A. S. DE MEDEIROS

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SMOOTH SINGULAR SOLUTIONS OF HYPERPLANE FIELDS (II)

BY A. S. DE MEDEIROS (1)

0. Introduction

In this paper we extend to real analytic, and to C^∞ equations, the results established in [4], concerning the existence of singular solutions of a holomorphic total differential equation on $\mathbf{C}^n$, *i. e.* holomorphic solutions passing through a point where the coefficients of the equation vanish. Naturally, in the real case, we ask for singular solutions of the same class of differentiability of the equation.

Although the methods we make use of are basically the same developed in [4], we encounter, this time, problems of a more delicate quantitative nature (*see* e. g. Theorem 2.3.1) due, of course, to the greater complexity of the structure of real bilinear spaces compared to that of the complex ones. (Theorems 2.3.1 and 2.3.2, for example, could be stated in terms of multilinear algebra only.)

Our main result (Theorem A) describes precisely the maximal dimension of the singular solutions of an analytic, or a C^∞, total differential equation on $\mathbf{R}^{2m}$ passing through a point where its exterior differential has maximal rank.

In order to prove this theorem we have modified the proof of Theorem A of [4] so as to linearize the problem. For this reason we start, in paragraph 2, with the linear total differential equations. In that paragraph we define the characteristic frequencies, a finite sequence of real numbers associated to the given equation, which turns out to be an invariant of the equation determining the maximal dimension of the singular solutions stated in Theorem A.

We remark that we shall not include in this introduction a general precise description of the results for they are somehow technical. We provide, however, in the beginning of each section an explanation about the relevant facts contained there.

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The definitions and notation are introduced along the various sections and are printed in a different character. Most of them appear in paragraph 1 which consists of a résumé of the basic facts settled in [4].

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1. Preliminaries

We shall adopt the same notation and definitions presented in [4]. However, for sake of clarity, we make below a brief rewiew of these elements.

We denote by $\Lambda^1(\mathbf{R}^n)$ (resp. $\mathscr{X}(\mathbf{R}^n)$) the set of germs of analytic differential 1-forms (resp. analytic vector fields) on $\mathbf{R}^n$ vanishing at the origin. Given $\omega \in \Lambda^1(\mathbf{R}^n)$ we use indistinctly the expressions: *a singular solution of the total differential equation* $\omega=0$, a *singular integral manifold of* ω, and a *singular solution of the hyperplane field defined by* ω, to mean a germ of differentiable submanifold, at the origin of $\mathbf{R}^n$, such that ω pulls back to zero on it.

The set of linear differential 1-forms on $\mathbf{R}^n$ is denoted by $\Lambda_1^1(\mathbf{R}^n)$. If $\omega \in \Lambda_1^1(\mathbf{R}^n)$ then, the linear subspaces of $\mathbf{R}^n$ that are singular solutions of $\omega=0$ are called *isotropic subspaces of* ω.

Let $J_0^1(\omega)$ be the linear part of $\omega \in \Lambda^1(\mathbf{R}^n)$. We denote by $b_\omega(x,y)$ and $g_\omega(x)$ the bilinear form $J_0^1(\omega)(x).y$ and the quadratic form $J_0^1(\omega)(x).x$ respectively. A necessary condition for M to be a singular solution of $\omega=0$ is that its tangent space, $T_0 M$, at 0 be an isotropic subspace of b_ω (and consequently of g_ω).

The set of forms ω in $\Lambda^1(\mathbf{R}^n)$ such that $d\omega(0)$ has maximal rank is denoted by $\tilde{\Lambda}^1(\mathbf{R}^n)$. When n is even to each form $\omega \in \tilde{\Lambda}^1(\mathbf{R}^n)$ there corresponds a unique $X \in \mathscr{X}(\mathbf{R}^n)$ such that $\omega = i(X)\,d\omega$. We point out that $L_X \omega = \omega$. This vector field is denoted by $X(\omega)$.

We define $\operatorname{Spect}(\omega)$ to be $\operatorname{Spect}(J_0^1(X(\omega)))$. The elements of $\operatorname{Spect}(\omega)$ are called *eigenvalues of* ω.

Let $\theta_0 = \sum_1^m dx_i \wedge dy_i$ be the canonical symplectic form of $\mathbf{R}^{2m}$. We deduce from Darboux's theorem that any ω in $\tilde{\Lambda}^1(\mathbf{R}^{2m})$ can be put in the following *reduced form*

$$\omega = dH + \frac{1}{2} i(I)\,\theta_0$$

where $H(0)=0$, $I(x)=x$ is the radial vector field on $\mathbf{R}^{2m}$, and $i(I)\,\theta_0$ is the interior product of I and θ_0. It follows then that $X(\omega)$ can be written in the form

$$X(\omega) = X_H(\omega) + \frac{1}{2} I$$

where $X_H(\omega)$ is the hamiltonian vector field induced by the function H in the symplectic space $(\mathbf{R}^{2m}, \theta_0)$. Of course this implies that

$$\operatorname{Spect}(\omega) = \frac{1}{2} + \operatorname{Spec}(J_0^1(X_H(\omega)))$$

Finally we recall that $X \in \mathscr{X}(\mathbf{R}^n)$ is not *u-resonant* if there exists no relation of the type

$$\sum_{i=1}^{n} m_i \lambda_i = 1 \qquad \text{with} \quad m_i \in \mathbf{N} \quad \text{and} \quad \sum_{i=1}^{n} m_i > 2$$

among the eigenvalues λ_i of $J_0^1(X)$. This means that for any nonzero homogeneous differential 1-form of degree greater than one we necessarily have $L_A \omega \neq \omega$, where $A = J_0^1(X)$.

2. Linear forms

2.1. Aproximation by simpler forms. — Let $\tilde{\Lambda}_1^1(\mathbf{R}^n)$ denote the set of linear differential 1-forms ω such that $d\omega$ has maximal rank.

Roughly speaking, we describe in this section perturbations of $\omega \in \tilde{\Lambda}_1^1(\mathbf{R}^{2m})$ that have a very simple structure and that inherit, as far as isotropic subspaces are concerned, the relevant characters of ω itself.

The result we state below as the normal decomposition lemma is an immediate consequence of Williamson's theorem discussed in [1], and the reduced form of ω mentioned in the last paragraph.

Normal decomposition lemma. — Given $\omega \in \tilde{\Lambda}_1^1(\mathbf{R}^{2m})$ there exists a decomposition $\mathbf{R}^{2m} = \bigoplus_i E_i$ such that ω reduces, by a linear change of coordinates, to a direct sum of forms of the type $dH + (1/2)\, i(I)\, \theta_0$ on these subspaces. Moreover the functions H are in the normal forms listed in [1].

We recall, according as described in [1], that the subspaces E_i are invariant subspaces of $X_H(\omega)$ associated to a partition of its Jordan blocks into groups of the following six types:

- Type 1: A pair of blocks with real eigenvalues $\pm a$; $a \neq 0$.
- Type 2: A quadruple of blocks with complex eigenvalues $\pm a \pm bi$; $a, b \neq 0$.
- Type 3: A pair of blocks with eigenvalue zero.
- Type 4: One block of even order with eigenvalue zero.
- Type 5: A pair of blocks of odd order with purely imaginary eigenvalues $\pm bi$; $b \neq 0$.
- Type 6: A pair of block of even order with purely imaginary eigenvalues $\pm bi$; $b \neq 0$.

We shall denote by $\nu(\omega)$ the number of subspaces of type 5 in the normal decomposition of ω. (Note that 2ν is simply the number of blocks of odd order with purely imaginary eigenvalues in the normal Jordan form of $X_H(\omega)$.)

If $\nu \neq 0$ let $E_1, \ldots, E_\nu$ be such subspaces and $\pm b_1 i, \ldots, \pm b_\nu i$, $b_j > 0$, be the corresponding eigenvalues. For each $1 \leqslant k \leqslant \nu$ let $\varepsilon_k = \pm 1$ be the sign of the signature of $g_\omega \mid E_k$. (The signature of a quadratic form is the difference between the number of positive and negative squares in its diagonal form.) We define, following the current nomenclature of hamiltonians' theory, the *characteristic frequencies of* ω to be the finite sequence of real numbers $\alpha_k = \varepsilon_k b_k$; $k = 1, \ldots, \nu$. We shall suppose from now on that these frequencies are enumerated in nondecreasing order *i. e.* $\alpha_1 \leqslant \ldots \leqslant \alpha_\nu$.

If $\nu = 0$ we say that ω *is frequency free*.

Now we proceed to the description of the particular perturbations we shall make use of to ensure the existence of high dimensional singular solutions of $\omega = 0$.

It is well known that, generically, the eigenvalues of a hamiltonian vector field are all simple. Accordingly, the same conclusion holds for the eigenvalues of the differential 1-forms in $\tilde{\Lambda}_1^1(\mathbf{R}^{2m})$. Unfortunatly, this result, yet general, is not sharp enough for our purpose, once small perturbations in the frequencies have shown to cause abrupt lowering in the maximal dimensions of the isotropic subspaces.

In order to make precise the perturbations we need, we define $\omega \in \tilde{\Lambda}_1^1(\mathbf{R}^{2m})$ *to be simple if* $X_H(\omega)$ is diagonalizable and nonsingular (*i. e.* $\det(X_H(\omega)) \neq 0$)

By suitably perturbing ω on each subspace E_i of its normal decomposition it is not difficult to prove the following

PROPOSITION 2.1.1. — *Every* $\omega \in \tilde{\Lambda}_1^1(\mathbf{R}^{2m})$ *can be arbitrarily approached by simple forms having the same characteristic frequencies as* ω *itself.*

2.2. EXISTENCE OF HIGH DIMENSIONAL ISOTROPIC SUBSPACES. — Given $\omega \in \tilde{\Lambda}_1^1(\mathbf{R}^{2m})$ we shall consider the decomposition $\mathbf{R}^{2m} = \mathring{E}(\omega) \oplus E'(\omega)$, where $\mathring{E}$ (resp. E') is the sum of all subspaces, in the normal decomposition of ω, associated to eigenvalues with zero (resp. nonzero) real parts. Of course this yields a decomposition of ω, henceforth denoted by $\omega = \mathring{\omega} \oplus \omega'$.

If $\dim E' = 2m'$ it follows from Corollary 1.3.1 of [4] that ω' has an isotropic subspace of the maximal dimension m'. Consequently, ω has an isotropic subspace of dimension $m' + \rho$, where ρ is the index of Witt of $b_{\mathring{\omega}}$.

Now let $\omega \in \tilde{\Lambda}_1^1(\mathbf{R}^{2m})$ be given and let ω_k be any sequence of simple forms, like in Proposition 2.1.1, approaching ω. It is easily seen that, for sufficiently large k,

(i) $\dim E'(\omega_k) = 2(m - \nu(\omega))$

(ii) $\mathring{\omega}_k = \sum_{i=1}^{\nu} \alpha_i (x_i\, dx_i + y_i\, dy_i) + (1/2)\, i(\mathrm{I})\, \theta_0$, where $\alpha_1, \ldots, \alpha_\nu$ are the characteristic frequencies of ω and $(x_1, \ldots, x_\nu, y_1, \ldots, y_\nu)$ are the coordinates on $\mathring{E}(\omega_k)$.

In particular the index of Witt of $b_{\mathring{\omega}_k}$ depends only on ω and will be denoted by $\rho(\omega)$. In the case ω is frequency free we set $\rho(\omega) = 0$.

It follows from the above discussion that each ω_k has an isotropic subspace of dimension $m - \nu(\omega) + \rho(\omega)$. By taking on account the compactness of the Grassmann manifolds we have established the following

LEMMA 2.2.1. — *Let* $\omega \in \tilde{\Lambda}_1^1(\mathbf{R}^{2m})$. *Then, there exists an isotropic subspace of* ω *of dimension* $m - \nu(\omega) + \rho(\omega)$.

2.3. THE MAIN RESULT FOR LINEAR FORMS. — We proceed now to the characterization of $\rho(\omega)$ in terms of the characteristic frequencies of ω. This will be a consequence of the more general result stated in

THEOREM 2.3.1. — *Let* $(x_1, \ldots, x_\nu, y_1, \ldots, y_\nu)$ *denote the coordinates in* $\mathbf{R}^{2\nu}$ *and let* $\alpha_1 \leqslant \ldots \leqslant \alpha_\nu$ *be real numbers. Then, the linear differential* 1-*form*

$$\omega = \sum_{i=1}^{\nu} \alpha_i (x_i\, dx_i + y_i\, dy_i) + \frac{1}{2} i(\mathrm{I})\, \theta_0$$

has an r-dimensional isotropic subspace if, and oly if, the following inequalities hold

$(*)$ $\quad \alpha_k + \alpha_{r-k+1} \leqslant 0 \quad$ *and* $\quad \alpha_{\nu-k+1} + \alpha_{\nu-r+k} \geqslant 0,\;$ *for all* $k = 1, \ldots, r$.

Proof. — Let D be the $\nu \times \nu$ diagonal matrix $\operatorname{diag}(\alpha_1, \ldots, \alpha_\nu)$. If we consider in $\mathbf{R}^{2\nu}$ the complex structure determined by the operator $\mathrm{J} = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$, the matrix $\begin{pmatrix} \mathrm{D} & 0 \\ 0 & \mathrm{D} \end{pmatrix} + (1/2)\,\mathrm{J}$ of the bilinear form b_ω, represents the linear transformation M of $\mathbf{C}^\nu$ given by $\mathrm{M} = \mathrm{D} + (i/2)\,\mathrm{Id}$.

We point out that the sesquilinear form B on $\mathbf{C}^\nu$ represented by M is such that $b_\omega = \operatorname{Re} \mathrm{B}$.

The proof clearly follows from the lemmae below

LEMMA I. — *There exists an r-dimensional isotropic subspace for* ω *if, and only if,* D *is unitarily equivalent (in* $\mathbf{C}^\nu$*) to a (hermitian) matrix whose first* $r \times r$ *principal submatrix has only purely imaginary entries.*

For further references we shall report to such a matrix as an *r-imaginary matrix.*

LEMMA II. — *Let* $\alpha_1 \leqslant \ldots \leqslant \alpha_\nu$ *be real numbers. Then, the matrix* $\operatorname{diag}(\alpha_1, \ldots, \alpha_\nu)$ *is unitarily equivalent (in* $\mathbf{C}^\nu$*) to an r-imaginary matrix if, and only if, the inequalities* $(*)$ *are satisfied.*

Proof of Lemma I. — Suppose there exists a unitary matrix U such that $\mathrm{U}^* \mathrm{D} \mathrm{U}$ is r-imaginary. Under the linear change of coordinates U, the matrix of the sesquilinear form B will become $\mathrm{U}^* \mathrm{B} \mathrm{U} = \mathrm{U}^* \mathrm{D} \mathrm{U} + (i/2)\,\mathrm{Id}$ which is itself an r-imaginary matrix. Hence, the restriction of B to the real subspace spanned by the first r basic vectors has zero real part and since $b_\omega = \operatorname{Re} \mathrm{B}$ this subspace is an r-dimensional isotropic subspace.

Conversely, let E be an r-dimensional isotropic subspace of ω. Since E is isotropic for J there exists (*see* Theorem III.2.B of [3]) a ν-dimensional isotropic subspace $\bar{\mathrm{E}}$ of J such that $\mathrm{E} \subset \bar{\mathrm{E}}$. By extending an orthonormal basis of E to an orthonormal basis of

$\bar{E}$ and by adjoining to this set of vectors their images under J we obtain an orthonormal basis of $\mathbf{R}^{2\nu}$. The change of basis transformation T on $\mathbf{R}^{2\nu}$ represents a unitary transformation U on $\mathbf{C}^\nu$. In fact, T is a real orthogonal canonical transformation *i.e.* $T^t T = \mathrm{Id}$ and $T^t J T = J$. So that $JT = TJ$ which is the condition for representing a linear complex transformation U and, since T is orthogonal, this transformation is unitary.

Finally the matrix $U^* D U$ which corresponds, in $\mathbf{C}^\nu$, to the matrix $T^t \operatorname{diag}(D, D) T$ is r-imaginary for the latter has the form $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ and A has the first $r \times r$ principal submatrix equal to zero.

Proof of Lemma II. — First we observe that condition (*) holds if, and only if,

(**) there exist real numbers $b_1 \leqslant \ldots \leqslant b_r$ such that

$$b_k + b_{r-k+1} = 0 \quad \text{and} \quad \alpha_k \leqslant b_k \leqslant \alpha_{\nu-r+k}, \qquad \forall k = 1, \ldots, r$$

This fact is an elementary exercise in real numbers.

We start by proving the sufficiency.

Suppose that (**) holds. All we have to show is that there exists an r-imaginary matrix H such that $\operatorname{Spect}(H) = \{\alpha_1, \ldots, \alpha_\nu\}$.

If $r = \nu$ we have necessarily $\alpha_k = b_k$ and in this case we can define H to be the block diagonal matrix $H = \operatorname{diag}(A_k)$, where

$$A_k = \begin{cases} \begin{pmatrix} 0 & \alpha_k i \\ -\alpha_k i & 0 \end{pmatrix} & \text{if } 1 \leqslant k < \dfrac{\nu+1}{2} \\ \text{the zero } 1 \times 1 \text{ matrix if } k = (\nu+1)/2 \text{ is an integer.} \end{cases}$$

If $r < \nu$ we proceed by induction on ν.

It is easy to show that there exist real numbers $\mu_1, \ldots, \mu_{\nu-1}$ such that

(i) $\alpha_1 \leqslant \mu_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_{\nu-1} \leqslant \mu_{\nu-1} \leqslant \alpha_\nu$

(ii) $\mu_k \leqslant b_k \leqslant \mu_{(\nu-1)-r+k}$, $\forall k = 1, \ldots, r$.

By the induction hypothesis and (ii) above there is an r-imaginary matrix $\hat{H}$ of order $\nu - 1$ such that $\operatorname{Spect}(\hat{H}) = \{\mu_1, \ldots, \mu_{\nu-1}\}$. Let $\Delta = \operatorname{diag}(\mu_1, \ldots, \mu_{\nu-1})$, Theorem 4.3.10 of [6] asserts that, if condition (i) above is satisfied then, there exist $y \in \mathbf{R}^{\nu-1}$ and $a \in \mathbf{R}$ such that the matrix $H' = \begin{pmatrix} \Delta & y \\ y^t & a \end{pmatrix}$ has the eigenvalues $\alpha_1, \ldots, \alpha_\nu$. If $\hat{U}$ is a unitary transformation such that $\hat{U}^* \Delta \hat{U} = \hat{H}$ then, $U = \begin{pmatrix} \hat{U} & 0 \\ 0 & 1 \end{pmatrix}$ is unitary and $H = U^* H' U$ is r-imaginary. Since $\operatorname{Spect}(H) = \operatorname{Spect}(H')$ the sufficiency is established.

The converse is a direct consequence of Theorem 4.3.15 of [6] (that relates the eigenvalues of a hermitian matrix to those of a principal submatrix) and the fact that the eigenvalues of an $r \times r$ r-imaginary matrix occur in pairs $\pm\lambda$. This, of course, finishes the proofs of both Lemma II and the Theorem. ■

We finish this section with

THEOREM 2.3.2. — *Let $\omega \in \tilde{\Lambda}_1^1(\mathbf{R}^{2m})$ and let ν be the number of its characteristic frequencies. Then, the maximal dimension of the isotropic subspaces of ω is precisely $m-\nu+\rho$, where $\rho=0$ if $\nu=0$ and, otherwise, ρ is the greatest integer r $(1 \leqslant r \leqslant \nu)$ for which the characteristic frequencies $\alpha_1 \leqslant \ldots \leqslant \alpha_\nu$ satisfy the inequalities*

$$\alpha_k+\alpha_{r-k+1} \leqslant 0 \quad \text{and} \quad \alpha_{\nu-k+1}+\alpha_{\nu-r+k} \geqslant 0, \qquad \forall k=1, \ldots, r.$$

Before we prove the theorem we need the following

TECHNICAL LEMMA. – *Let $\mathbf{R}^{2m}=\mathbf{R}^{2m_1} \oplus \mathbf{R}^{2m_2}$ and let $\theta_0=\theta_0^1 \oplus \theta_0^2$, where θ_0^i $(i=1,2)$ denotes the canonical symplectic form on $\mathbf{R}^{2m_i}$. If $T=T_1 \oplus T_2: \mathbf{R}^l \rightarrow \mathbf{R}^{2m}$ is an injective linear transformation such that $T^* \theta_0=0$ then, rank $(T_1 \mid \operatorname{Ker}(T_2')) \geqslant l-m_2$, where T_2' is the first coordinate of T_2 relatively to the decomposition $\mathbf{R}^{2m_2}=\mathbf{R}^{m_2} \oplus \mathbf{R}^{m_2}$.*

Proof. – Let $l'=\operatorname{rank}(T_1 \mid \operatorname{Ker}(T_2'))$ and suppose that $l'<l-m_2$.

We denote by $(x_1, \ldots, x_{2m_1})$ and $(u_1, \ldots, u_{m_2}, v_1, \ldots, v_{m_2})$ the coordinates on $\mathbf{R}^{2m_1}$ and $\mathbf{R}^{2m_2}$ respectively.

Now, let $\operatorname{Im}^*(T)$ be the subspace of $(\mathbf{R}^l)^*$ generated by the coordinates of T. By successively extending a basis of $\operatorname{Im}^*(T_2')$ to basis of $\operatorname{Im}^*(T_1 \oplus T_2')$ and $\operatorname{Im}^*(T)$, we obtain a complete set of coordinates $(u_{i_1}, \ldots, u_{i_r}, x_{j_1}, \ldots, x_{j_s}, v_{k_1}, \ldots, v_{k_t})$ on $\mathbf{R}^l$ having the following properties:

(1) Ker (T_2') is the subspace $u_{i_1}=\ldots=u_{i_r}=0$.

(2) T_1 does not depend on $v_{k_1}, \ldots, v_{k_t}$.

(3) $l' \geqslant s$.

On the other hand, since $l'<l-m_2$ and $l=r+s+t$, we deduce from (3) that $r+t>m_2$. This implies, in particular, that $L=\{i_1, \ldots, i_r\} \cap \{k_1, \ldots, k_t\} \neq \varnothing$.

Let $T_2=T_2' \oplus T_2''$. By rearranging the coordinates on $\mathbf{R}^l$ we may assume that T_2' and T_2'' have the following forms

$$T_2'=(u_1, \ldots, u_{r'}, u_{r'+1}, \ldots, u_r, U_{r+1}, \ldots, U_\tau, U_{\tau+1}, \ldots, U_{m_2})$$

$$T_2''=(v_1, \ldots, v_{r'}, V_{r'+1}, \ldots, V_r, v_{r+1}, \ldots, v_\tau, V_{\tau+1}, \ldots, V_{m_2})$$

where $r'=\#(L)$, $\tau=r-r'+t$ and U_μ, V_ν are linear functions.

From the equation $T^* \theta_0=0$ we deduce $T_2^* \theta_0^2=-T_1^* \theta_0^1$ and, since T_1 does not depend on the v-coordinates, the expression in the differentials $du_i \wedge dv_i$ $(i=1, \ldots, r')$, in the expansion of $T_2^* \theta_0^2$, must vanish. Then, by taking on account the particular form of T_2, we conclude that $-\sum_1^{r'} du_i \wedge dv_i$ is precisely the term in the differentials $du_i \wedge dv_i$ $(i=1, \ldots, r')$ in the expansion of $\eta=\sum_{\tau+1}^{m_2} dU_k \wedge dV_k$. Hence, we must have $\operatorname{rank}(\eta) \geqslant 2r'$. On the other hand, from the very definition of η, we have $\operatorname{rank}(\eta) \leqslant 2(m_2-\tau)$ and then $m_2-\tau \geqslant r'$ which contradicts the fact that $l'<l-m_2$. ■

Proof of the Theorem. — In view of Lemma 2.2.1 and Theorem 2.3.1 it suffices to show that $m-\nu+\rho$ is an upper bound for the dimensions of the isotropic subspaces of ω.

We proceed by induction on m.

If $m=1$, either ω is frequency free or ω satisfies the hypothesis of Theorem 2.3.1 and, in both cases, the statement is trivially true.

Now suppose it is true for any $\hat{m}<m$.

We consider the decompositions $\mathbf{R}^{2m}=\mathring{E}\oplus E'$ and $\omega=\mathring{\omega}\oplus\omega'$ (*see* Section 2.1) and we set $\dim E'=2m'$.

Let $T:\mathbf{R}^l\to\mathbf{R}^{2m}$ be a linear parametrization of an l-dimensional isotropic subspace of ω. We shall distinguish the following two cases:

(i) $m'\neq 0$.

Since, by Lemma 2.2.1, there exists an isotropic subspace of ω' of dimension m', we can choose symplectic coordinates $u_1, \ldots, u_{2m'}$, on E' so that $u_1=\ldots=u_{m'}=0$ is such a subspace.

We write $T=\mathring{T}\oplus T'$ and denote by T'_j, $j=1, \ldots, 2m'$ the components of T'. Now let K be the kernel of $(T'_1, \ldots, T'_{m'})$. Clearly $\mathring{T}\,|\,K$ is a parametrization of an isotropic subspace of $\mathring{\omega}$, on $\mathbf{R}^{2(m-m')}$, of dimension $l'=\operatorname{rank}(\mathring{T}\,|\,K)$ and then, by the technical lemma above, we necessarily have $l'\geqslant l-m'$.

By the induction hypothesis, and the fact that ω and $\mathring{\omega}$ have exactly the same frequencies, it follows that $l-m'\leqslant l'\leqslant(m-m')-\nu+\rho$ so that $l\leqslant m-\nu+\rho$ as desired.

(ii) $m'=0$.

If all subspaces in the normal decomposition of ω are 2-dimensional, we are in the conditions of Theorem 2.3.1 and there is nothing to prove.

Now let α be a frequency such that the corresponding subspace $\bar{E}$ is of dimension $2\bar{m}$, $\bar{m}=2k+1\geqslant 3$.

By the normal decomposition lemma, there exist coordinates $(x_1, \ldots, x_{2k+1}, y_1 \ldots y_{2k+1})$ on $\bar{E}$ such that ω can be written in the form $\omega=\hat{\omega}\oplus(\bar{\omega}_1+\bar{\omega}_2)$ where

$$\bar{\omega}_1=\alpha(x_{k+1}\,dx_{k+1}+y_{k+1}\,dy_{k+1})+\frac{1}{2}(x_{k+1}\,dy_{k+1}-y_{k+1}\,dx_{k+1})$$

and where $x_1=\ldots=x_k=y_{k+2}=\ldots=y_{2k+1}=0$ is an isotropic subspaces of $\bar{\omega}_2$.

By arguing just like in (i) above we arrive to an isotropic subspace of $\hat{\omega}\oplus\bar{\omega}_1$, on $\mathbf{R}^{2(m-\bar{m}+1)}$, of dimension $l'\geqslant l-(\bar{m}-1)$, and the induction is completed in exactly the same way we have done before in (i). ■

Remark. — The characteristic frequencies may be computed directly from the given form ω without any help of a previous change of coordinates. In fact, if M denotes the matrix of b_ω then, $(1/2)(M^t-M)^{-1}(M^t+M)$ (the matrix of $X_H(\omega)$) furnishes all the elements required to determine the frequencies.

3. Analytic and C^∞ forms

3.1. THE MAIN RESULT ON EVEN DIMENSION. — Given $\omega \in \tilde{\Lambda}^1(\mathbf{R}^{2m})$ we define the invariants: $\nu(\omega)$, *characteristic frequencies*, and $\rho(\omega)$ to be the corresponding invariants of its linear part $J_0^1(\omega)$. In fact we may, and we do, extend this definition to forms that are only differentiable at 0.

In what follows $\Lambda_\infty^1(\mathbf{R}^n)$ denotes the set of germs of C^∞ differential 1-forms on $\mathbf{R}^n$ vanishing at the origin.

We are going to prove the following

THEOREM A. — *Let* $\omega \in \Lambda^1(\mathbf{R}^{2m})$ (*resp.* $\omega \in \Lambda_\infty^1(\mathbf{R}^{2m})$) *be such that* $d\omega(0)$ *has maximal rank. Then, there exists an analytic* (*resp. a* C^∞) *singular solution of* $\omega=0$ *of dimension* $m-\nu(\omega)+\rho(\omega)$. *Furthermore this dimension is maximal among those of the singular solutions.*

Proof. — Clearly the theorem is proved once the existence is established, for, the maximality is a direct consequence of Theorem 2.3.2.

We follow the proof of Theorem A of [4].

We recall that: $X=X(\omega)$; N is the X-invariant submanifold whose tangent space, $T_0 N$, at 0 is the invariant subspace of $J_0^1(X)$ associated to the eigenvalues $\lambda \in \operatorname{Spect}(\omega)$ such that $\operatorname{Re}\lambda \geqslant 1/2$; $\bar{\omega}=\omega \mid N$ and $\bar{X}=X \mid N$.

Now we go one step forward, by showing that the diffeomorphism in Poincaré-Dulac's theorem normalizing $\bar{X}$, referred in [4], does linearize $\bar{\omega}$. (If $\omega \in C^\infty$ we call upon the C^∞ version of Poincaré-Dulac's theorem (*see* 5.24.F of [2]).)

Suppose we have carried out that change of coordinates. Let $\bar{\omega}=\bar{\omega}_1+\bar{\omega}_2 \dots$ and $\bar{X}=\bar{X}_1+\bar{X}_2+\dots$, where $\bar{\omega}_k$ (resp. $\bar{X}_k$) is the k-homogeneous part of $\bar{\omega}$ (resp. $\bar{X}$). The decomposition $\mathbf{R}^{2m}=\mathring{E}(\omega_1) \oplus E'(\omega_1)$ induces the decomposition $T_0 N=\mathring{E}(\omega_1) \oplus F$, where $F=E'(\omega_1) \cap T_0 N$. We know from [4] that, for $k \geqslant 2$, $\bar{X}_k$ has a zero component in $\mathring{E}(\omega_1)$. On the other hand, since F is an isotropic subspace of ω_1 (*see* Corollary 1.3.1 of [4]), we conclude that $\bar{\omega}_1=\mathring{\omega}_1$. These considerations together with the classical formula $L_{\bar{X}_k}\bar{\omega}_1=i(\bar{X}_k)\,d\bar{\omega}_1+d(i(\bar{X}_k)\,\bar{\omega}_1)$ show that $L_{\bar{X}_k}\bar{\omega}_1=0$ for $k \geqslant 2$.

At this point we are ready to show (by induction on k) that $\bar{\omega}_k=0$, for $k \geqslant 2$.

In fact the relations $L_{\bar{X}_k}\bar{\omega}_1=0\ (\forall k \geqslant 2)$ and $L_{\bar{X}}\bar{\omega}=\bar{\omega}$ imply

$$(*) \qquad L_{\bar{X}_1}\bar{\omega}_k+L_{\bar{X}_2}\bar{\omega}_{k-1}+\dots+L_{\bar{X}_{k-1}}\bar{\omega}_2=\bar{\omega}_k, \quad \text{for all } k \geqslant 2.$$

If $k=2$ we have directly $L_{\bar{X}_1}\bar{\omega}_2=\bar{\omega}_2$ and, since $\bar{X}_1$ is not u-ressonant, we conclude that $\bar{\omega}_2=0$.

Suppose now that $\bar{\omega}_2=0, \dots, \bar{\omega}_k=0$. It follows from $(*)$ that $L_{\bar{X}_1}\bar{\omega}_{k+1}=\bar{\omega}_{k+1}$ and again $\bar{\omega}_{k+1}=0$.

In other words we have shown that

(i) $\bar{\omega}$ is linear in the analytic case.

(ii) $\bar{\omega}=J_0^1(\bar{\omega})+\eta$, where $\eta(p)=o(p^\infty)$, in the C^∞ case.

From the equation $L_{\bar{X}}\bar{\omega}=\bar{\omega}$ we deduce $L_{\bar{X}}\eta=\eta$. We claim that this implies $\eta=0$. (Which establishes, definitively, the linearity of $\bar{\omega}$.)

In fact, that equation is equivalent to $(X_t)^*\eta=e^t\eta$, where X_t denotes the flow of X. Hence, by taking norms, we have

$$|\eta(p)|\leqslant e^{-t}|\eta(X_t(p))||D_pX_t|$$

On the other hand, there is a neighborhood V(0) such that

(*a*) there exist K, $\mu>0$ for which

$$|X_t(p)|\leqslant K e^{\mu t}|p|, \qquad \forall t\leqslant 0 \quad \text{and} \quad p\in V,$$

for, $\operatorname{Re}\lambda>0$, $\forall\lambda\in\operatorname{Spec}(J_0^1(\bar{X}))$ (*see* Theorem 9.1 of [5]).

(*b*) there exist K', $\mu'>0$ such that

$$|D_pX_t|\leqslant K' e^{-\mu' t}, \qquad \forall t\leqslant 0 \quad \text{and} \quad p\in V,$$

once, D_pX_t satisfies the linear differential equation

$$\dot{Y}=A(p,t)Y, \qquad \text{where} \qquad A(p,t)=D_{X_t(p)}X.$$

Since $\eta(p)=o(p^\infty)$, we deduce from (a) and (*b*) above that there exists a neighborhood V(0) such that for each $k\in\mathbf{Z}^+$ one can find a constant $C_k>0$ verifying the following inequality

$$(1) \qquad |\eta(p)|\leqslant C_k e^{(k\mu-\mu'-1)t}, \qquad \forall t\leqslant 0 \quad \text{and} \quad p\in V.$$

By choosing k such that $k\mu-\mu'-1>0$ and by taking, in (1), $\lim\limits_{t\to-\infty}$, we conclude finally that $\eta(p)=0$, $\forall p\in V$, as desired.

The theorem obviously follows from the above linearization of $\bar{\omega}$ and Theorem 2.3.1 of the last section. ■

3.2. Comments on the odd dimensional, and lower rank, cases. — We shall denote by $\tilde{\Lambda}^1_\infty(\mathbf{R}^n)$ the set of $\omega\in\Lambda^1_\infty(\mathbf{R}^n)$ such that $d\omega(0)$ has maximal rank.

Given ω in $\tilde{\Lambda}^1(\mathbf{R}^{2m+1})$ (resp. in $\tilde{\Lambda}^1_\infty(\mathbf{R}^{2m+1})$) we can define the invariants $\nu(\omega)$, *characteristic frequencies* and $\rho(\omega)$, to be the corresponding invariants of the form $\omega|E$, where E is the ($2m$-dimensional subspace) image of the skew-symmetric matrix of $d\omega(0)$.

An immediate consequence of Theorem A is the following

Theorem B. — *Let* $\omega\in\tilde{\Lambda}^{-1}(\mathbf{R}^{2m+1})$ (*resp.* $\omega\in\tilde{\Lambda}^1_\infty(\mathbf{R}^{2m+1})$) *then, there exists an analytic* (*resp. a* C^∞) *singular solution of* $\omega=0$ *of dimension* $m-\nu+\rho$.

We point out that, similarly to what happens in the complex case, there could exist a singular solution of dimension $(m-\nu+\rho)+1$, but none of a greater dimension.

We would like also to mention that the choice of $E=Im(d\omega(0))$, which lies behind the statement of Theorem B, is due exclusively to its intrinsic character. Of course to