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On the characteristic polynomials of orbital varieties


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ON THE CHARACTERISTIC POLYNOMIALS
OF ORBITAL VARIETIES

BY ANTHONY JOSEPH

1. Introduction

1.1. This paper is a sequel to [19] and we shall retain the same notation though with
some modifications following ([7], [8]). Varieties and vector spaces will be nearly always
defined over the complex field C, and often considered as being over the real field R.

1.2. Let G be a complex connected simply connected Lie group with g its Lie algebra
and having the triangular decomposition g = n ⊕ h ⊕ n⁻. Set b = h ⊕ n and let B, N, H
be the subgroups of G corresponding to b, n, h. Identify g with g* through the Killing
form. Choose u ∈ g* which is ad-nilpotent and let O_u (or simply, O) denote the G orbit
in g* containing u. We call O a nilpotent orbit. We call a component C of O \cap n an
orbital variety.

1.3. Let L be a simple highest weight module. It is easy to show that the associated
variety \mathcal{V}(L) [resp. the vanishing cycle \mathcal{S}(L)] of L is a union (resp. sum) of orbital
varieties. Calculating this sum turns out to be a difficult task especially when one
discovers that \mathcal{V}(L) is not necessarily irreducible ([19], 10.1 and Tanisaki [30] or [20],
8.6-8.8).

1.4. A basic construction in [19] assigned to each orbital variety C a polynomial \( p_C \)
in S(h) called the characteristic polynomial of C. Let W denote the Weyl group for the
pair g, h. It was shown ([19], 3.1) that the \( p_C \), as C runs over the components of O \cap n,
span a W submodule of S(h). It was conjectured ([19], 3.3) that this is just the
representation of W assigned to O by the Springer correspondence. This would have
had the important consequence that the \( p_C \) would be linearly independent, as C runs
over all orbital varieties. Shortly afterwards this conjecture was proved by Hotta [14]
but only by a tedious comparison of matrix coefficients and detailed knowledge of the
Springer correspondence.

1.5. A factorization theorem which obtains from a basic result on Goldie rank
polynomials ([19], Sect. 5) relates the characteristic polynomial \( p_{\mathcal{S}(L)} \) assigned to \mathcal{S}(L)

\( p_{\mathcal{S}(L)} \) to the Goldie rank polynomial associated to Ann L. The latter can be computed (up
to a scalar) from the Jantzen matrix ([18], 5.1) which is of course now known to be
given by the Kazhdan-Lusztig polynomials.

1.6. In [19], 9.8, we conjectured that \( p_C \) can be computed from an analogue of the
Jantzen matrix introduced by Kazhdan and Lusztig ([22], Sect. 7) in their study of the
Springer correspondence. This would have implied Hotta’s result and indeed is practi-
cally equivalent to it. In any case combining Hotta’s result with a knowledge of the \( p_C \)
gives via the remarks in 1.5 an effective way of computing the associated varieties \( \nu'(L) \).

1.7. Following these ideas Kashiwara and Tanisaki [21], and independently Ginsburg [11]
showed that \( \nu'(L) \) can be computed by directly combining these two not
quite identical versions of the Jantzen matrix. Later Borho, Brylinski and MacPherson
([7], [8]) reproved Hotta’s result by using and developing a further interpretation of
characteristic polynomials which they called character polynomials of cone bundles
(associated to orbital varieties). Finally Rossmann [25] gave a particularly nice interpre-
tation of the (inverse) of the Kazhdan-Lusztig geometric analogue of the Jantzen matrix,
obtaining a new and simpler proof of the Springer correspondence. He also established
the conjecture for \( p^*_C \) discussed in 1.6 though here was constrained to use Hotta’s result.

1.8. The above mentioned results might be thought to imply a thorough understanding
of characteristic polynomials. Yet apart from Rossmann, the authors use a baggage of
heavy machinery from sheaf theory and consequently depart far from the elementary
approach which was a key point of [19]. Consequently one could well question, and as
we shall see with good reason, if a full understanding had been reached.

1.9. Identify the flag variety \( X := G/B \) with the set \( \mathcal{B} \) of all Borel subgroups of \( G \). Let
\( \mathcal{B}_u \) denote the variety of all Borel subgroups whose Lie algebra contains \( u \), equivalently
the fixed point variety for the action of one parameter group defined by \( u \). Let \( A_u \)
denote the component group of \( u \), namely the quotient of \( \text{Stab}_G u \) by its connected
component. Then \( A_u \) permutes the irreducible components of \( \mathcal{B}_u \) and after
Spaltenstein [28] the \( A_u \) orbits of components in \( \mathcal{B}_u \) are in bijection with the components
of \( O_u \cap \mathfrak{n} \). When [19] was being written we had also considered a dimension function \( q_Z \)
which could be assigned to a component of \( \mathcal{B}_u \) and indeed to any subset \( Z \) of \( X \). It
was easily seen that \( q_Z \) was independent of the choice of an irreducible component of \( \mathcal{B}_u \)
in an \( A_u \) orbit and moreover it appeared to often coincide with a \( p_C \). This was discussed
with Bernstein and with Borho during their visit to the Weizmann Institute (June
1982). Bernstein immediately observed that \( q_Z \) is asymptotically a polynomial (as a
consequence of coherent sheaf theory on the projective variety \( G/B \)). He further showed
that the top degree part \( \text{gr} \ q_Z \) of \( q_Z \) is the image of the fundamental class of \( Z \) in the
cohomology ring of \( \mathcal{B} \) identified via Borel’s construction [5] with the W harmonic
polynomials on \( \mathfrak{h}^* \) and hence coincides (Corollary 6.7) with the degree polynomial \( c_Z \)
of \( Z \). Via the very natural Hotta-Springer specialization theorem [15] it is then immedi-
ate that the \( \text{gr} \ q_Z : Z \) a component of \( \mathcal{B}_u \) span the Springer representation assigned to
\( O_u \) (and by a counting argument using [28] that the distinct \( q_Z \) form a basis).

1.10. An obvious and important goal is to interrelate the \( q_Z \) and the \( p_C \)
polynomials. One may interpret the Borho, Brylinski, MacPherson work ([7], [8]) as in
particular achieving this goal. However, these authors use a very roundabout analysis using on the one hand the deep consequences of the decomposition theorem for perverse sheaves giving in particular the irreducibility of Springer’s representation and on the other hand the rather messy arguments of [19] to relate these polynomials.

1.11. In this paper we prove a second factorization theorem which relates the $p_c$ to degree polynomials $c_\nu$ for the components of the $\mathcal{H}_\nu$. This uses the Berline-Vergne integration method [4] which can be understood with only a knowledge of elementary differential geometry. Their method was used in a particularly imaginative way by Rossmann [25] to relate Harish-Chandra’s character formula to the geometry of the flag variety obtaining notably an action of the Weyl group which recovers the Springer correspondence. For our convenience we use Rossman’s formulation of the integration formula. The main distinction with Rossmann’s work is that we have to make do with just one complex structure on $G/B$ (whereas Rossmann uses $G/B \times G/B$) and this leads to a less natural theory. (A similar cutting down was also necessary in the proof of the first factorization theorem ([19], Sect. 5 and [18], Sect. 4) and this was particularly unnatural.) We also need to interpret (Sect. 5) our characteristic polynomials as weighted Lelong numbers. This immediately gave a result which we had long thought to be true, namely that the characteristic polynomials (and hence the Goldie rank polynomials) are sums with rational coefficients $\geq 0$ of products of distinct positive roots. Actually (once one knows this to be true!) it turns out that a natural generalization of [19], 2.9, also gives this result using the original definition of characteristic polynomials (Sect. 8). Moreover the latter method is quite an effective computational tool in low rank. Again, it gives an important corollary concerning when certain completely prime ideals are induced (8.7). By a result of Vogan ([32], Prop. 7.12) this also gives information on the possible unitarity of representations of complex groups. Although this information is not new (for example, Enright lectured at the Weizmann Institute in June 1982 on a proof using the Dirac operator) the present analysis is particularly neat.

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2. Some basic constructions and notations

2.1. Let $R$ denote the set of non-zero roots for the pair $(g, h)$ and $R^+$ (resp. $S$) the set of positive (resp. simple) roots corresponding to the triangular decomposition defined
in 1.2. Set \( n = |R^+| = \dim_c n = \dim_c G/B \). Let \( P(R) \) [resp. \( P(R^+), P(R)^{++} \)] denote the set of integral (resp. dominant, dominant and regular) weights. Let \( \{ e_\alpha : \alpha \in \Phi \} \) be a Chevalley basis for the complex Lie algebra \( \mathfrak{g} \) and let \( z \mapsto \bar{z} \) denote complex conjugation with respect to that basis. Define a Cartan involution \( \theta \) for \( \mathfrak{g} \) by composing complex conjugation with the product of the Chevalley and principal antiautomorphisms. Set \( \mathfrak{t} = \{ x \in \mathfrak{g} \mid \theta(x) = x \}, \mathfrak{p} = \{ x \in \mathfrak{g} \mid \theta(x) = -x \} \), \( \mathfrak{t} = \mathfrak{t} \cap \mathfrak{h}, a = \mathfrak{p} \cap \mathfrak{h} \) which of course are real subspaces of \( \mathfrak{g} \). Let \( K, A, T \) be the real Lie subgroups of \( G \) corresponding to \( \mathfrak{t}, \mathfrak{a}, \mathfrak{t} \). Set \( t^+ = \{ x \in \mathfrak{t}^* \mid x(h) = 0, \forall h \in \mathfrak{t} \} \) which we view as a subspace of \( \mathfrak{t} \) via the Killing form on \( \mathfrak{t} \) (which is a real compact semisimple Lie algebra). Similarly \( b^+ \subset \mathfrak{g}^* \) identifies with \( n \).

2.2. Note that \( \{ ih_\alpha : \alpha \in \Phi \} \) and \( \{ i(e_\alpha + e_{-\alpha}), (e_\alpha - e_{-\alpha}) : \alpha \in \Phi \} \) form \( \mathbb{R} \) bases for \( t \) and \( t^+ \) respectively. Define a linear map \( (1+\theta) : n \to t^+ \) by \( (1+\theta)(z) = z + \theta(z) \).

Writing
\[
z = \sum_{\alpha \in \Phi^+} z_\alpha e_\alpha,
\]
we obtain
\[
(1+\theta)(z) = \sum_{\alpha \in \Phi^+} x_\alpha (e_\alpha - e_{-\alpha}) + iy_\alpha (e_\alpha + e_{-\alpha})
\]
where \( x_\alpha = \text{Re} z_\alpha, y_\alpha = \text{Im} z_\alpha \). Consequently \( 1 + \theta \) is bijective. It translates a complex subvariety of \( n \) into a real even dimensional subvariety of \( t^+ \) with no homological boundary. (Indeed the set of singular points is a closed subvariety of complex codimension \( \geq 1 \) and any smooth point is interior.)

2.3. Recall that the cotangent bundle \( T^*(X) \) of the flag variety \( X = G/B \) identifies with \( G \times \mathfrak{n} \). Since \( 1 + \theta \) is bijective and commutes with the action of \( T \) we obtain a map \( \Theta : K \times t^+ \to G \times \mathfrak{n} \) by setting \( \Theta(k, x) = (k, (1+\theta)^{-1}(x)) \). Via the Iwasawa decomposition \( G = KAN \) it follows that \( \Theta \) is bijective and gives a further realization of \( T^*(X) \) (due to Borel). Writing \( g = kan, x = (1+\theta)anz \in \mathfrak{n} \), we have \( \Theta^{-1}(g, z) = (k, x) \) and moreover if \( gz \in \mathfrak{n} \), then \( (1+\theta)(gz) = kx \). The map \( \pi : (g, z) \mapsto gz \) of \( G \times \mathfrak{n} \) onto \( G \mathfrak{n} \) [resp. \( \pi' : (k, x) \mapsto kx \) of \( K \times t^+ \) onto \( K t^+ \)] is called the moment map. The above calculation shows for every subset \( Y \subset K \times t^+ \) one has \( \pi'(Y) \subset t^+ \) if \( \pi(\Theta(Y)) \subset n \). Of course we may also identify \( G \times \mathfrak{n} \) (resp. \( K \times t^+ \)) with the subvariety \( \{ g \mathfrak{B}, \mathfrak{g} \mathfrak{n} \mid g \in G \} \) of \( G/B \times \mathfrak{g} \) (resp. \( \{ k \mathfrak{T}, k t^+ \mid k \in K \} \) of \( K/T \times \mathfrak{t}^* \)) and then \( \pi \) (resp. \( \pi' \)) is just projection onto the second factor so is a closed mapping as \( X \) is projective. However, whilst \( \pi \) is birational obviously \( \pi' \) is not. The latter will not cause difficulty here because we only apply \( \pi' \) to the images of conormals of \( B \) orbits.

2.4. Recall the Bruhat decomposition
\[
X = \coprod_{w \in W} X(w), \quad X(w) = BwB/B.
\]
Let $Y(w)$ denote the conormal corresponding to the $B$ orbit $X(w)$, that is

$$Y(w) : = \{ (bwB, b(n \cap wn) \mid b \in B \} \subset G/B \times g.$$ 

Let $O(w)$ denote the unique dense nilpotent orbit in the irreducible algebraic set $G(n \cap wn)$ and let $C(w)$ denote the Zariski closure of $\pi(Y(w)) \cap O(w)$ in $O(w)$. By the remark in 2.3 one has $\pi(Y(w)) = C(w)$ where the latter denotes the Zariski closure in $n$. Also $C(w) = \{ b(n \cap wn) \mid b \in B \}$ which is irreducible. The embedding $x \mapsto (x, 0)$ of $X(w)$ into $Y(w)$ carries $X(w)$ into $Y(w)$. In particular $(y, 0) \in Y(w)$ if and only if $y \leq w$ (Bruhat order). We remark that the map $w \mapsto O(w)$ of $W$ into the set of nilpotent $G$ orbits in $g^*$ was introduced by Steinberg [29]. It is known to be surjective ([29] Sect. 3); but except in $gl(n)$ and some other special cases its fibres are unknown. Again every orbital variety is some $C(w)$ ([19], Sect. 9); but it is not known which $C(w)$ are distinct.

2.5. Given $M$ a complex (or real) smooth irreducible algebraic variety, viewed as a $C^\infty$ manifold, let $F(M)$ denote the space of complex valued $C^\infty$ functions on $M$ and $D^1(M)$ the space of $C^\infty$ vector fields on $M$. Set $D_0(M) = F(M)$, $D_*^M = \bigoplus_{m=0}^{\infty} D^m(M)$, where for each $m = 1, 2, \ldots$, $D^m(M)$ denotes the space of exterior differential $m$ forms on $M$. Let $d : D^m(M) \to D^{m+1}(M)$ denote the usual exterior derivative and for each $x \in D^1(M)$ define the contraction map $c(x) : D^{m+1}(M) \to D^m(M)$ by $(c(x) \omega) (x_1 \wedge x_2 \ldots \wedge x_m) = \omega(x \wedge x_1 \wedge \ldots \wedge x_m)$. Then the $x$-equivariant derivative $d_x : D_*^M \to D_*^M$ is defined as $d_x = d + c(x)$ and satisfies $d_x^2 = L_x$ where $L_x$ is the Lie derivative. Consequently we have a cohomology theory on $x$ invariant forms, an important observation due to Berline and Vergne [4]. An element $\sigma \in D_2(M)$ is called symplectic if it is closed and non-degenerate (forcing $M$ to be even dimensional and in this respect all our varieties will be considered as being real). Given that $\sigma \in D_2(M)$ is closed we say that $x \in D^1(M)$ is Hamiltonian if there exists a function (called a Hamiltonian function) $f$ such that $c(x) \sigma + d_x f = 0$. Observe this implies that $d_x (\sigma + f) = 0$ and hence that $d_x (\sigma + f)^m = 0$, $\forall m = 1, 2, \ldots$, where the product is in the even part of $D_*^M$. Since this is a commutative subalgebra of $D_*^M$ we have omitted $\wedge$. Given $m \in M$ let $T_m(M)$ denote the tangent space of $M$ at $m$.

Given $f : M \to N$ a $C^\infty$ map of $C^\infty$ manifolds, let $\varphi_* : T_m(M) \to T_{\varphi(m)}(N)$ denote the Jacobian of $\varphi$ defined by $(\varphi_* x) (f) = x(f \circ \varphi)$ and $\varphi^*$ the pull-back $D_*^N \to D_*^M$ defined through $\langle \varphi^* (\omega) \mid x \rangle = \langle \omega \mid \varphi_* (x) \rangle$. One checks that $d$ commutes with the pull-back $\varphi^*$ and so if $\sigma \in D_*(N)$ is closed, then so is $\varphi^* (\sigma) \in D_*(M)$. Again take $y \in D^1(N)$ and suppose $x \in D^1(M)$ satisfies $\varphi_* (x) = y$, $\forall m \in M$ (for example if $y \in \text{Lie} H$, for an equivariant map $\varphi$ of $H$ manifolds) then $\varphi^* (c(y) \omega) = c(x) \varphi^* (\omega)$, for all $\omega \in D_*(N)$. Consequently, if $y$ is Hamiltonian relative to $\sigma \in D_2(N)$ with Hamiltonian function $f$, then $x$ is Hamiltonian relative to $\varphi^* (\sigma)$ with Hamiltonian function $f \circ \varphi$.

Finally, if $M$ is a homogeneous space, say $M = K/T$ then $D^1(M)$ is generated over $F(M)$ by an image of $\mathfrak{t} = \text{Lie} K$ (in this case $\mathfrak{t}$ itself).

2.6. Fix $i\lambda \in \mathfrak{t}^*$ regular [i.e. $(\lambda, \alpha) \neq 0$, $\forall \alpha \in \mathfrak{R}$] and extend $i\lambda$ to an element of $\mathfrak{t}^*$ via linearity and the condition $\lambda(t^i) = 0$. Following Borel [5] we define a $K$ invariant closed
2-form $\sigma_\lambda$ on $K/T$ through

$$\sigma_\lambda(x, y)(kT) = \langle [x, y], k\lambda \rangle$$

$\forall x, y \in \mathfrak{t}, k \in K$. Moreover $x \in \mathfrak{t}$ is Hamiltonian with respect to $\sigma_\lambda$ having Hamiltonian function $f^{(1)}_{\lambda, h}(kT) = \langle x, k\lambda \rangle$. The form $\sigma_\lambda$ is non-degenerate for $\lambda$ regular.

2.7. Consider the Cartesian co-ordinates $x_\alpha, y_\alpha : \alpha \in \mathbb{R}^+$ on $t^\perp$ defined in 2.2. Consider $h \in \mathfrak{t}$ as a vector field on $t^\perp$ (or on $\mathfrak{t}$). Noting the relation $[h, e_\alpha] = x(h)e_\alpha$ and recalling that $x(h)$ is pure imaginary for all $\alpha \in \mathbb{R}^+$ we obtain

$$h = \sum_{\alpha \in \mathbb{R}^+} \alpha(h) \left( z_{\alpha} \frac{\partial}{\partial z_{\alpha}} - \bar{z}_{\alpha} \frac{\partial}{\partial \bar{z}_{\alpha}} \right),$$

$$= \sum_{\alpha \in \mathbb{R}^+} i \alpha(h) \left( y_{\alpha} \frac{\partial}{\partial x_{\alpha}} - x_{\alpha} \frac{\partial}{\partial y_{\alpha}} \right).$$

Set $t_\alpha = \{ h \in \mathfrak{t} | i\alpha(h) \neq 0, \forall \alpha \in \mathbb{R} \}$. From now on fix $h \in t_\alpha$. Observe that

$$\tau_h := \sum_{\alpha \in \mathbb{R}^+} \left( \frac{1}{i\alpha(h)} \right) dx_{\alpha} \wedge dy_{\alpha}$$

is a $T$ invariant symplectic form on $t^\perp$ which pulls back via the decomposition $\mathfrak{t} = \mathfrak{t} \oplus t^\perp$ to a $T$-invariant closed two-form on $\mathfrak{t}$ which we shall also denote by $\tau_h$. Moreover, with respect to $\tau_h$ one checks that $h$ is a Hamiltonian vector field on $t^\perp$ (or on $\mathfrak{t}$) with Hamiltonian function

$$f^{(2)} = -\frac{1}{2} \sum_{\alpha \in \mathbb{R}^+} (x_{\alpha}^2 + y_{\alpha}^2).$$

2.8. Recall that we have an embedding $(k, x) \mapsto (kT, kx)$ of $K \times T t^\perp$ into $K/T \times \mathfrak{t}$. Under this map the closed two-form $\Sigma_{\lambda, h} := \sigma_\lambda + \tau_h$ defined on $K/T \times \mathfrak{t}$ by 2.6, 2.7 pulls back to a closed two-form on the cotangent bundle $T^* (X) = K \times T t^\perp$ and which we shall also denote by $\Sigma_{\lambda, h}$. Moreover with respect to the diagonal action of $K$ on $K/T \times \mathfrak{t}$ the above map is $K$ equivariant and so from 2.5 it easily follows that each $h \in \mathfrak{t}$ viewed as a vector field on $T^* (X)$ is Hamiltonian with respect to $\Sigma_{\lambda, h}$ and has Hamiltonian function

$$f_{\lambda, h}(k, a) = \langle h, k\lambda \rangle - \frac{1}{2} \sum_{\alpha \in \mathbb{R}^+} (x_{\alpha}^2 + y_{\alpha}^2)$$

where $x_\alpha, y_\alpha : \alpha \in \mathbb{R}^+$ denote the co-ordinates of the image of the point $ka \in \mathfrak{t}$ under the projection onto $t^\perp$ defined by the decomposition $\mathfrak{t} = \mathfrak{t} \oplus t^\perp$.

2.9. Compared to Rossmann ([25], I, Sect. 2) we seem to be forced into less structure here. Thus $\Sigma_{\lambda, h}$ is only $T$ invariant, we only claim the vector fields coming $t$ are Hamiltonian and that $\Sigma_{\lambda, h}$ is closed (but not necessarily symplectic).
3. Factorization via integration

3.1. Let \( Y(w) \subset T^*(X) \) or simply, \( Y \) be the conormal of a \( B \) orbit (2.4) viewed as a real subvariety of \( K \times T^\perp \). Then \( Y \) (or \( \bar{Y} \)) is \( 2n \) dimensional and has no boundary in homology. We wish to consider integrals of the form

\[
I_h^\gamma (h) := \frac{1}{n!} \int_{Y} e^{f_{h, k}(h, a) \Sigma_{k, h}} : \forall \lambda \in t^*, h \in t
\]

where \( f_{h, k} \) and \( \Sigma_{k, h} \) are defined by 2.8. Since \( K/T \) is compact the only possible non-convergence comes from integration over \( t^* \). However, here the convergence is ensured by the exponential decay implied by our choice of \( f^{(2)} \). We conclude that this (and similar integrals below) converge.

To evaluate this and similar integrals we use an integration lemma of Berline and Vergne [4] in the form proposed by Rossmann ([25], 3.1). For completeness we state the result needed below.

Let \( M \) be a real \( C^\infty \) manifold [eventually either \( T^*(X) \) or \( T^\perp \) viewed as real manifolds] and \( \Gamma \) an irreducible \( 2m \) dimensional real algebraic subvariety (eventually always coming from a complex algebraic variety) with no boundary in homology; but possibly with singularities. Fix \( h \in \mathscr{O}^1 (M) \), \( \omega \in \mathscr{O}_* (M) \), \( \varphi \in \mathscr{O}_1 (M) \). Assume

1. \( h \) has finitely many zeros on \( M \).
2. \( \Gamma \) is tangential to \( h \).
3. \( d_h \omega = 0 \).
4. \( \varphi \) is \( h \) invariant and \( \varphi(h) \) vanishes only on the zeros of \( h \).

Take any zero \( p \) of \( h \). Assume there exists a co-ordinate system \((x_j)\) on \( M \) around \( p \) such that

\[
\varphi(h) = \sum_j x_j^2 + o(\sum_j x_j^2)
\]

and

5. The ball \( B_r(p) \) of radius \( r > 0 \) around \( p \) is \( h \) invariant.

Condition (3) generally forces \( \omega \) to be inhomogeneous. Indeed it relates in particular the \( 2m \) component \( \omega_{2m} \) to its zero component \( \omega_0 \). Here (and elsewhere) we shall use

\[
\int_{\Gamma} \omega \to \text{mean} \int_{\Gamma} \omega_{2m}.
\]

Finally to ensure convergence we assume that \( M \) admits an increasing sequence of compact submanifolds \( M_s : s \in \mathbb{N} \) stable under \( h \) whose union is \( M \). [In \( T^*(X) \) take the inverse images under \( \pi \) of closed balls in \( t \) of radius \( s \) centered at the origin.] Writing \( \theta = \varphi/\varphi(h) \) we further assume that

\[
\lim_{s \to \infty} \int_{\varnothing(\Gamma \cap M_s)} \theta(d\theta)^l \omega = 0, \quad \forall l = 0, 1, 2, \ldots
\]
The integration lemma asserts the truth of the following formula

$$\int_{\Gamma} \omega = (-1)^n \sum_p \omega_0(p) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{2m}} \int_{\Gamma \cap B_\varepsilon(p)} (d\varphi)^n.$$

We sketch briefly the proof. Let $M'$ denote the complement in $M$ of the zeros of $h$. On $M'$ the one-form $\theta = \varphi/\varphi(h)$ is defined and satisfies $d_h \theta = 1 + d\theta$ by definition of $d_h$ and $d_h^2 \theta = 0$ by (4) and 2.5. Using (3) one checks the key identity $\omega = d_h(\theta (d_h \theta)^{-1} \omega)$ defined on $M'$. Set $\Gamma_s = \Gamma \cap M_s$ and choose $s$ sufficiently large to ensure that $B_\varepsilon(p) \subset M_s, \forall p$. Then by (1)

$$\int_{\Gamma_s} \omega = \lim_{\varepsilon \to 0} \sum_p \int_{\Gamma_s \setminus \Gamma_s \cap B_\varepsilon(p)} d_h(\theta (d_h \theta)^{-1} \omega)$$

$$= - \lim_{\varepsilon \to 0} \sum_p \int_{\delta (\Gamma_s \cap B_\varepsilon(p))} \theta (d_h \theta)^{-1} \omega + \int_{\partial \Gamma_s} \theta (d_h \theta)^{-1} \omega$$

by the equivariant Stokes' lemma ([4], 1.4) using that $\Gamma$ has no boundary in homology and (2), (6) to ensure that $B_\varepsilon(p) \cap \Gamma$ and $\Gamma_s$ are tangential to $h$ needed for the hypothesis of that lemma. Expanding $(d_h \theta)^{-1}$, using (5) to replace $\varphi(h)$ by $\varepsilon^2$ and applying Stokes' lemma we obtain

$$\int_{\delta (\Gamma \cap B_\varepsilon(p))} \theta (d_h \theta)^{-1} \omega = \sum_{j=0}^{m} \left( \frac{(-1)^{j-1}}{\varepsilon^{2j}} \left( \int_{\Gamma \cap B_\varepsilon(p)} (d\varphi)^j \omega_2(m-j) \right) \right),$$

up to terms that can be ignored in the limit $\varepsilon \to 0$. As noted in Rossmann ([25], Sect. 3) only the term with $j = m$ is non-vanishing in the limit [essentially because the volume of $\Gamma \cap B_\varepsilon(p)$ goes like $\varepsilon^{2m}$]. By (7) the integral over $\partial \Gamma_s$ vanishes in the limit $s \to \infty$. This proves the lemma.

3.2. Let $C$ be a (complex) quasi-affine irreducible subvariety of $\mathfrak{n}$ viewed as a real irreducible subvariety of $t^\perp$ (via 2.2) of dimension $2m : m = \dim_C C$. Assume that $C$ is $T$ invariant.

Fix $h \in t^\perp$, and consider the 1 form

$$\varphi_h := \sum_{a \in \mathbb{R}^n} \frac{1}{\alpha(h)} (x_a dy_a - y_a dx_a)$$

on $t^\perp$. From the formula (2.7) for $h$ viewed as a vector field we conclude that $\varphi_h$ is $h$ invariant and

$$\varphi_h(h) = \sum_{a \in \mathbb{R}^n} (x_a^2 + y_a^2).$$

For $\varepsilon > 0$, let $B_\varepsilon$ denote the closed ball in $t^\perp$ of radius $\varepsilon$ centered at the origin (with respect to the above Euclidean co-ordinates).
Viewed as a function on $t^r$, it is clear that $\varphi_h$ and hence $(d\varphi_h)^m$ are rational in $h$.

**Lemma.** — The integral

$$J_C(h) := \frac{1}{m!} \int_C \exp \left( -\frac{1}{2} \sum_{x \in \mathbb{R}^+} (x^2 + y^2) \right) \tau_h^m$$

converges and coincides with the rational function

$$\lim_{\varepsilon \to 0} \frac{(-1)^m}{\varepsilon^{2m}} \int_{C \cap B_{\varepsilon}} (d\varphi_h)^m.$$

on $t^r$.

The above integral is computed by the method described in 3.1. First we check conditions (1)-(7). Since $h$ has only a zero at the origin condition (1) holds. Since $C$ is $T$ invariant any $h \in t$ is tangential to $C$ so condition (2) holds. Let $\omega_h \in \mathcal{D}_*(t^r)$ be the form

$$\omega_h = \exp(f^{(2)} + \tau_h).$$

Then by 2.7 one has $d_\varepsilon \omega_h - 0$ so condition (3) holds. Condition (4) with $\varphi_h$ above have already been verified. In the coordinates $(x_a, y_a)$ condition (5) holds trivially and condition (6) is easily checked. Condition (7) is a trivial consequence of the exponential factor $\exp f^{(2)}$ in $\omega_h$, taking $M_\varepsilon$ to be the closed ball of radius $s$ with respect to the Euclidean metric defined by $x_\alpha, y_\alpha : \alpha \in \mathbb{R}^+$.

Since

$$J_C(h) = \int_C \omega_h$$

the conclusion of 3.1 gives for all $h \in t^r$,

$$J_C(h) = \lim_{\varepsilon \to 0} \frac{(-1)^m}{\varepsilon^{2m}} \int_{C \cap B_{\varepsilon}} (d\varphi_h)^m$$

as required.

3.3. Recall that the Euclidean volume $\text{Vol} B^m$ of the $2m$ ball $B^m_\varepsilon$ of radius $\varepsilon$ is just $(\varepsilon^{2m} \pi^m)/m!$. Thus had we been able to replace $t/\alpha(h)$ by $-(1/\pi)$, the rational function in the conclusion of 3.2 would take the form

$$\lim_{\varepsilon \to 0} \frac{\text{Vol}(C \cap B_{\varepsilon})}{\text{Vol} B^m_\varepsilon}$$

which is just the Lelong number of $C$ at the origin. It obviously vanishes if $0 \notin \overline{C}$ and takes the value 1 if $C$ is non-singular at 0. More generally it is determined by the multiplicity of an appropriate Hilbert-Samuel polynomial. (See 5.4.)
In section 4 we obtain an expression for the above rational function which may be viewed as a weighted Lelong number. For the moment we observe that up to the factor \((-i\pi)^n\) it may be written in the form

\[
\prod_{s \in K} \alpha(h) \left( \sum_{u \in K^+} \prod_{u} \alpha(h) a_u \right)
\]

where \(\prod_{u}\) is the product of the roots in \(R^+ \setminus U\) and

\[
a_u = \lim_{t \to 0} \frac{1}{Vol B_e} \int_{C \cap H + S_{e}} \prod_{u} (dx_u \wedge dy_u).
\]

Despite the similarity with our previous expression, \(a_u\) is not itself a Lelong number and in fact does not even take integer values (see Sect. 8). Again there does not seem to be any way of computing \(a_u\) from Hilbert-Samuel polynomials.

Nevertheless \(a_u \geq 0\), so we conclude that the numerator in (\(\star\)) is a positive sum of products of distinct positive roots.

3.4. Fix \(w \in W\) and set \(C^0(w) = \pi(Y(w)) \cap O(w)\). Since \(\pi(Y(w)) = \{ b | (n \cap w n) \} | b \in B\) and has the same closure as \(C(w)\) it follows from ([27], Chap. 1. 5, thm. 6) that \(C^0(w)\) contains an open subset of \(C(w)\). For each \(u \in C^0(w)\) set \(Z_u^0(w) = \pi^{-1}(u) \cap Y(w)\) which we may identify with a subvariety of \(X\) and hence of \(B\). Under the second identification one has \(Z_u^0(w) \subset B_u = \pi^{-1}(u)\). Let \(Z_u(w)\) denote the closure of \(Z_u^0(w)\) in \(B_u\). Let \((\text{Stab}_G)u_0\) denote the identity component of \(\text{Stab}_G u\) and set \(A_u = (\text{Stab}_G u)/(\text{Stab}_G u_0)\) which is a finite group acting on the set of irreducible components of \(B_u\). In what follows we shall sometimes omit \(w\) and we use a left superscript to denote conjugation of \(B\) by an element of \(G\).

**Lemma.** For all \(u \in C^0(w)\),

(i) \(\dim C Z_u^0(w) = \dim C B_u = \dim C X - (1/2) \dim C O(w)\).

(ii) \(Z_u(w)\) is a union of irreducible components of \(B_u\) belonging to a single \(A_u\) orbit.

There exists \(C^0(w) \subset C^0(w)\) Zariski open in \(C(w)\) such that

(iii) \(Z_u(w)\) is irreducible, for all \(u \in C^0(w)\).

(iv) The image of the fundamental class of \([Z_u(w)]\) of \(Z_u(w)\) in \(H_*(B, C)\) is independent of the choice of \(u \in C^0(w)\).

(i) One has (dropping \(C\)) that

\[
\dim B_u \geq \dim Z_u \geq \dim Y - \dim C, \quad \text{by [27], Chap. 1. 6, thm. 7}
\]

\[
= n - \dim C = n - \frac{1}{2} \dim O = \dim C B_u, \quad \text{by [28].}
\]

(ii) Observe that \(Y \subset G \times \tilde{B} C\) and so \(Z_u \subset G \times \tilde{B} C \cap \pi^{-1}(u)\). Yet

\[
G \times \tilde{B} C \cap \pi^{-1}(u) = \{ gB | g^{-1} u \in C \}
\]
which from the discussion in the fourth paragraph of [28] is exactly an \( A_u \) orbit of an irreducible component of \( \mathcal{B}_u \). By (i), this proves (ii).

For (iii) observe that
\[
Z^0_u = \{ B' \in \mathcal{B}_u \mid B' = bwB, \ b \in B \}.
\]

Following Steinberg we define
\[
S_u(O) = \{ (v, B, gwn) \mid v \in O \cap gB \cap gnw, \ g \in G \} \subset O \times \mathcal{B} \times \mathcal{B}.
\]

Consider the projection of \( S_u(O) \) onto \( O \) defined by the first factor. By [19], 9.3 (ii) the inverse image \( M \) of \( u \) is an \( A_u \) orbit (for the diagonal action) of an irreducible component of \( \mathcal{B}_u \times \mathcal{B}_u \). Take \( g \in G/B \) such that \( u \in g \cap n \cap wn \). Then the inverse image of \( B \) in \( M \) under the first projection equals \( \{ gwb \mid u \in g \cap wn \} = Z^0_{g^{-1}u} \). (Note here that \( g^{-1}u \in C^0 \).) If \( B \) belongs to only one component of \( \mathcal{B}_u \) which holds on an open set \( \Omega \) of \( \mathcal{B}_u \) then the asserted property of \( M \) implies that \( Z^0_{g^{-1}u} \) and hence \( Z^0_{g^{-1}u} \) is irreducible. It follows that \( Z_u \) is irreducible on the set obtained from \( \{ gb \in \Omega \mid g^{-1}u \in \cap wn \} \) by taking the inverse image of \( p_1 : G \rightarrow G/B \) and applying the quotient map \( g \mapsto g^{-1}u \). This is just \( p_2(p_1^{-1}(\Omega)) \cap C^0 \) and is open in \( C^0 \). Hence (iii).

(iv) Since \( \pi^{-1} = \pi^{-1}(gu) \) we have \( \pi^*(G \times_B C \cap \pi^{-1}(u)) = G \times_B C \cap \pi^{-1}(gu) \). It follows that \( g \) takes the irreducible components of \( G \times_B C \cap \pi^{-1}(u) \) (which are also irreducible components of \( \mathcal{B}_u \)) to the irreducible components of \( G \times_B C \cap \pi^{-1}(gu) \). Since \( G \) is connected this action does not change the images of their fundamental classes in \( H_*(\mathcal{B}, \mathbb{C}) \). Moreover since \( G \times_B C \cap \pi^{-1}(u) \) is a single \( A_u \) orbit of irreducible components, and of course \( A_u \) is a subquotient of \( G \), all these images are the same. Combined with (ii) and (iii) this proves (iv).

3.5. To apply 3.4 to the computation of \( I^\vee \_X(h) \) we first note that it is possible to interchange freely our two pictures concerning \( T^*(X) \) and the moment map. This is because \( \pi(Y) \subset n \) and we have a commutative diagram of maps
\[
\begin{array}{ccc}
Y & \xrightarrow{\pi} & C \\
\theta \uparrow & & \downarrow (1+\theta) \\
Y' & \xrightarrow{\pi'} & C'
\end{array}
\]
defined in 2.2, 2.3.

3.6. Let \( \mathcal{H} \) denote the set of all \( W \) harmonic polynomials on \( S(h) \). We recall that \( \mathcal{H} \) is spanned by homogeneous polynomials on \( h^* \) and that one has a direct sum decomposition \( S(h) = \mathcal{H} \oplus I_+ \), \( S(h) \) where \( I_+ \) denotes the augmentation ideal of \( S(h) \).

Let \( Z \) be a closed irreducible subvariety of \( K/T \). After Borel ([5] see also [25], II, Sect. 7) the integral
\[
c_Z(\lambda) := \int_Z \exp \sigma_\lambda
\]
depends only on the image of the fundamental class \([Z]\) of \(Z\) in \(H_* (\mathcal{B}, \mathbb{C})\) and the map

\[
[Z] \mapsto \left( \lambda \mapsto \int_Z \exp \sigma_\lambda \right)
\]

extends linearly to an isomorphism of \(H_* (\mathcal{B}, \mathbb{C})\) onto \(\mathcal{H}^*\).

Now fix \(w \in \mathcal{W}\). Set \(r = \dim_C X - (1/2) \dim_C O(w)\). From the above and 3.4 we obtain

**Corollary.** — The function

\[
\lambda \mapsto c_w (\lambda) : = \int_{Z_u (w)} \exp \sigma_\lambda = \frac{1}{r!} \int_{Z_u (w)} \sigma_\lambda^r
\]

is independent of the choice of \(u \in C^00 (w)\) and is a \(\mathcal{W}\) harmonic polynomial of degree \(r\).

3.7. We may now prove the main result of this section. Fix \(w \in \mathcal{W}\) and set \(Y = Y (w), C = C (w), Z_u = Z_u (w)\). Define \(I^k, J^k\) as in 3.1, 3.2. Set \(r = \dim_C Z_w\).

**Proposition.** — The integral \(I^k_Y (w)\) converges and as a function of \(\lambda\) it satisfies

\[
I^k_Y (w) (h) = c_w (\lambda) J_{C (w)} (h) + O (\lambda^{r+1}).
\]

One has

\[
I^k_Y (h) := \frac{1}{n!} \int_Y e^{f_{\lambda, h} (k, a)} \sigma_\lambda^k
\]

\[
= \sum_{s=0}^{n} \frac{1}{(n-s)! s!} \int_{\pi(Y)} \left[ \int_{Z_u} e^{f_{\lambda, h} (k, a)} \sigma_\lambda^s \right]^s \tau_h^{n-s}
\]

\[
= \sum_{s=0}^{n} \frac{1}{(n-s)! s!} \int_{\pi(Y)} \left[ e^{f^{(2)} (w)} \int_{Z_u} e^{f_{\lambda, h} (k, a)} \sigma_\lambda^s \right]^s \tau_h^{n-s}
\]

We claim that in the above sum only the term \(s = r\) contributes to order \(\leq r\) in \(\lambda\). Indeed for a term to be non-vanishing we must have \(n-s \leq n-r\) because \(n-r\) is half the real dimension of \(\pi(Y) = C^0\) and terms not satisfying this will make no contribution because of integration with respect to \(\tau_h\). To show that \(s \leq r\), it is enough to recall that \(\sigma_\lambda\) and \(f_{\lambda, h}^{(1)}\) are linear in \(\lambda\).

Finally the only dependence on \(\lambda\) comes from the integral over \(Z_w\). Hence we can take \(u \in C^00\) since the complement of \(C^00\) in \(\pi(Y) = C^0\) is of measure zero (because \(C^00\) is Zariski open in the irreducible algebraic set \(C^0\)). Since \(\exp f_{\lambda, h}^{(1)} (k, T) = \exp \langle h, k \lambda \rangle\) and \(\sigma_\lambda^s\) is homogeneous of degree \(r\) in \(\lambda\), expanding the exponential we conclude by 3.6 that

\[
I^k_Y (h) = c_w (\lambda) J_{\pi(Y)} (h) + O (\lambda^{r+1}).
\]

Since \(\pi(Y)\) and \(C\) differ only by sets of measure zero, the required conclusion is obtained.
3.8. The proof of the above factorization result follows closely the analysis of Rossmann ([25], Sect. II, 7), with some technical differences arising because here we are dealing with conormals of B orbits in X rather as in Rossmann with conormals of G orbits in X × X.

4. Comparison of factors

4.1 (Notation 3.7). We now compute \( \Pi^2_h(h) \) in another way. This will allow us to compare the factors \( c_{\sigma^s} J_{c(w)} \). A similar comparison technique occurred in the thesis work of D. King [23]. It was used in an essential way in the work of D. Barbasch and D. Vogan [3] in [19], Sect. 5, in [19], 5.1 and in Rossmann ([25], Sects II, 7-9). Our method of computation follows Rossmann using RL (notation 3.2).

4.2 First we construct a \( K \) invariant Riemannian metric \( \mathcal{G} \) on \( K \times_T t^\perp \) as the pull-back of a \( K \times K \) invariant Riemannian metric on \( K/T \times I \) under the \( K \) equivariant map defined in 2.8. The latter will just be the product of metrics defined on \( K/T \) and \( I \). The existence of such a metric is pretty obvious; but we shall describe it anyway.

Given \( \lambda_0 \in \mathfrak{t}^* \) a dominant integral weight, let \( V_{\lambda_0} \) denote the simple \( \mathfrak{g} \) module with highest weight vector \( v_{\lambda_0} \) of weight \( \lambda_0 \). The map \( g \mapsto [gv, \lambda_0] \) of \( G \) into \( \mathcal{P} V_{\lambda_0} \) factors to a map of \( G/B \) into \( \mathcal{P} V_{\lambda_0} \), which is an embedding if \( \lambda_0 \) is regular. Take \( \lambda_0 \) regular. The pull-back of the Fubini-Study metric on \( \mathcal{P} V_{\lambda_0} \) which is Riemannian and GL(\( V_{\lambda_0} \)) invariant ([12], pp. 30, 31) gives the required metric on \( K/T = G/B \). It is the metric which is the real part (see [12], p. 28) for the Kähler form on \( K/T \) for which \( \sigma_0 \) is the imaginary part. It is obviously non-degenerate at the base point \( T \) (for \( \lambda_0 \) regular).

On \( I \) we take the constant metric defined by the negative of the Killing form (which is negative definite on \( I \) and \( I \) invariant). The pull-back of the product metric (which is non-degenerate) is \( K \) invariant, positive and non-degenerate at any point \( p \in (T, t^\perp) \).

4.3. Fix \( h \in \mathfrak{t}_r \), considered as a vector field on \( T^*(X) \). Then \( (k, a) \in K \times_T t^\perp \) is a zero of \( h \) if and only if \( (ad k^{-1}) h \in \mathfrak{t} \) and \( [(ad k^{-1}) h, a] = 0 \). Since \( h \) is regular, so is \( (ad k^{-1}) h \) and thus the latter condition implies that \( a = 0 \). Again since \( h \) is regular one has \( K h \cap \mathfrak{t} = W h \) (where the groups act by the adjoint action). We conclude that the zeros of \( h \) on \( T^*(X) \) form the finite set \( \{(y, 0) \mid y \in W \} \). Observe also that \( f_{h, k}(k, a) \) takes the value \( \exp<h, y > \) at \( (y, 0) \).

4.4. Fix \( h \in \mathfrak{t}_r \). The function \( x \mapsto \mathcal{G}(h, h)(x) \) on \( K \times_T t^\perp \) is \( h \) invariant, takes positive values and vanishes exactly on the finitely many zeros of \( h \). Thus for \( \epsilon > 0 \) the set \( P_\epsilon := \{ x \in K \times_T t^\perp \mid \mathcal{G}(h, h)(x) < \epsilon^2 \} \) is \( h \) (even \( T \)) invariant and for \( \epsilon \) sufficiently small decomposes into a disjoint union of \( h \)-invariant open neighbourhoods \( B_\epsilon(y) : y \in W \) of the zeros of \( h \) in \( K \times_T t^\perp \) viewed as a real \( 4n \) dimensional \( C^\infty \) manifold. Take the \( h \) invariant partition of unity defined by these neighbourhoods and patch together an \( h \) invariant 1 form \( \varphi_h \) on \( T^*(X) \) by taking \( \varphi_h(\xi) = \mathcal{G}(h, \xi) \) outside \( P_\epsilon \) and on the above neighbourhood of \( (y, 0) \) an \( h \) invariant 1 form to be constructed later (4.6); but which
we assume for the moment satisfies

\[ \varphi_h(x) = \|x\|^2 + o(\|x\|^2) \]

in terms of local (Euclidean) co-ordinates around \((y, 0)\). We can of course assume that around \((y, 0)\) that

\[ \mathcal{G}(h, h)(x) = \|x\|^2 + o(\|x\|^2) \]

and so we could have taken \(\varphi_h(\xi) = \mathcal{G}(h, \xi)\) everywhere. However, this is not the most convenient choice.

4.5. **Proposition.** — For each \(w \in \mathbb{W}\) the integral \(I_Y(w)\) converges on \(\mathfrak{h}_r\) to the function

\[ h \mapsto \sum_{y \in \mathbb{W}} (-1)^n \exp \langle h, y \lambda \rangle \lim_{\epsilon \to 0} \int_{\epsilon^2 Y(w) \cap B_h(y)} (d\varphi_h)^n. \]

We apply 3.1 to the quadruple \((h, Y(w), \omega_h, \varphi_h)\) where \(h \in \mathfrak{t}_r, \omega_h^\lambda = \exp(f_{\lambda, h} + \Sigma_{\lambda, h}) \) and \(\varphi_h\) is defined in 4.4. Condition (1) of 3.1 was verified in 4.3. Condition (2) holds because \(Y(w)\) is \(\mathbb{T}\) invariant hence tangential to \(h\). Moreover, \(Y(w)\) has no homological boundary. Again \(d_h \omega_h^\lambda = 0\) by the construction of 2.6-2.8. This verifies condition (3). Condition (4) holds by construction. Condition (5) is just \((\star)\) of 4.4. Condition (6) follows from the \(h\) invariance of \(Y(w)\) and the \(B_h(y) : y \in \mathbb{W}\). Condition (7) follows from the exponential factor in \(\omega_h\) and the specific choice of \(\varphi_h\) outside \(P_e\). Since

\[ I_Y(w) = \int_{Y(w)} \omega_h^\lambda \]

the conclusion of 3.1 gives

\[ I_Y(w)(h) = \sum_{y \in \mathbb{W}} (-1)^n \exp \langle h, y \lambda \rangle \lim_{\epsilon \to 0} \int_{\epsilon^2 Y(w) \cap B_h(y)} (d\varphi_h)^n, \]

as required.

4.6. We now compute the limits occurring in 4.5. This follows Rossmann ([25], I, Sect. 3) with some slight differences coming from our use of conormals of \(B\) orbits. Using Rossmann's notation we let \(E_{u_k}(X(w))\) denote the Euler number of the variety \(X(w)\) at the point \(y\). This is defined as follows. As it is a local concept it is enough to consider an irreducible variety \(U\) of \(\mathbb{C}^n\) of complex dimension \(d\) and that the point in question to be the origin \(0\). Let \(V \subset (\mathbb{C}^n)^*\) be the corresponding conormal variety. Let \(q = (q_1, \ldots, q_d), p = (p_1, \ldots, p_d)\) denote co-ordinates in \(\mathbb{C}^n\) and \((\mathbb{C}^n)^*\) respectively, and set \(B_\epsilon = \{(q, p) \mid |q|^2 + |p|^2 \leq \epsilon^2\}\). Then

\[ E_{u_0}(U) : = (-1)^n - d \lim_{\epsilon \to 0} \int_{V \cap B_\epsilon} \left[ \frac{1}{2\pi i} \sum_{j=1}^{n} (d\tilde{q}_j \wedge dq_j - d\tilde{p}_j \wedge dp_j) \right]^n. \]
As in the case of the Lelong number (see 5.3) a limiting argument shows that $E u_0(U)$ may be expressed as an integral over the tangent cone to $V$ at $(0, 0)$ and is an integer, vanishes if $0 \notin \mathcal{U}$, equals 1 if $0 \in \mathcal{U}$ is a non-singular point. All this is discussed in Rossmann ([25], I, Appendix). We expect the $E u_0(X(w))$ to be non-negative. For the moment one knows that the inverse of the $A$ matrix (see 4.7) has entries $\geq 0$. This is because by [25], II, Section 12 and [21] these determine multiplicities of certain characteristic cycles. Then a Kazhdan-Lusztig type inversion property should relate these two sets of integers. In any case positivity is not needed here.

Let $l(w)$ denote the reduced length of $w \in W$. We have the

**Lemma.** — For all $y, w \in W$ one has

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{(Y(w) \cap B_\varepsilon(y))} (d\varphi_h)_y = \frac{(2\pi i)^n (-1)^{l(w)+l(y)}}{\prod_{a \in A^+} (\alpha, h)} E u_0(X(w)).
\]

Fix $y \in W$ such that $(y, 0) \in Y(w)$. Choose local (complex) co-ordinates $q_\alpha : \alpha \in -y^{-1} R^+$ around $y B \in G/B$ and extend these to local canonical co-ordinates $q_\alpha, p_\alpha : \alpha \in -y^{-1} R^+$ around $z_y : = (y, 0) \in T^*(G/B)$ exactly as in Rossmann ([25], I, Sect. 3). One has $(\exp h)(q_\alpha, p_\alpha) = (e^{\alpha(h)} q_\alpha, e^{-\alpha(h)} p_\alpha)$ and so $(\exp h)(\tilde{q}_\alpha, \tilde{p}_\alpha) = (e^{-\alpha(h)} \tilde{q}_\alpha, e^{\alpha(h)} \tilde{p}_\alpha)$ since $\alpha(h)$ is pure imaginary.

We should eventually like to apply ([25], I, Lemma 3.4). For this we must show that for any differentiable arc $(q(t), p(t))$ on $Y$ contained in $Y$ for $t > 0$ and passing through $z_y = (q(0), p(0)) = (0, 0)$ at $t = 0$ one has $p'(0). q'(0) = 0$ (where prime denotes differentiation and dot the obvious scalar product obtained by summing over $-y^{-1} R^+$).

Consider the canonical one-form

\[
\psi = p \cdot dq
\]

on $T^*(G/B)$. The one-form $\psi$ vanishes on any conormal, hence on $Y$. This translates to $p(t). q'(t) = 0$. Differentiating with respect to $t$ and setting $t = 0$ gives $p'(0). q'(0) = 0$ as required.

Now in the neighbourhood $B_\varepsilon(y)$ of $z_y$ in $T^*(X)$ defined in 4.4 we take

\[
\varphi_h = \sum_{\alpha \in -y^{-1} R^+} \frac{1}{\alpha(h)} (\tilde{q}_\alpha + p_\alpha)(dq_\alpha + dp_\alpha).
\]

(Here we recall that $h \in t_\gamma$.) Then on $Y$

\[
\varphi_h(h) = \sum (\tilde{q}_\alpha + p_\alpha) (q_\alpha + p_\alpha)
\]

\[
= \|q + p\|^2 = \|q\|^2 + \|p\|^2 + O (\|q\|^2 + \|p\|^2)
\]

by [25], I, Lemma 3.4, which we can now apply. Hence $\varphi_h$ satisfies (\heartsuit) of 4.4. From the above transformation properties of $q_\alpha, p_\alpha$ we also see that $\varphi_h$ is $h$ invariant. Thus the above formula for $\varphi_h$ can then be used in computing the left hand side of (\heartsuit\heartsuit). Since
$d\psi = \sum dp_\alpha \wedge dq_\alpha$ vanishes on $Y$ we have

$$(dp_\alpha)_n = \frac{(-1)^{(n)} + n}{\prod_{\alpha \in R^+} \alpha(h)} \left( \sum_{\alpha \in R^+} (dp_\alpha \wedge dq_\alpha - dp_\alpha \wedge dp_\alpha) \right)^n.$$ 

Recalling that $X(w)$ has complex dimension $l(w)$, substitution gives the required conclusion.

4. 7. Let us define the matrix $A$ with entries

$$A(w, y) = (-1)^{(w) - 1} \text{E}_{u_y}(X(w)).$$

Combining 4. 5 and 4. 6 we obtain

**COROLLARY.** — For each $w \in W$, one has

$$l_{Y (w)}(h) = \frac{-2 \pi i}{\prod_{\alpha \in R^+} \alpha(h)} \sum_{y \in W} A(w, y) e^{(h, y)}$$

on $i_e$.

4. 8. We need one further observation to apply this to 3. 7 to obtain the result announced in 4. 1. Namely that the Euler numbers $\text{E}_{u_y}(X(w))$ coincide with those computed by conormals of $G$ orbits in $X \times X$. This is because of the well-known relation between $B$ orbits in $X$ and $G$ orbits in $X \times X$. (It is also noted in [25], I, Sect. 4.) Now the involution $(x, y) \mapsto (y, x)$ on $X \times X$ carries the $G$ conormal defined by $w$ to the $G$ conormal defined by $w^{-1}$ and the point defined by $y$ to that defined by $y^{-1}$. Consequently $\text{E}_{u_{y^{-1}}}(X(w^{-1})) = \text{E}_{u_y}(X(w))$ and so $A(w, y) = A(w^{-1}, y^{-1})$. [We remark that a similar property holds for Verma module multiplicities for an essentially similar reason ([16], 5. 4).] Set $m = \dim_C C(w)$. We have already seen (3. 2) that $J_{C(w)}(h)$ can be written in the form

$$(\star) \quad J_{C(w)}(h) = \frac{-2 \pi i}{\prod_{\alpha \in R^+} \alpha(h)} p_{C(w)}(h)$$

for some homogenous polynomial $p_{C(w)}$ of degree $n - m$. Comparison of 3. 7 and 4. 7 shows (as in say [19], 5. 1) that

**THEOREM.** — For all $w \in W$ one has

$$p_{C(w)} = c_w^{-1}$$

up to a non-zero scalar.

4. 9. It follows from the work of Rossmann (explicitly [25], II, Cor. 5. 2, Sect. 10, eq. (9)) that $A(w, y)$ is exactly the entries of the matrix used in [19], Conj. 9. 8. It implies as in Rossmann ([25], II, Sect. 8) that $c_w$ and $p_{C(w)}$ are determined (up to scalars)
by the formula given in [19], Conj. 9.8. Of course we already know this for $c_w$ from Rossmann ([25], II, Thm. 8.2).

Finally, we remark that because of the irreducibility of Springer's representation, Schur's lemma implies that the scalar in the conclusion of the theorem depends only on $O(w)$. Here we must also use the fact that $C(w) = C(w')$ implies the equality $p_C(w) = p_C(w')$ (and not just their proportionality as in the case of the Goldie rank polynomials $q^\circ$ occurring in [18], Sect. 5. 5, Remark 1).

The fact that the above scalar depends only on the orbit $O(w)$ was also noted by Rossmann ([25], II, Sect. 10, eq. (13)) in an equivalent form. At first this may seem an innocuous fact hardly worthy of special note, in fact it is extremely curious and the demanding reader should really question its validity. The point is that the $c_w$ (or the $p_C(w)$) are overdetermined by the $A(w, y)$ coefficients (they have to coincide for several different $w$) and so this independence reflects a property of these coefficients. This was already true for the Goldie rank polynomials ([18], Sect. 5) and reflected a property of the entries $a(w, y)$ of the inverse Jantzen matrix. However, in the latter case the resulting polynomials $q^\circ$ which are directly determined by this matrix can be proportional and not equal. This fact forces the $A$ and $a$ matrices to be distinct [outside $\text{sl}(n)$]. For example in type $B_2$ there is one left-cell $\mathcal{C}$ of $W$ for which the $q^\circ : w \in \mathcal{C}$ (must be proportional but) differ by a factor of 2. Exactly here we get a distinction in the $A$ and $a$ matrices and which eventually leads to an extra nilpotent orbit not occurring as an associated variety of the integral fibre of primitive ideals ([17], Sect. 9).

5. Characteristic polynomials and weighted Lelong numbers

5.1. We now relate the rational function $J_c(h) : h \in \mathfrak{t}$, obtained from the conclusion of 3.2 to the rational function $r_c$ defined in [19], 2.4. Such a relation may already be anticipated from the discussion in 3.3.

5.2. Our strategy is to recompute $J_c(h)$ using again 3.1; but this time by first introducing new variables. Here we identify $t$, $t^*$ through the Killing form and make the substitution $h = -i \mu$ with $\mu$ a dominant integral regular weight. Thus $\mu(\alpha)$ is a positive integer for all $\alpha \in \mathbb{R}^+$. Now introduce new (complex) co-ordinate functions $\xi_\alpha : \alpha \in \mathbb{R}^+$ on $\mathfrak{n}$ by taking $\xi_\alpha$ to be an $\mu(\alpha)$ th root of $z_\alpha$. As we shall see we can apply 3.1 as previously.

5.3. Before going further let us recall the way to compute the Lelong number at a point $w$ in a complex irreducible affine algebraic variety $C$ viewed as real variety of twice its complex dimension.

Let $I(C)$ denote the ideal of definition of $\mathbb{C} \subset \mathbb{A}^n(C)$. Let $A(C)$ (or simply, $A$) denote the algebra of regular functions on $C$ which of course identifies with $\mathbb{C}[z_1, z_2, \ldots, z^n]/I(C)$. Given $w \in C$, let $m_w$ (or simply, $m$) denote the maximal ideal corresponding to $w$. For all $s \in \mathbb{N}$ set $\text{gr}^s m(A)_s = (m^s + I(C))/(m^{s+1} + I(C))$, where
\[ m^0 = A. \] We define

\[ \text{gr}^s(A) = \bigoplus_{s=0}^{\infty} \text{gr}^s(A), \]

which is a graded ring.

We can describe \( \text{gr}^s(A) \) in a second way which makes clear that it is finitely generated. Namely for each \( f \in I(C) \setminus \{0\} \) we let \( \text{gr} f \) denote the lowest degree term in the expansion of \( f \) as a homogeneous polynomial in the variables \( z_j - w_j : j = 1, 2, \ldots, n \) and set \( \text{gr} I(C) = \{ \text{gr} f | f \in I(C) \} \). Then \( C[z_1, z_2, \ldots, z^n]/\text{gr} I(C) \) is isomorphic as a graded ring to \( \text{gr}_m(A) \). This construction also makes it evident that \( \text{Spec} \text{gr}_m(A) \) is just the tangent cone \( T_m(C) \) at \( m \) ([27], p. 79).

Since \( \text{gr}_m(A) \) is finitely generated the function

\[ s \mapsto \dim_C \text{gr}_m(A), \]

is polynomial for all \( s \) sufficiently large (the Hilbert-Samuel polynomial, [2], 11.2). Furthermore this function takes the form

\[ f(s) = \frac{\epsilon s^{m-1}}{(m-1)!} + \text{lower order terms in } s, \]

where \( m = \dim_C C \) and \( \epsilon \) is a positive integer, depending on \( w \) and \( C \), called the multiplicity of \( f \). We denote it by \( \text{mult}_w(C) \) and call it the multiplicity of \( C \) at \( w \).

Now view \( C \) as a real \( 2m \) dimensional variety by setting \( x_j = \text{re} \ z_j, y_j = \text{im} \ z_j : j = 1, 2, \ldots, n \) and substituting for \( x_j, y_j \) in the expression for \( I(C) \). For each \( \epsilon > 0 \) let \( B_\epsilon(w) \) denote the closed \( (2n) \) ball centered at \( w \in C \) of radius \( \epsilon \) (with respect to the Euclidean norm on \( \mathbb{R}^{2n} \) defined by the above co-ordinates). Let \( B_\epsilon^m \) denote the closed \( 2m \) ball of radius \( \epsilon \) (centered at the origin). Set

\[ \tau = \sum_{j=1}^{n} dx_j \wedge dy_j. \]

Then \( (1/m!) \tau \) is the volume element for \( 2m \) dimensional analytic subvariety \( D \) of \( \mathbb{R}^{2n} \) and we may define

\[ \text{Vol} D = \frac{1}{m!} \int_D \tau = \int_D \exp \tau. \]

(For more details see [12], p. 32.)

Consider the function

\[ L(C, w, \epsilon) = \frac{\text{Vol}(C \cap B_\epsilon(w))}{\text{Vol} B_\epsilon^m}. \]
Theorem. — The function \( L(C, w, \varepsilon) \) takes positive values and is increasing in \( \varepsilon \). Furthermore
\[
\lim_{\varepsilon \to 0} L(C, w, \varepsilon) = \text{mult}_w(C).
\]

It is called the Lelong number of \( C \) at \( w \in \overline{C} \).

It is difficult to find a clear and simple proof of this remarkable result in the literature, let alone to find out to whom it is due. A rough proof occurs in [12], pp. 390-391. For the purposes of this section we only need the case when \( w = 0 \) and \( I(C) \) is already graded [equivalently with \( A(C) \) and \( T_0(C) \) are isomorphic as graded rings]. Then the limiting process is trivial and we may also view \( C \) as a projective variety of dimension \( m - 1 \). In this case the result can be read off from [24], 5.22 and 6.25. The limiting process is also considered in Thie ([31], Sect. 2, 3) though in more generality than needed here or in 4.6.

5.4. From now on we shall assume that \( w \) is the origin (which we can do without loss of generality except with respect to the remarks concerning whether \( A(C) \) is a graded ring). Introduce an \( n \)-tuple \( k = (k_1, k_2, \ldots, k_n) \) of positive integers. A monomial \( z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n} \) will be said to have \( k \)-degree \( s \) if \( \sum k_i l_i = s \). A polynomial will be said to be \( k \)-homogeneous if it is a sum of monomials of the same \( k \)-degree. An ideal \( I \subset C \langle z_1, z_2, \ldots, z_n \rangle \) will be said to be \( k \)-homogeneous if it is generated by \( k \)-homogeneous polynomials.

We make a brief digression to discuss a generalization of the Lelong number which does not work. For each \( s \) let \( m^{(s)} \) denote the ideal generated by the \( k \)-homogeneous polynomials of \( k \)-degree \( \geq s \). Exactly as in 5.3 we may introduce a \( k \)-graded ring by setting \( \text{gr}(A)_{(s)} = (m^{(s)} + I(C))/m^{(s+1)} + I(C) \) and
\[
\text{gr}(A) = \bigoplus_{s=0}^{\infty} \text{gr}(A)_{(s)}.
\]

Exactly as before \( \text{gr}(A) \) is finitely generated and we let \( \text{mult}_0^k(C) \) denote the multiplicity of the associated Hilbert-Samuel polynomial. Now set
\[
\tau_k = \sum_{j=1}^{n} \frac{1}{k_j} dx_j \wedge dy_j
\]
and define \( L(C, 0, \varepsilon, k) \) as before; but replacing \( \tau_1 \) by \( \tau_k \). One can show that \( L(C, 0, \varepsilon, k) \) satisfies the first conclusion of theorem 5.3 and one can ask if
\[
(\star) \quad \lim_{\varepsilon \to 0} L(C, 0, \varepsilon, k) = \text{mult}_0^k(C) ?
\]
Actually, this fails, already for the variety defined by the equation \( z_1^2 = z_2^3 \). In this case \( \text{mult}_0(C) = 2 \). Close to the origin \( C \) resembles 2 discs approaching the \( z_2 \)-plane \( z_1 = 0 \). It follows that only the integration over \( dx_2 \wedge dy_2 \) contributes and so we conclude that left hand side of \((\star)\) is just \( 2/k_2 \). This equals the right hand side if \( 2k_1 \leq 3k_2 \); but
otherwise the right hand side equals $3/k$. However we shall show that $(\star)$ holds if $I(C)$ is $k$-graded. In this case the limiting process is trivial; but otherwise it does not seem that this more general result is on obvious consequence of theorem 5.3 or can be proved in a similar way to that result.

5.5. Now return to the situation described in 5.2. Recall in particular that $C$ is a complex $T$ stable irreductible quasi-affine subvariety of $n$ and that $ih = \mu \in P(R)^{++}$.

**Proposition.** — The integral

$$J_K = \int_C \left[ \exp \frac{-1}{2} \sum_{\alpha \in R^+} (x^2 + y^2) \right] \tau^n$$

converges to $(-2\pi i)^n \text{mult}_K(C)$, where $k$ is the $n$-tuple defined by $k_\alpha = \mu(\alpha) : \alpha \in R^+$.

To compute the integral we substitute for the co-ordinates $z^\alpha$ discussed in 5.2. This must be done (as usual) in three places. First in $C$ which means that in the ideal of definition $I \subset C[z] = C[z_\alpha : \alpha \in R^+]$ of $C$ we must replace $z_\alpha$ by $\xi_\alpha$ to obtain a new ideal $I' \subset C[w]$. However, doing this leads to overcounting. Indeed for each point $w \in C$ we obtain

$$\prod_{\alpha \in R^+} \mu(\alpha)$$

solutions in the $\xi_\alpha$ variables (since $C$ is algebraically closed). This introduces the above factor as a denominator in the expression for $J_K$.

Second and third, we must introduce this substitution in the integrand and in $\tau_k$. However, instead we can simply recompute $h$ as a vector field in these new co-ordinates. A brief calculation gives

$$h = \sum_{\alpha \in R^+} \left( \frac{\xi_\alpha}{\partial z_\alpha} - \frac{\bar{\xi}_\alpha}{\partial \bar{z}_\alpha} \right)$$

$$= \sum_{\alpha \in R^+} \left( b_\alpha \frac{\partial}{\partial a_\alpha} - a_\alpha \frac{\partial}{\partial b_\alpha} \right)$$

where $\xi_\alpha = a_\alpha + ib_\alpha$. Now we can transport the above change of variables into the relation $d_a \omega_h = 0$ (with $\omega_h$ as in 3.2) without upsetting its validity. Now consider the 1-form

$$\varphi = i \sum_{\alpha \in R^+} (a_\alpha \partial b_\alpha - b_\alpha \partial a_\alpha)$$

which is obviously $h$ invariant. Furthermore

$$\varphi(h) = \sum_{\alpha \in R^+} (a_\alpha^2 + b_\alpha^2).$$

Let $B_\varepsilon$ denote the ball centered at the origin of radius $\varepsilon$ with respect to the co-ordinates $a_\alpha, b_\alpha$.  

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As in 3.2 we can now check that RL applies. By its conclusion we obtain

\[
\left( \prod_{z \in R^+} \mu(z) \right) J^\mu = \lim_{\varepsilon \to 0} \left( -1 \right)^m \int_{C \cap B} (d\varphi)^m.
\]

Since

\[d\varphi = 2i \sum_{z \in R^+} (da_n \wedge db_n) = 2i \tau_1 \]

we conclude from 5.3 that

\[J^\mu = \left( -2 \pi i \right)^m \frac{\text{mult}_0(C)}{\prod_{z \in R^+} \mu(z)} \]

where \(\text{mult}_0(C)\) is computed with respect to the \(\xi_a\) variables. It remains to show that

\[\frac{\text{mult}_0(C)}{\prod_{z \in R^+} \mu(z)} = \text{mult}_0^*(C)\]

where the right hand side is computed with respect to the \(z_a\) variables. Let \(p\) denote the denominator of the left hand side.

It is clear that \(C[w]\) is a free rank \(p\) module over \(C[z]\) with generators

\[g_i = \prod_{z \in R^+} \xi_{l_z} : l_z \in \{1, \ldots, k_z\}.
\]

Set \(A = C[z]/I, A' = C[\xi_z]/I'\). Then the image of the \(g_i\) generate \(A'\) over \(A\). If this were not a free generation, we should have

\[\sum_i g_i P_i \in I'
\]

for some \(P_i\) not all in \(I\). Given \(z \in C\) let \(\xi_{l_z} : l_z \in \{1, 2, \ldots, k_z\}\) denote the \(p\) possible co-ordinates in the \(\xi_z\) variables corresponding to \(z\). Now \(\det g_i(\xi_{l_z}) \neq 0\) by the linear independence of the characters of the appropriate product of cyclic groups. Then from (\(\S\)) we conclude that \(P_i(z) = 0, \forall i\) and since \(z\) was arbitrary \(P_i \in I, \forall i\) which is a contradiction. Hence, \(A'\) is a free rank \(p\) module over \(A\).

Since \(I(C)\) is \(k\)-homogeneous we may identify \(\text{gr} A_{(i)}\) with the \(k\)-homogeneous polynomials of \(k\)-degree \(s\) in the \(z_a\) variables and \(\text{gr} A'_{\xi}\) with the homogeneous polynomials of degree \(s\) in the \(\xi_a\) variables. Taking account of our first observation we conclude that

\[p \sum_{j=1}^{s+p} \dim \text{gr} A_{(j)} \geq \sum_{j=1}^{s} \dim \text{gr} A'_{\xi} \geq p \sum_{j=1}^{s} \dim \text{gr} A_{(j)}
\]

which implies (\(\S\)).
5.6. Comparison of 3.2 and 5.5 shows that the above analysis establishes (★) of 5.4 when \( I(C) \) is \( k \)-graded.

5.7. We now obtain the main result of this section.

**Theorem.** — For each orbital variety \( C \cap n \) the polynomial \( p_C \) defined by 4.8 (★) and 3.2 coincides with the characteristic polynomial of \( C \) defined in [19], 2.4.

From the definition of \( r_C \) in [19], 2.4 it is immediate that

\[
r_C(\mu) = m! \operatorname{mult}^k(C)
\]

where \( k \) is the \( n \)-tuple defined in 5.5. Recalling that the characteristic polynomial of \( C \) is just \( r_C \) times the product of the positive roots, the result obtains from 5.5 and substitution in 5.4 (★) using (as usual) that \( P(R)^{++} \) is Zariski dense in \( h^* \).

5.8. Combining 4.8, 5.7 and the remarks in 4.9 we obtain a proof of [19], Conj. 9.8. This implies Hotta's theorem [14] and avoids the use of any sheaf theory which one should always try to do.

### 6. Dimension polynomials

Section 6.1-6.6 are derived from very rough notes taken in discussions with Bernstein during June 1982. The first part 6.1-6.5 would probably be considered quite standard, whilst the second 6.6 is a little less so. The results are of computational interest and go beyond the considerations of fixed point varieties.

6.1. (Notation 2.1, 4.2). Let \( Z \) be a subset of the flag variety \( X \). Given \( \lambda \in \mathbb{P}(R)^+ \), we have a map \( g \mapsto g_{v_z} \) of \( G \) into \( V_z \). Set \( V_Z = \{ zv_z : z \in Z \} \) and \( q_Z(\lambda) = \dim_C C V_Z \). We call \( q_Z \) the dimension function of \( Z \). Here \( C V_Z \) denotes the \( C \)-linear span of \( V_Z \) which is obviously a bizarre thing to consider. Yet \( V_Z^\perp = \{ f \in V_z^* \mid f(a) = 0, \forall a \in V_Z \} \) is a subspace of \( V_z^* \) so we way anticipate one should consider \( V_z^*/V_z^\perp \).

6.2. Let \( \pi_0 : G \to G/B \) denote the natural projection. For each subvariety \( V \) of \( G \), let \( A(V) \) denote the ring of regular functions on \( V \). We view each \( \lambda \in \mathbb{P}(R) \) as a linear function on \( b \) by setting \( \lambda(n) = 0 \) and we let \( \chi_{\lambda} \) denote the character on \( B \) obtained from \( \lambda \) via the exponential map. We then have a \( G \)-equivariant sheaf \( \mathcal{O}(\lambda) \) on \( X \) defined by local sections through

\[
\Gamma(U, \mathcal{O}(\lambda)) = \{ f \in A(\pi_0^{-1}(U)) \mid f(gb) = \chi_{\lambda}(b) f(g), \forall b \in B, g \in G \}.
\]

The structure sheaf \( \mathcal{O}_X \) on \( X \) identifies with \( \mathcal{O}(0) \). The sheaf \( \mathcal{O}(\lambda) \) is a locally-free \( \mathcal{O}_X \) module of rank 1 and may be identified with a \( G \)-equivariant line bundle on \( X \).

Suppose \( \lambda \in \mathbb{P}(R)^+ \). Then each \( f \in V_z^* \) defines a global section on \( \mathcal{O}(\lambda) \) through the map \( g \mapsto \langle g v_z, f \rangle \) of \( G \) into \( C \). The Bott-Borel-Weil theorem [35], a very simple proof of which can be found in [9], asserts that the map \( V_z^* \to \Gamma(X, \mathcal{O}(\lambda)) \) so obtained is bijective.
For any subset $Z \subset X$ we have $q_Z = \dim V^*/V_Z^*$. It follows that $q_Z = q_Z^*$ and so we can assume that $Z$ is a closed subvariety of $X$ without loss of generality. This will be done in the sequel.

6.3. Let $Z \subset X$ be a closed subvariety. The ideal sheaf $\mathcal{I}_Z$ of $Z$ is defined by local sections through

$$\Gamma(U, \mathcal{I}_Z) = \{ f \in \Gamma(U, \mathcal{O}_X) \mid f(Z \cap U) = 0 \}.$$

Then the structure sheaf $\mathcal{O}_Z$ of $Z$ identifies with the quotient sheaf $\mathcal{O}_X/\mathcal{I}_Z$ and so is obviously a coherent $\mathcal{O}_X$ module.

For each $\lambda \in P(R)$ define the sheaf $\mathcal{O}_Z(\lambda) := \mathcal{O}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{O}_Z$ on $Z$. Right exactness of the tensor product gives a surjective map $\mathcal{O}(\lambda) \twoheadrightarrow \mathcal{O}_Z(\lambda)$.

Since $\mathcal{O}(\lambda)$ is ample for $\lambda \in P(R)^{++}$ and since the $m$ fold tensor product $\mathcal{O}(\lambda)^m$ of $\mathcal{O}(\lambda)$ over $\mathcal{O}_X$ identifies with $\mathcal{O}(m\lambda)$, we conclude from Serre’s theorem (concerning the Hilbert-Samuel polynomial [31], p. 15.3) applied to the projective variety $X$ that the map $\Gamma(X, \mathcal{O}(\lambda)) \to \Gamma(Z, \mathcal{O}_Z(\lambda))$ with kernel $\{ f \in \Gamma(X, \mathcal{O}(\lambda)) \mid f(Z) = 0 \}$ induced on global sections is surjective for $\lambda$ very dominant. This gives the

**Lemma.** — For $\lambda \in P(R)$ very dominant

$$\Gamma(Z, \mathcal{O}_Z(\lambda)) = \Gamma(X, \mathcal{O}(\lambda))/\{ f \in \Gamma(X, \mathcal{O}(\lambda)) \mid f(Z) = 0 \} \cong V^*_X/V_Z^*.$$

In particular $q_Z(\lambda) = \dim \Gamma(Z, \mathcal{O}_Z(\lambda))$ for $\lambda$ very dominant.

6.4. Let $K(X)$ [resp. $K_G(X)$] denote the Grothendieck group of locally free (resp. and $G$ equivariant) sheaves (or vector bundles) on $X$. Let $R(B)$ denote the Grothendieck group generated by the algebraic $B$ modules over $\mathbb{C}$. We have an isomorphism $ZP(R) \cong R(B)$ via the map $\lambda \mapsto \chi_\lambda$ and an isomorphism $R(B) \to K_G(X)$ via the map $F \to G \times_B F$. Also (for the flag variety $X$) the forgetful homomorphism $K_G(X) \to K(X)$ is surjective. (For more details on this, see [8], Sects. 2, 4.)

Finally, by [6], p. 106, every coherent sheaf on the projective variety $X$ admits a finite free resolution by locally-free sheaves of length $\leq n = \dim_X X$.

6.5. Let $\mathcal{F}$ be a coherent sheaf on $X$. Define the Euler characteristic $\chi(\mathcal{F})$ of $\mathcal{F}$ by

$$\chi(\mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \dim \mathcal{H}^i(X, \mathcal{F}).$$

Let $\rho$ denote the half sum of the positive roots and let $\Pi$ denote Weyl’s dimension function on $\mathfrak{h}^*$ namely

$$\Pi(\lambda) = \prod_{\alpha \in \mathfrak{h}^{++}} \frac{\langle \lambda, \alpha \rangle}{\langle \rho, \alpha \rangle},$$

which we recall is $W$ harmonic. It is immediate from the Bott-Borel-Weil theorem [9] that

$$\chi(\mathcal{O}(\lambda)) = \Pi(\lambda + \rho), \quad \forall \lambda \in P(R).$$

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Set $\mathcal{F}(\lambda) = \mathcal{O}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{F}$. Since $\mathcal{O}(\lambda)$ is locally free, the functor $\mathcal{F} \rightarrow \mathcal{F}(\lambda)$ is exact. So from a locally free resolution of $\mathcal{F}$ we obtain locally free resolution of $\mathcal{F}(\lambda)$. Let $[\mathcal{F}]$ denote the image of $\mathcal{F}$ in the Grothendieck group of coherent sheaves on $X$. By 6.4 there exists a finite set $I(\mathcal{F}) \subset P(R)$ and integers $n^\mathcal{F}_\mu$ such that

$$[\mathcal{F}] = \sum_{\mu \in I(\mathcal{F})} n^\mathcal{F}_\mu [\mathcal{O}(\mu)]$$

and by tensoring with $\mathcal{O}(\lambda)$ that

$$[\mathcal{F}(\lambda)] = \sum_{\mu \in I(\mathcal{F})} n^\mathcal{F}_\mu [\mathcal{O}(\mu + \lambda)], \quad \forall \lambda \in P(R).$$

Finally we remark that as in 6.3 one has $H^i(X, \mathcal{F}(\lambda)) = 0$, $\forall i > 0$ for $\lambda \in P(R)$ very dominant.

Given $Z$ as in 6.3 we use $n^Z_\mu$ [resp. $I(Z)$] to denote $n^\mathcal{F}_\mu$ [resp. $I(\mathcal{F})$] with $\mathcal{F} = \mathcal{O}_Z$. In view of 6.3 we obtain the

**Proposition.** — Let $Z \subset X$ be a closed subvariety. Then

$$q_Z(\lambda) = \sum_{\mu \in I(Z)} n^Z_\mu \prod (\mu + \lambda + \rho),$$

for all $\lambda \in P(R)$ very dominant. The latter extends to a $W$ harmonic polynomial on $Z$.

6.6. Recall that $K(X)$ inherits a ring structure by tensoring over $\mathcal{O}_X$ and that $H^*(X) = H^*(X, \mathbb{C})$ admits a ring structure via the cup product. After Borel [5] the first Chern class $c_1(\mathcal{O}(\lambda)) \in H^2(X)$ of $\mathcal{O}(\lambda)$ identifies with $\sigma_1$ and the map $\lambda \mapsto \sigma_1$ extends to ring homomorphism of $\mathfrak{S}(\mathfrak{h})$ onto $H^*(X)$ with kernel $I_+$ (notation, 3.6). Here the cup product is just the exterior or wedge product on exterior forms. Since there are no terms in odd dimension, $H^*(X)$ is commutative and we omit the wedge. We also recall that the isomorphism $\mathcal{H} = \mathfrak{S}(\mathfrak{h})/I_+ \cong H^*(X)$ which results, is an isomorphism of graded rings up to doubling of dimensions. Given $a \in H^*(X)$ non-zero, let $\text{gr } a$ denote its lowest degree component.

Let $\eta : H_*(X) \rightarrow H^*(X)$ denote the isomorphism induced by Poincare duality. If $Z$ is a closed subvariety in $X$ of complex dimension $m$ we denote by $[Z]$ its fundamental class in $H_{2m}(X)$. One has

$$\eta [Z_1 \cap Z_2] = \eta [Z_1] \eta [Z_2]$$

where $[Z_1 \cap Z_2]$ is viewed as an algebraic cycle (i.e. multiplicities are counted).

Let $\text{ch} : K(X) \rightarrow H^*(X)$ denote the functorial homomorphism defined by the Chern character. For the flag variety $\text{ch}$ is an isomorphism (see [8], 4.1 for example). The following result was pointed out to me by J. Bernstein.

**Theorem.** — Let $Z$ be a closed subvariety of $X$. Then

$$\text{gr } \text{ch } [\mathcal{O}_Z] = \eta [Z].$$
We first establish the assertion when $Z$ is irreducible and of complex codimension 1. Then $Z$ is given by a single homogeneous equation which for sufficiently dominant can be assumed to define a global section $f$ of $\mathcal{O}(\lambda)$. This gives an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}(\lambda) \rightarrow \mathcal{O}_Z \rightarrow 0.$$  

We conclude that $[\mathcal{O}] = [\mathcal{O}(\lambda)] - [\mathcal{O}_X]$ and so applying the Chern character map we obtain

$$\text{ch}[\mathcal{O}] = \text{ch} [\mathcal{O}(\lambda)] - \text{ch} [\mathcal{O}_X] = e^{\chi_1 (\mathcal{O}(\lambda))} - e^{\chi_1 (\mathcal{O}(0))},$$  

by say ([10], p. 56)

and so $\text{gr ch}[\mathcal{O}] = c_1 (\mathcal{O}(\lambda)) = \eta | (Z)$ by say ([10], Prop. 2, p. 141).

For the general case it suffices to show that

$$\text{gr ch}[\mathcal{O}_Z \cap Z] = \text{gr ch}[\mathcal{O}_{Z_1}] \text{ gr ch}[\mathcal{O}_{Z_2}]$$

for closed subvarieties $Z_1$, $Z_2$ of $X$ intersecting properly with $Z_1$ a divisor, as the result then obtains by induction on codimension and primary decomposition. (The result for $Z = X$ is trivial.)

Now recall that $X$ is smooth and assume that

$$m := \text{codim } (Z_1 \cap Z_2) = \text{codim } Z_1 + \text{codim } Z_2$$

(proper intersection) and that $\text{codim } Z_1 = 1$ ($Z_1$ is a divisor). Let $\text{Tor}^f_1 (\mathcal{F}, \ldots)$ denote the $i$th derived functor of the tensor product functor $\mathcal{F} \otimes_{\mathcal{O}_X} -$. By Krull's theorem ([27], Thm. 5, p. 58) we have the primary decomposition

$$[Z_1 \cap Z_2] = \sum_{j=1}^{s} n_j [V_j]$$

where the $V_j$ are irreducible, $\dim V_j = \dim (Z_1 \cap Z_2) = \dim Z_2 - 1$. Then by Serre ([26], p. 145-146, Prop. 1 (3) and remarque 2) and because $Z_1$ is a divisor, which implies the vanishing of the higher Tors, one has

$$\sum_{i=0}^{\infty} (-1)^i [\text{Tor}^i_1 (\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2})] = \sum_{j=1}^{s} n_j \mathcal{O}_{V_j}$$

Now take resolutions of $\mathcal{O}_{Z_i}$, $i = 1, 2$ by locally free sheaves $\mathcal{F}^i$. Then we can write

$$[\mathcal{O}_{Z_i}] = \sum_{j=1}^{n_i} [\mathcal{F}^i_j]$$

and we obtain by the bi-additivity of the alternating
sum of the \( \text{Tor}_i^X (\cdot, \cdot) \) that
\[
\sum_{i=0}^{\infty} (-1)^i [\text{Tor}_i^X (\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2})],
\]
\[
= \sum_{i, j, k} (-1)^i n_j^{(1)} n_k^{(2)} [\text{Tor}_i^X (\mathcal{F}_j, \mathcal{F}_k)],
\]
\[
= \sum_{j, k} n_j^{(1)} n_k^{(2)} [\mathcal{F}_j \otimes_{\mathcal{O}_X} \mathcal{F}_k].
\]

Applying the Chern character map this and (1) and (2) give
\[
\text{ch} [\mathcal{O}_{Z_1 \cap Z_2}] = \sum n_j^{(1)} n_k^{(2)} \text{ch} [\mathcal{F}_j] \text{ch} [\mathcal{F}_k]
\]
\[
= \text{ch} [\mathcal{O}_{Z_1}] \text{ch} [\mathcal{O}_{Z_2}].
\]

Applying \( \text{gr} \) proves the required assertion.

6.7. We can now relate the dimension \( q_Z \) and degree
\[
\lambda \mapsto c_Z (\lambda) = \int_Z \exp \sigma_\lambda
\]
polynomials for any subvariety \( Z \) of \( X \). Here \( Z \) can be assumed closed without loss of
generality. Both are \( W \) harmonic polynomials on \( \mathfrak{h}^* \). For any polynomial \( p \in S (\mathfrak{h}) \) let
\( \text{gr} p \) denotes its \textit{leading} homogeneous part.

\textbf{Corollary.} — \textit{Let \( Z \) be a subvariety of \( X \). Then \( c_Z = \text{gr} q_Z \).}

Set \( m = \dim_e Z \). Then
\[
c_Z = \frac{1}{m!} \int_Z \sigma_\lambda^n = \frac{1}{m!} \int_X \sigma_\lambda^n \eta [Z],
\]
by the definition of Poincaré duality. By 6.5 we can write
\[
[\mathcal{O}_Z] = \sum_{\mu \in I (Z)} n^Z_\mu [\mathcal{O} (\mu)]
\]
and then by 6.6,
\[
\eta [Z] = \text{gr} ( \sum_{\mu \in I (Z)} n^Z_\mu e^\mu),
\]
\[
= \sum_{\mu \in I (Z)} n^Z_\mu \sigma_\mu^{n-m} (n-m)!.
\]

Here we have noted that terms of degree \(< n-m \) must cancel. Now by [18], 2.3 (i) the
first non-vanishing term in the expansion
\[
\sum_{\mu \in I (Z)} n^Z_\mu e^\mu
\]
is always W harmonic. The remarks in 6. 6 concerning the map \( \mu \mapsto \sigma_\mu \) then imply that

\[
(2) \quad \sum_{\mu \in T(Z)} n^2_\mu \mu^l = 0, \quad \forall l < n - m.
\]

Now we may substitute the expression for \( \eta[Z] \) in (1) since higher order terms are eliminated by the definition of the integral. Given \( p \in S(\h) \), let \( p^\mathbb{g} \in \mathcal{H} \) denote its harmonic part defined by the direct sum decomposition in 3.6. Let (notation 6.5) \( \omega_X \) denote the image of \( \Pi \) in \( \mathbb{H}^* (X) \). Then identifying \( \mathfrak{h}, \mathfrak{h}^* \) through the Cartan inner product gives

\[
c_Z(\lambda) = \frac{1}{m! (n-m)!} \sum_{\mu \in T(Z)} n^2_\mu \int_X \sigma^m_\mu \sigma^{n-m}_\mu,
\]

and so

\[
(3) \quad c_Z(\lambda) = \frac{1}{m! (n-m)!} \left( \sum_{\mu \in T(Z)} n^2_\mu \lambda^m \mu^{n-m} \right) (p) \int_X \omega_X.
\]

[Here we have used that \( \Pi(p) = 1 \) to obtain the correct normalization.) Now for all \( \nu \in \mathfrak{h}^* \) one knows that

\[
(4) \quad \sum_{w \in W} (-1)^{l(w)} w \nu^\mathbb{g} \in \mathcal{H}.
\]

and so

\[
(5) \quad (\nu^\mathbb{g})^\mathbb{g}(p) = \frac{1}{|W|} \sum_{w \in W} (-1)^{l(w)} (w \nu, p) = (p^\mathbb{g})^\mathbb{g}(\nu).
\]

Taking \( \nu = p \) in (4) and use of the Weyl denominator formula gives that

\[
(6) \quad (p^\mathbb{g})^\mathbb{g} = \frac{n!}{|W|} \prod_{\alpha \in \mathbb{R}^+} \alpha.
\]

From 6.5 and (5), (6) we obtain

\[
q_Z(\lambda) = \frac{|W|}{n! \prod_{\alpha \in \mathbb{R}^+} (\alpha, p)} \left( \sum_{\mu \in T(Z)} n^2_\mu ((\mu + \lambda + p)^\mathbb{g}) (p) \right)
\]
and by (2) (using that $\mathcal{H}$ is graded, so $\operatorname{gr}$ and $b$ commute) we conclude that as a polynomial in $\lambda$

$$
\operatorname{gr} q_Z(\lambda) = \frac{|W|}{(n-m)! m!} \prod_{\alpha \in P^+} (\alpha, \rho) \left( \sum_{\mu \in I(\mathcal{Z})} n^2_\mu \lambda^m \mu^{n-m} \rho \right) \\
= \frac{|W|}{\prod_{\alpha \in P^+} (\alpha, \rho)} \int_X \omega_X \cdot c_Z(\lambda).
$$

Consider the special case when $Z$ is reduced to the base point $B \in G/B$. For any embedding $i_k : X \subset P V_k : \lambda \in P(R)^{++}$ one has $\operatorname{deg} B = 1$, whilst $q_B(\lambda) = 1, \forall \lambda \in P(R)^{++}$ trivially. We conclude that the scale factor in the above expression equals 1. This establishes the corollary.

6.8. Combined 4.8, 5.7 and 6.7 achieve the goal described in the introduction. Indeed fix $w \in W$ and let $p_{C(w)}$ denote the characteristic polynomial of the orbital variety $C(w)$. Choose $u \in C(w)$ such that $Z_u(w) := \pi^{-1}(u)$ is irreducible [which is possible by 3.4 (iii)] and let $q_{Z_u(w)}$ denote the dimension polynomial of $Z_u(w) \subset G/B$.

**Theorem.** — Up to non-zero scalar depending only on $O(w)$, one has

$$
p_{C(w)} = \operatorname{gr} q_{Z_u(w^{-1})}, \quad \forall w \in W.
$$

We remark that by 6.6, the right hand side above represents the image of $[Z_u(w^{-1})]$ in $H_{2m}(X) : m = \dim Z_u(w^{-1})$. Consequently by the Hotta-Springer specialization theorem [15] these polynomials span the Springer module $H_{2m}(\mathcal{A} \lambda_u)$ associated to the nilpotent orbit $O(w)$ containing $u$. Of course this is also true of the degree polynomial; but we had felt that the dimension polynomial is easier to visualize and to compute. In particular when $Z$ is a Schubert variety the dimension polynomial is given by the Demazure character formula [1]. However, as the referee was quick to point out, these degree polynomials can be deduced from ([34], 4.1) or by setting $Y = 0$ in [33], p. 51, proposition. Moreover, at least for the moment, these calculations are easier than the proof of the Demazure character formula. By Rossmann ([25], II, Thm. 8.2) the degree polynomials for the irreducible components can also be computed from the matrix $A$ with entries $A(w, y) : w, y \in W$ of Euler numbers. (For the moment the latter are not known completely.)

7. Comparison with the results of Borho, Brylinski, MacPherson ([7], [8])

7.1. Fix $w \in W$ and define $O(w), C(w)$ as before. Fix $u \in C(w)$ and set $D_u(w) = \pi^{-1}(u) \cap G \times_n C(w)$. One easily checks that $GD_u(w) = G \times_n C(w)$. The latter is referred to as an orbital cone bundle in [7], 2.1, and it is shown that the characteristic
classes of the distinct orbital cone bundles for \(O(w)\) form a basis for Springer's representation in \(H_{2m}(\mathcal{R}_w)\) : \(m = n - \dim C(w)\). We shall in particular obtain another proof of this assertion. The key observation is the following curious fact.

7.2. Lemma. — For all \(w \in W\) and all \(u \in C^0(w)\) one has \(A_u Z_u(w^{-1}) = \overline{D_u(w)}\).

Set \(E_u(w) = \{ g \in G \mid g^{-1} u \in B(\mathfrak{n} \cap \mathfrak{w} \mathfrak{n}) \}\). This is just the inverse image in \(G\) of the closed subvariety \(B(\mathfrak{n} \cap \mathfrak{w} \mathfrak{n}) = C(w) = C^0(w)\) under the map \(g \mapsto g^{-1} u\). Under the natural projection \(\pi_0 : G \to G/B\) one has \(\pi_0(E_u(w)) = \overline{D_u(w)}\). From Spaltenstein ([28], fourth paragraph) we further see that \(\overline{D_u(w)}\) is a single \(A^w\) orbit of components of \(\mathcal{R}_w\). By 3.4 (ii) this also holds for \(A_u Z_u(w^{-1})\) so it suffices to show that \(Z_u^0(w^{-1}) \subset D_u(w)\). In fact

\[
Z_u^0(w^{-1}) = \pi^{-1}(u) \cap Y^0(w^{-1})
= \{ bw^{-1} B \mid b^{-1} u \in \mathfrak{n} \cap w^{-1} \mathfrak{n} \}
= \{ bw^{-1} B \mid (bw^{-1})^{-1} u \in \mathfrak{n} \cap \mathfrak{w} \mathfrak{n} \}
\subseteq \pi_0(E_u(w)) = \overline{D_u(w)},
\]

as required.

7.3. Fix \(w \in W\) and let \(p_C\) denote the characteristic polynomial of \(C(w)\). Let \(c_D\) denote the degree polynomial of \(D_u(w)\).

Corollary. — For each orbital variety \(C\), one has \(c_D = p_C\), up to a non-zero scalar, depending only on \(O\).

This follows from 7.1, 4.8 and 5.7 if we recall that by 3.4 and 3.5 \(c_s = c_y\) if \(Z_u(x)\) and \(Z_u(y)\) lie in the same \(A_u\) orbit. This latter fact can also be seen using the dimensional polynomial and 6.7.

7.4. The above result can be read off from [8], 4.7 and [7], Theorem 3.2. It implies the result we wished to prove in 7.1. Although it would seem to be more natural than 4.8 and 5.7 where we are obliged to make explicit reference to \(w \in W\), this is not really so. Indeed the proof by Borho, Brylinski and MacPherson of 7.3 is extremely roundabout as we explained in the introduction. A main difficulty is that \(p_C\) is not obviously \(W\)-harmonic (and need not be if \(C\) is not orbital) whereas the degree polynomial \(c_D\) is always \(W\)-harmonic for any closed subvariety of \(G/B\).

8. A positivity lemma and some examples

8.1. It follows from 5.7 and 3.3 that the characteristic polynomial of an orbital variety \(C\) is a sum with positive coefficients of products of distinct positive roots. Moreover by [19], 5.2, this also holds for the Goldie rank polynomials, a fact which we had long since thought to be true; but which is completely unobvious from say [18], 5.1 even knowing explicitly the Jantzen matrix via the Kazhdan-Lusztig polynomials. [Similar remarks apply to the degree polynomials which by 4.8, 5.7 also have this property when \(Z\) is an irreducible component of a fixed point set \(\mathcal{R}_w\) (however it is not known if this
Here we give a second proof of this positivity property of $p_C$.

8.2. Let $C$ be any closed irreducible $H$ stable subvariety of $n$ and define its characteristic polynomial $p_C$ as in [19], 2.3. Take any $\alpha \in R^+$ and let $m_\alpha$ denote the subspace of $n$ generated by the $e_\beta : \beta \in R^+ \setminus \{\alpha\}$. Consider the intersection $C \cap m_\alpha$ as an algebraic cycle.

**Lemma:**

$$p_C \cap m_\alpha = \begin{cases} p_C & \text{if } C \subset m_\alpha, \\ \alpha p_C & \text{otherwise.} \end{cases}$$

This is proved exactly as in [19], 2.9; apart from the sign error occurring there!

8.3. Retain the above notation. Fix any ordering $\alpha_1, \alpha_2, \ldots, \alpha_n$ of the positive roots. Set $m_i = m_{\alpha_i}$. Let $L$ denote a chain $C = C_1, C_2, \ldots, C_n$ of $H$ stable closed irreducible subvarieties of $n$ by taking $C_{i+1}$ to be an irreducible component of $C_i \cap m_i$ and let $\mathcal{L}$ denote the set of all such chains. Let $n_{ij}$ denote the multiplicity of $C_{i+1}$ in $C_i \cap m_i$.

Set

$$\Delta^C_L = \left(\prod_{i=1}^{n-1} n_{ij}\right) \prod_{i \mid C_i = C_{i+1}} \alpha_i$$

which is a product of codim$_C C$ distinct positive roots with a positive integer coefficient. From the definition [19], 2.3 of $p_C$ it follows that $p(\alpha_0)$ is the product of the positive roots. This gives the

**Corollary.** — For each $H$ stable closed, irreducible subvariety $C$ of $n$ one has

$$p_C = \sum_{L \in \mathcal{L}} \Delta^C_L.$$  

In particular $p_C$ is the sum of products of distinct codim$_C C$ positive roots with non-negative integer coefficients.

Notice that the result cannot depend on the ordering of $R^+$ which is not a priori obvious but which is a familiar phenomenon in intersection theory.

8.4. Take $g = sl(n+1)$ with $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ the standard choice of simple roots. For all $i < j = \{1, 2, \ldots, n+1\}$ set

$$\alpha_{ij} = \sum_{k=i}^{j-1} \alpha_k \quad \text{and} \quad e_{ij} = e_{\alpha_{ij}}$$

Take $n = 3$ and consider the orbital variety $C$ with ideal of definition $I(C) = \langle e_{12}, e_{23}, e_{13}, e_{24}, e_{23}e_{14} \rangle$. Then $C$ is a complete intersection and by say
[8], 4.16 one has
\[ p_c = \alpha_1 \alpha_3 (\alpha_1 + 2 \alpha_2 + \alpha_3). \]

On the other hand, taking the ordering \{ \alpha_1, \alpha_3, \alpha_2, \ldots \} of \( R^+ \) we see that \( \mid \mathcal{A} \mid = 2 \) and correspondingly \( p_c \) is the sum of the two terms \( \alpha_1 \alpha_3 (\alpha_1 + \alpha_2), \alpha_1 \alpha_3 (\alpha_2 + \alpha_3) \). Again \( C \) can be viewed as a codimension 1 subvariety \( m_{\alpha_1} \cap m_{\alpha_3} \). Then the integrals occurring in 3.3(*) correspond to taking \( U \) a subset of cardinality 1 in \( \{ \alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 \} \). By the symmetry each of these integrals must be equal, whilst their sum is just \( \text{mult}_0 (C) = 2 \). Hence they are all equal to 1/2 and we obtain from 3.3(*) that
\[ p_c = \frac{1}{2} \alpha_1 \alpha_3 \{ \alpha_2 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_1 + \alpha_2 + \alpha_3 \} \]
\[ = \alpha_1 \alpha_3 (\alpha_1 + 2 \alpha_2 + \alpha_3) \]
as required.

Now take \( n=4 \) and consider the orbital variety \( C \) for which
\[ I(C) = \langle e_{12}, e_{34}, e_{45}, e_{35}, e_{13} e_{24} - e_{23} e_{14}, e_{14} e_{25} - e_{24} e_{15}, e_{13} e_{25} - e_{23} e_{15} \rangle \]
which is not a complete intersection (and so cannot be computed from [8], Prop. 4.15). Applying 8.3 we obtain
\[ p_c = \alpha_1 \alpha_3 \alpha_4 (\alpha_3 + \alpha_4) [ (\alpha_1 + \alpha_2) (\alpha_1 + 2 \alpha_2 + 2 \alpha_3 + \alpha_4) + (\alpha_2 + \alpha_3) (\alpha_2 + \alpha_3 + \alpha_4) ]. \]

Up to the scale factor of 1/12 this agrees as it should in \( \text{sl}(5) \) [and conjecturally in \( \text{sl}(n + 1) \)] with the Goldie rank polynomial occurring in the third line of [17], Thm. 11.4 (iii).

If we wish to apply 3.3(*) we have to compute 15 integrals which by symmetry occur in 3 sets for which each member of a given set has the same value. View \( C \) as a subvariety \( C' \) of \( m_{\alpha_1} \cap m_{\alpha_3} \cap m_{\alpha_4} \cap m_{\alpha_3 + \alpha_4} \). Then the 15 possible subsets \( U \) of \( R^+ \setminus \{ \alpha_1, \alpha_3, \alpha_4, \alpha_3 + \alpha_4 \} \) break up into a corresponding collection of three sets. These have, for example, members \( \{ \alpha_2, \alpha_1 + \alpha_3 \}, \{ \alpha_2, \alpha_1 + \alpha_2 + \alpha_3 \}, \{ \alpha_2, \alpha_2 + \alpha_3 \} \) respectively. If the corresponding integrals are denoted by \( I_1, I_2, I_3 \) then comparison of 3.3(*) with our previous expression for \( p_c \) gives \( I_1 = 0, I_2 = 1/6, I_3 = 1/3 \), that is for example
\[ \lim_{\epsilon \to 0} \frac{1}{\text{Vol} B_\epsilon^2} \int_{C \cap B_\epsilon} (dx_{a_2} \wedge dy_{a_2})(dx_{a_2 + a_3} \wedge dy_{a_2 + a_3}) = \frac{1}{3}. \]

However, we did not attempt to verify this directly.

8.5 One can ask if it is possible to verify directly that \( p_c \) as defined by 4.8(*) satisfies 8.2 by direct computation. Ironically, this is extremely difficult, though it is intuitively plausible. The trouble is that it fails in general. Namely if \( C \) is a complex irreducible locally closed algebraic subvariety \( \mathbb{A}^n(C) \) and \( P \) is a hyperplane, then the integral \( \int_{P \cap C} \) over the algebraic cycle \( P \cap C \) can be quite unrelated to \( J_c \) (consider the
case when \( C \) is defined by the equation \( z_1^2 = z_2^3 \) and intersect with the hyperplane \( z_2 = 0 \). One needs, for example, that \( C \) and \( P \) are stable under a non-trivial action of \( C^* \) (as in 8.2) so one can apply 5.5.

One may also ask if it is possible to show directly that \( p_C \) as defined by 4.8(*) satisfies [19], 2.6. Here we recall that this and [19], 2.9, led to the action of \( W \) on the space spanned by characteristic polynomials of orbital varieties.

In fact the analogue of [19], 2.6, can be proved by an analysis along the lines of Sect. 4 where \( G \) is replaced by the parabolic subgroup \( P_\alpha \supset B \) defined by \( \alpha \in S \). We leave the details to the reader; but remark that to make a precise comparison with [19], 2.6, one needs to check that the factor \( z \) occurring in [19], 2.6, is just the degree of the moment map \( (p, v) \mapsto v \) of \( P_x \times_B m_\alpha \) onto \( P_x m_\alpha \).

8.6. By 8.3 and [19], 5.2, every Goldie rank polynomial \( \tilde{p}_w \) can be expressed as a positive linear combination of products of distinct positive roots. Let \( \text{Supp} \tilde{p}_w \subset B \) denote the simple roots which occur in such an expression for \( \tilde{p}_w \). We show that this positivity has an important consequence for when a primitive ideal corresponding to a regular integral central character can be completely prime.

Let \( L(\lambda) : \lambda \in B^* \) denote the simple highest weight module with highest weight \( \lambda - \rho \) and set \( J(\lambda) = \text{Ann}_{U(g)} L(\lambda) \) which is primitive ideal. Now fix \( \lambda \) regular \( \{i.e. (\lambda, \alpha) \neq 0, \forall \alpha \in R\} \) and integral \( \{i.e. \lambda \in P(R)\} \). For each \( B' \subset B \), let \( w_{B'} \) denote the unique longest element of the Weyl group \( W_{B'} \) generated by the \( s_\alpha : \alpha \in B' \).

**Theorem.** — Assume that \( J(w \lambda) : w \in W \) is completely prime. Set \( B' = \text{Supp} \tilde{p}_w \). Then \( 2(\alpha, \lambda)/(\alpha, \alpha) = 1 \) for all \( \alpha \in B' \) and \( J(w \lambda) = J(w_{B'} \lambda) \).

Let \( P(R)^+ \) denote the dominant elements of \( P(R) \) and set

\[
\tilde{p}_w(\mu) = \text{rk} U(g)/J(w \mu) : \mu \in P(R)^+,
\]

where \( \text{rk} \) denotes Goldie rank. Now set \( n^w_\alpha := 2(\alpha, \mu)/(\alpha, \alpha) : \alpha \in B \) which is a non-negative integer for all \( \mu \in P(R)^+ \). The above positivity property of \( \tilde{p}_w \) implies that \( \tilde{p}_w(\mu) \leq \tilde{p}_w(\mu') \) if \( n^w_\alpha \leq n^{\mu'}_\alpha, \forall \alpha \in B \) with a strict inequality if \( n^w_\alpha < n^{\mu'}_\alpha \) for some \( \alpha \in B' \). We conclude that \( n^w_\alpha = 1 \), for all \( \alpha \in B' \) and that \( \tilde{p}_w(\mu) = 0 \) if \( n^w_\alpha = 0 \) for some \( \alpha \in B' \). Set \( R' = \mathbb{Z} B' \cap R, \ R'^+ = R' \cap R^+ \) and let \( p_{B'} \) denote the product of roots in \( R'^+ \). It is well-known that the second conclusion above implies that \( p_{B'} \) divides \( \tilde{p}_w \). (By the polynomial character of Goldie rank, the above vanishing corresponds to the Borho-Jantzen degeneration to the \( \alpha \)-wall and then from the reflection functor across the \( \alpha \)-wall we obtain \( s_\alpha \tilde{p}_w = -\tilde{p}_w \) for all \( \alpha \in B' \)—hence the required assertion.) Yet \( \tilde{p}_w \) is a \( W \) harmonic polynomial and \( p_{B'} \) is the highest degree \( W \) harmonic polynomial satisfying \( \text{Supp} p_{B'} = B' \). Hence \( \tilde{p}_w = p_{B'} \) up to a non-zero scalar. Again from [18], 5.1, and well-known properties of the Jantzen matrix (or Kazhdan-Lusztig polynomials) one has \( p_{B'}(\mu) = \text{rk} U(g)/J(w_{B'} \mu) : \mu \in P(R)^+ \) up to an overall non-zero scalar. By [18], 5.5, this implies the assertion of the theorem.

**Remarks.** — The second assertion of the theorem should extend to the non-integral (regular) case; but the arguments run into difficulties because one cannot translate in an arbitrary fashion to the walls. Indeed if we let \( B_\alpha \) denote the corresponding simple root
system, then it is false that the $2(\mu, \alpha)/(\alpha, \alpha): \alpha \in B_\mathcal{L}$ take arbitrary integer values on $\lambda + P(R)$. In particular the first assertion of the theorem is always false if $\text{Supp} \tilde{p}_w$ cannot be conjugated into $B$. The result fails miserably for $\lambda$ non-regular. The difficulty in the proof is that $\text{Supp} \tilde{p}_w$ contains not only those $\alpha \in B$ for which $\tilde{p}_w(\mu)=0$ when $n^\alpha_w=0$, but also those $\alpha \in B$ for which $n^\alpha_w=0$.

8.7. Retain the above notation. Then $L(w_B w_\mathcal{L} \lambda)$ is just the module induced from the one dimensional representation $C_{w_B w_\mathcal{L}} \lambda^{-\rho}$ of the parabolic subalgebra $p_B$: defined by $B':=-B_B B'$. We obtain the

COROLLARY. — Assume $\lambda \in P(R)^+$, If $J(w \lambda): w \in W$ is completely prime, then $J(w \lambda)$ is an induced ideal.

APPENDIX

Index of notation

Symbols appearing frequently are listed below in order of appearance (or where they are first defined).

1.1. $C, \mathbb{R}$.
1.2. $G, g, n, h, n^-, b, B, N, H$.
1.4. $W$.
1.9. $B, B_\mathcal{L}$.
2.1. $R, R^+, S, n, P(R), P(R)^+, P(R)^{++}, e_\alpha, h_\alpha, \theta, t, p, t, a, K, A, T, t^\perp, b^\perp$.
2.2. $z_\alpha, x_\alpha, y_\alpha$.
2.3. $X, T^*(X), \Theta, \pi, \pi'$.
2.4. $X(w), Y(w), O(w), C(w)$.
2.5. $\mathcal{D}_s, d, c(x), d_x$.
2.6. $\sigma_\alpha, f^{(1)}_{\lambda, x}$.
2.7. $\tau_{\lambda, \alpha} f^{(2)}$.
2.8. $\Sigma_{h, r} f_{h, r}$.
3.1. $I^*_y(h)$.
3.2. $J_\mathcal{L}(h), B_c$.
3.4. $C^0(w), Z^0(w), Z_\alpha(w), A_\mathcal{L}$.
3.6. $\mathcal{H}, I_{++}, c_Z, c_\mathcal{L}$.
4.1. $V_\lambda, v_\lambda$.
4.6. $E u_\mathcal{L}(Y)$.
4.7. $A(w, y)$.
4.8. $p_C$.
5.3. $\text{mult}_w(C)$.
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ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE