^0 \omega_2).$$

As we have seen, the involution  $\tau$  (1.4) has 8 fixed points on the affine variety  $A$ . In fact, it has 8 other fixed points at infinity given by the branch points of  $\mathcal{D}_\varepsilon$  on  $\mathcal{D}_\varepsilon^0$ . This is the object of the

PROPOSITION 2.8. — *The involution  $\tau$  (1.4) on the Abelian surface  $T$  coming from the one defined on the affine variety  $A$  has 8 fixed points at infinity.*

*Proof.* — This is easily proved as follows. From the asymptotic expansions (2.9), one can see that the functions  $m_1, m_2, \gamma_1, \gamma_2$  remain invariable by the transformation

$$(t, \alpha, \beta) \mapsto (-t, -\alpha, \beta)$$

whereas  $m_3, \gamma_3$  change into  $-m_3, -\gamma_3$ . Then the involution  $\tau$  (1.4) is transformed at infinity into an involution  $\sigma_\varepsilon$  (2.10) on  $\mathcal{D}_\varepsilon$ . Now the fixed points of  $\sigma_\varepsilon$  are given by the branch points of  $\mathcal{D}_\varepsilon$  on  $\mathcal{D}_\varepsilon^0$ . This concludes the proof of proposition 2.8.

D. PRYM VARIETY. — As is well known, if the period matrix of an abelian variety A is  $(\Delta, Z)$  with

$$\Delta = \begin{bmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{bmatrix}, \quad \delta_1 | \delta_2 | \dots | \delta_n \in \mathbb{N}, \quad Z \text{ symmetric, } \text{Im } Z > 0$$

then the period matrix of the dual abelian variety of A [i.e. the group  $\text{Pic}^0(A)$  of holomorphic line bundles on A with chern class zero] is  $(\delta_n \Delta^{-1}, \delta_n \Delta^{-1} Z \Delta^{-1})$ .

THEOREM 2.3. — *The abelian variety T is characterized as the dual Prym variety  $\text{Prym}(\mathcal{D}_\varepsilon/\mathcal{D}_\varepsilon^0)^*$  of the genus 3 curve  $\mathcal{D}_\varepsilon$  (2.13) which is a double cover of the elliptic curve  $\mathcal{D}_\varepsilon^0$  (2.11).*

*Proof.* — Let  $A_j, B_j (1 \leq j \leq 3)$  a basis of cycles of  $\mathcal{D}_\varepsilon$  (Fig. 2) such that:  $(A_j, A_k) = (B_j, B_k) = 0$ ,  $(A_j, B_k) = \delta_{jk}$ ,  $\sigma_\varepsilon(A_1) = A_3$ ,  $\sigma_\varepsilon(B_1) = B_3$ ,  $\sigma_\varepsilon(A_l) = -A_l$ ,  $\sigma_\varepsilon(B_l) = -B_l$ ,  $(1 \leq j, k \leq 3, l = 2, 3)$  for the involution  $\sigma_\varepsilon$  (2.10).

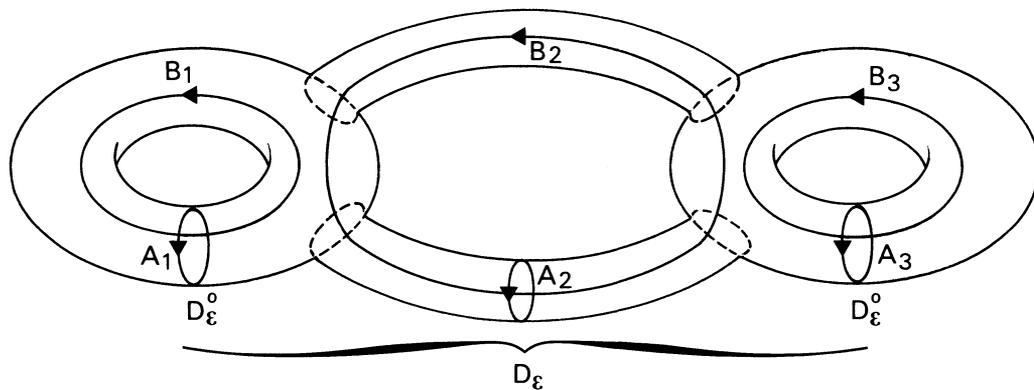


Fig. 2

From the double cover  $\varphi_\varepsilon$  (2.12), we can construct a subabelian variety of the Jacobi variety  $\text{Jac}(\mathcal{D}_\varepsilon)$  of  $\mathcal{D}_\varepsilon$ , called a Prym variety  $\text{Prym}(\mathcal{D}_\varepsilon/\mathcal{D}_\varepsilon^0)$ : the involution  $\sigma_\varepsilon$  on  $\mathcal{D}_\varepsilon$ , extends by linearity to a map  $\sigma_\varepsilon : \text{Jac}(\mathcal{D}_\varepsilon) \rightarrow \text{Jac}(\mathcal{D}_\varepsilon)$ . Up to points of order two,  $\text{Jac}(\mathcal{D}_\varepsilon)$  splits up into an even part  $\text{Jac}(\mathcal{D}_\varepsilon^0)$  and an odd part  $\text{Prym}(\mathcal{D}_\varepsilon/\mathcal{D}_\varepsilon^0)$ . The period matrix of this Prym variety is explicitly given by

$$\Omega = \begin{matrix} 2 \int_{A_1} \omega_1 & \int_{A_2} \omega_1 & 2 \int_{B_1} \omega_1 & \int_{B_2} \omega_1 \\ 2 \int_{A_1} \omega_2 & \int_{A_2} \omega_2 & 2 \int_{B_1} \omega_2 & \int_{B_2} \omega_2 \end{matrix}$$

where  $\omega_1$  and  $\omega_2$  are two holomorphic differentials on  $\mathcal{D}_\varepsilon$  (2.14). Let us call

$$U = \begin{bmatrix} 2 \int_{A_1} \omega_1 & \int_{A_2} \omega_1 \\ 2 \int_{A_1} \omega_2 & \int_{A_2} \omega_2 \end{bmatrix}, \quad V = \begin{bmatrix} 2 \int_{B_1} \omega_1 & \int_{B_2} \omega_1 \\ 2 \int_{B_1} \omega_2 & \int_{B_2} \omega_2 \end{bmatrix},$$

$$e'_1 = 2 \begin{bmatrix} \int_{A_1} \omega_1 \\ \int_{A_1} \omega_2 \end{bmatrix} \quad \text{and} \quad e'_2 = \begin{bmatrix} \int_{A_2} \omega_1 \\ \int_{A_2} \omega_2 \end{bmatrix}.$$

Observe that in the new basis  $e_1 = e'_1/2$ ,  $e_2 = e'_2$ , the period matrix  $\Omega$  takes the form

$$(2.16) \quad \Omega = (2\Delta^{-1}, 2\Delta^{-1}Z\Delta^{-1})$$

where

$$\Delta = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad Z = U^{-1}V\Delta \text{ symmetric and } \text{Im } Z > 0.$$

Consider now a basis  $dt_1, dt_2$  of holomorphic differentials on  $T$ , the map

$T \rightarrow \mathbb{C}^2/L_{\Omega^*} : p \mapsto \int_{p_0}^p \begin{pmatrix} dt_1 \\ dt_2 \end{pmatrix}$ , a basis  $a_j, b_j (j=1, 2)$  of  $H_1(T, \mathbb{Z})$ , the period matrix

$$\Omega^* = \begin{bmatrix} \int_{a_1} dt_1 & \int_{a_2} dt_1 & \int_{b_1} dt_1 & \int_{b_2} dt_1 \\ \int_{a_1} dt_2 & \int_{a_2} dt_2 & \int_{b_1} dt_2 & \int_{b_2} dt_2 \end{bmatrix}$$

and the lattice

$$L_{\Omega^*} = \left\{ \sum_1^2 l_j \int_{a_j} \begin{pmatrix} dt_1 \\ dt_2 \end{pmatrix} + n_j \int_{b_j} \begin{pmatrix} dt_1 \\ dt_2 \end{pmatrix} : l_j, n_j \in \mathbb{Z} \right\}$$

associated to  $\Omega^*$ . By the Lefschetz hyperplane theorem, the map  $H_1(\mathcal{D}_\varepsilon, \mathbb{Z}) \rightarrow H_1(T, \mathbb{Z})$  induced by the inclusion  $\mathcal{D}_\varepsilon \hookrightarrow T$  is surjective and consequently there are 4 cycles  $a_j, b_j (j=1, 2)$  on the curve  $\mathcal{D}_\varepsilon$  such that

$$\Omega^* = \begin{bmatrix} \int_{a_1} \omega_1 & \int_{a_2} \omega_1 & \int_{b_1} \omega_1 & \int_{b_2} \omega_1 \\ \int_{a_1} \omega_2 & \int_{a_2} \omega_2 & \int_{b_1} \omega_2 & \int_{b_2} \omega_2 \end{bmatrix}$$

and

$$L_{\Omega^*} = \left\{ \sum_1^2 l_j \int_{a_j} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + n_j \int_{b_j} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} : l_j, n_j \in \mathbb{Z} \right\}$$

where  $\omega_1 = dt_1|_{\mathcal{D}_\varepsilon}$ ,  $\omega_2 = dt_2|_{\mathcal{D}_\varepsilon}$  (proposition 2.5); hence the 4 cycles  $a_j, b_j (j=1, 2)$  in  $\mathcal{D}_\varepsilon$  which we look for are  $A_j, B_j (j=1, 2)$  and they generate  $H_1(T, \mathbb{Z})$  such that

$$\Omega^* = \begin{bmatrix} \int_{A_1} \omega_1 & \int_{A_2} \omega_1 & \int_{B_1} \omega_1 & \int_{B_2} \omega_1 \\ \int_{A_1} \omega_2 & \int_{A_2} \omega_2 & \int_{B_1} \omega_2 & \int_{B_2} \omega_2 \end{bmatrix}$$

is a Riemann matrix <sup>(1)</sup>. Since  $U = 2U^* \Delta^{-1}$  and  $V = 2V^* \Delta^{-1}$  we have  $2\Delta^{-1}U^{-1}V\Delta\Delta^{-1} = (U^*)^{-1}2V^*\Delta^{-1}$  and from (2.16), we deduce that  $\Omega^* = (\Delta, Z)$ . Consequently  $T$  and  $\text{Prym}(\mathcal{D}_\varepsilon/\mathcal{D}_\varepsilon^0)^*$  i.e. dual of  $\text{Prym}(\mathcal{D}_\varepsilon/\mathcal{D}_\varepsilon^0)$ , are two abelian varieties analytically isomorphic to the same complex torus  $\mathbb{C}^2/L_{\Omega^*}$ . By Chow's theorem,  $T$  and  $\text{Prym}(\mathcal{D}_\varepsilon/\mathcal{D}_\varepsilon^0)^*$  are then algebraically isomorphic. This completes the proof of theorem 2.3.

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<sup>(1)</sup> Otherwise there would exist a non trivial linear combination  $\sum l_j A_j + n_j B_j = \partial\rho$  with  $\rho$  a 2-chain on  $T$ . But  $dt_1$  and  $dt_2$  are closed differentials on  $T$ , so the Stokes' formula implies that

$$\sum_1^2 l_j \int_{A_j} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + n_j \int_{B_j} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \int_{\partial\rho} \begin{pmatrix} dt_1 \\ dt_2 \end{pmatrix} = 0.$$

This is a contradiction because the columns of  $\Omega^*$  must be linearly independent.

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