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LISA R. GOLDBERG

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FIXED POINTS OF POLYNOMIAL MAPS I. ROTATION SUBSETS OF THE CIRCLES

By Lisa R. GOLDBERG (1)

Abstract. — We give a combinatorial analysis of *rational rotation subsets of the circle*. These are invariant subsets that have well-defined rational rotation numbers under the standard self-covering maps of S^1. This analysis has applications to the classification of dynamical systems generated by polynomials in one complex variable.

0. Introduction

Late in the 1800's, Poincaré showed that every homeomorphism of the circle has a well defined *rotation number* which measures asymptotically, the average distance each point is moved by the map. Since its inception, this concept has played a fundamental role in the theory of dynamical systems in one and two dimensions.

This article focuses on dynamical systems generated by the standard d-fold self-coverings of the circle S^1. We give a combinatorial classification of *rational rotation subsets* of S^1. By definition, these are invariant subsets that have well defined rational rotation numbers. For $d=2$, these sets are always periodic cycles, and they arise in a variety of different contexts ([B], [Bu], [GH], [GLT], [V1], [V2]). Other points of view that are not, to my knowledge, in the literature, have been taught to me by Charles Tresser.

There is an important application of rotation sets to the problem of classifying dynamical systems generated by polynomials in a single complex variable. A repelling fixed point of a degree d polynomial admits a set of *external arguments* $\Theta=\{\theta_0, \ldots, \theta_{n-1}\}$ which constitute a degree d rotation subset of the circle [DH]. This application will be explored at length in a joint project with John Milnor, that makes up Part II of this work.

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1. Notation and Definition

Parametrize the unit circle S^1 by the interval [0, 1). Let $d \geqq 2$ and consider the d-fold covering map

$$f_d: \quad \theta \mapsto d\theta \bmod 1$$

Let m and n be non-negative integers satisfying $0 \leqq m \leqq n$. We will adopt the convention throughout that an indexed subset $\Theta = \{\theta_0, \ldots, \theta_{n-1}\}$ of S^1 satisfies

$$0 \leqq \theta_0 < \ldots < \theta_{n-1} < 1.$$

DEFINITION. – A finite subset $\Theta = \{\theta_0, \ldots, \theta_{n-1}\}$ of S^1 is a *degree d m/n-rotation set* if $f_d(\theta_i) = \theta_{i+m \bmod 1}$ for $i = 0, \ldots, n-1$.

In general the numbers m and n need *not* be relatively prime, so that $m/n = kp/kq$ for some $k \geqq 1$ with p and q relatively prime. In this case, we say that the *rotation number* of the set Θ is p/q. It follows that the set Θ is a union of k cyclic orbits which are regularly interspersed, each of which has the order type any orbit of the rotation $\theta \mapsto \theta + p/q \bmod 1$. Hence, each of these k cyclic subsets of Θ will be called a *degree d p/q-rotation cycle*.

Remark. – Most finite sets invariant under f_d are not rotation sets. Consider the 4-cycle generated by the angle 1/5 whose base 2 expansion is .00110011...

To begin our analysis, we isolate the special case of rotation number zero. Here a rotation set is any non-vacuous set of fixed points of the map f_d. There fixed points are precisely the angles $j/(d-1)$ with $0 \leqq j < d-1$.

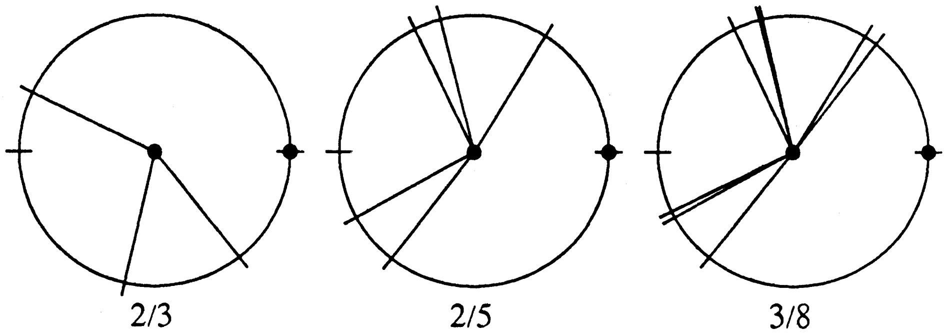

Fig. 1. – Three Quadratic Rotation Sets.

Henceforth, we will assume $0<p<q$.

LEMMA 1. — *For $q \geqq 2$, the q-cycles under f_d are in one-to-one correspondence with orbits of period q under the one-sided d-shift.*

Sketch Proof. — Label the d arcs obtained by removing the points $\{i/d\}$ from S^1 counterclockwise from 0 with the digits $0, 1, \ldots, d-1$. Let θ be a period q periodic point for f_d. If $\theta \in S^1$ is not a fixed point of f_d, let $\gamma(\theta) \in \{0, \ldots, d-1\}$ denote the label of the arc containing θ. Define the word

$$a = \gamma(\theta)\gamma(f_d\theta) \ldots \gamma(f_d^{0(q-1)}\theta).$$

The base d expansion of θ_1 is then given by $\theta = .aaaaa...$ □

2. Existence and Uniqueness of Rotation Sets

As we will see below, rotation sets with all possible rotation numbers exist in all degrees $d \geqq 2$; furthermore, quadratic rotation sets are completely classified by their rotation numbers. This is not true in higher degrees, as is indicated by examples in Figure 2. Two of the rotation sets in Figure 2 can be distinguished from the remaining three by the number of elements they contain, however a finer invariant is needed to distinguish all five examples. For each degree d rotation set, we will record the deployment of the elements with respect to the fixed points of the map f_d.

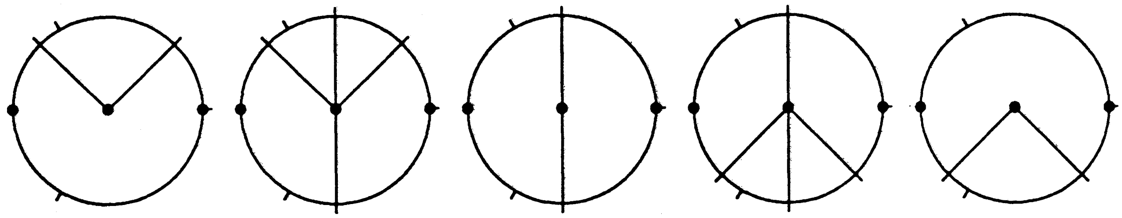

Fig. 2. — The Five Cubic Rotation Sets with Rotation Number 1/2.

DEFINITION. — Let

$$\Theta = \{\theta_0, \ldots, \theta_{n-1}\}$$

be a finite subset of S^1. The *degree d deployment sequence* of Θ is the nondecreasing sequence of non-negative integers $(s_2, \ldots, s_{d-1})$, where s_i is the number of θ_i's in the interval $[0, i/(d-1))$.

The cubic rotation sets in Figure 2 have deployment sequences

$$(2, 2), (3, 4), (1, 2), (1, 4), (0, 2)$$

respectively. (Thus the proportion s_1/s_2 of angles in the upper half-circle is 1, 3/4, 1/2, 1/4, 0 respectively.)

Remarks:

1. The last entry s_{d-1} is just the cardinality of Θ. In the case of a rotation set, it is always a product kq with $1 \leqq k \leqq d-1$. (Compare Corollary 6.)

2. This invariant contains no information for $d=1$. (It is just the single number (q).)

3. The degree d deployment sequence of a rotation set locates the components of the set with respect to the fixed points of f_d, not with respect to the f_d-preimages of 0. Therefore it does not, *a priori*, determine the base d expansions of the components.

LEMMA 2 (Uniqueness). — *A degree d rotation set is completely determined by its rotation number p/q together with its deployment sequence*

$$0 \leqq s_1 \leqq s_2 \leqq \ldots \leqq s_{d-1} = kq.$$

The proof depends on the interplay between the fixed points of f_d and its preimages of zero.

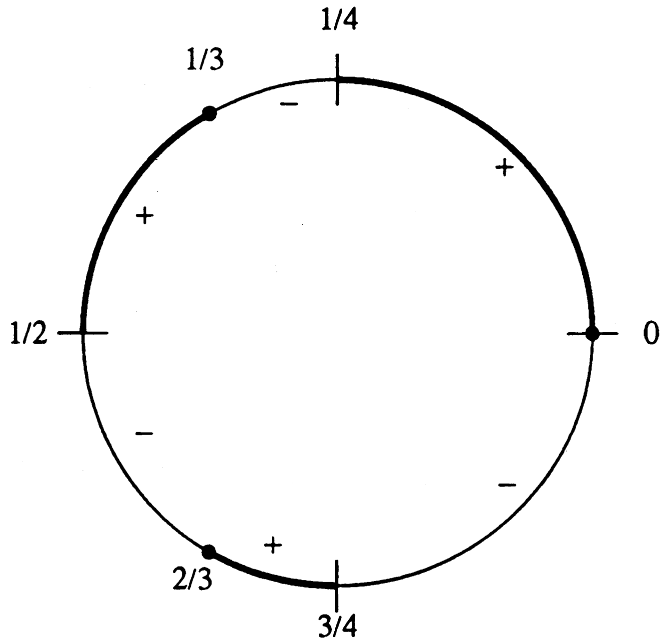

Fig. 3. — Intervals of Advancing and Retreating for f_4.

DEFINITION. — A point $\theta \in S^1$ is *advancing* if $f_d(\theta) > \theta$, *retreating* if $f_d(\theta) < \theta$. (Remember that all angles are reduced modulo 1 so as to lie in the half-open interval $[0, 1)$.

Proof of Lemma 2. — For $j=1, \ldots, d-1$, let U_j denote the arc $((j-1)/(d-1), j/(d-1))$. Each arc U_j contains exactly one f_d-preimage of zero j/d that divides it into a pair of subarcs

$$U_{j,\,\mathrm{adv}} = \left(\frac{j-1}{d-1}, \frac{j}{d}\right)$$

$$U_{j,\,\mathrm{ret}}=\left[\frac{j}{d},\frac{j}{d-1}\right).$$

These are labeled to reflect the fact that $(j-1)/(d-1)<\theta<f(\theta)<1$ on $U_{j,\,\mathrm{adv}}$ and $0<f(\theta)<\theta<j/(d-1)$ on $U_{j,\,\mathrm{ret}}$.

Let $\Theta=\{\theta_0, \ldots, \theta_{kq-1}\}$ be a degree $d\,kp/kq$-rotation set with deployment sequence $(s_1, s_2, \ldots, s_{d-1}=kq)$. Since Θ is a kp/kq-rotation set, f_d advances $\theta_0, \ldots, \theta_{kq-kp-1}$ and retreats the other θ_i's. *If* $0\leqq i\leqq kq-kp-1$, then

$$\theta_i\in U_{j,\,\mathrm{adv}}=\left(\frac{j-1}{d-1},\frac{j}{d}\right)\subset\left(\frac{j-1}{d},\frac{j}{d}\right)$$

and if $q-p\leqq i\leqq q-1$,

$$\theta_i\in U_{j,\,\mathrm{ret}}=\left(\frac{j}{d},\frac{j}{d-1}\right)\subset\left(\frac{j}{d},\frac{j+1}{d}\right)$$

so that the location of the θ_i's vis-a-vis the f_d preimages of 0 is determined. Now, as in Lemma 1, the action of f_d yields the base d expansions of the θ_i's. □

We now turn to the question of existence. An examination of the proof of Lemma 2 gives an algorithm for constructing angles from the data consisting of a rotation number p/q, and a candidate deployment sequence $(s_1, \ldots, s_{d-1}=k_q)$. It is not difficult to check that the angles θ_i resulting from this construction satisfy

$$0\leqq\theta_0\leqq\ldots\leqq\theta_{kq-1}<1.$$

However, these inequalities need not be strict, so the angles θ_i will not be distinct in general. We give below, a necessary and sufficient condition for strict inequality, and hence for existence of a set of a set angles fitting the given combinatorial data.

Let $\Theta=\{\theta_0, \theta_1, \ldots, \theta_{kq-1}\}\subset S^1$ be disjoint from the fixed points of f_d. The complement of Θ in S^1 consists of kq arcs $A_0, A_1, \ldots, A_{kq-1}$ labeled so that the arcs A_i is bounded by θ_i and $\theta_{i+1\,\mathrm{mod}\,kq}$. We define the *weight* $\omega(A_i)$ of the arc A_i to be the number of f_d fixed points is contains. Note that the *length*, $l(A_i)$ of A_i equals the difference $\theta_{i+1}-\theta_i$ when $i<kq-1$ and equals $1+\theta_0-\theta_{kq-1}$ when $i=kq-1$.

LEMMA 3. — *Let* $\Theta=\{\theta_0, \theta_1, \ldots, \theta_{kq-1}\}$ *be a degree d rotation set with rotation number* p/q *and complementary arcs* $A_0, A_1, \ldots, A_{kq-1}$. *Then the following equation holds:*

$$(*)\qquad d\,l(A_i)=l(A_{i+kp\,\mathrm{mod}\,kq})+\omega(A_i)$$

Furthermore, the map f_d *carries* A_i *homeomorphically onto* $A_{i+kp\,\mathrm{mod}\,kq}$ *if and only if the weight* $\omega(A_i)$ *is zero.*

Proof. — The image of an arc A_i under f_d covers the (disjoint) arc $A_{i+kp\,\mathrm{mod}\,kq}$ and then winds some number N times around the circle. It is easy to check that each of these circumnavigators of S^1 in A_i contains a unique fixed point of f_d. Therefore, $N=\omega(A_i)$. □

We can solve these linear equations (∗) for the angles $l(A_i)$ as functions of the critical weights $\omega(A_i)$. If we sum these equations over a residue class modulo k, we obtain the equation

$$(d-1)(l(A_i)+l(A_{i+k})+\ldots+l(A_{i+k(q-1)}))=\omega(A_i)+\omega(A_{i+k})+\ldots+\omega(A_{i+k(q-1)})$$

for each i between 0 and $k-1$. That is, the total angular width of these q sectors is directly proportional to the total weight. In particular, at least one of these q sectors must contain a fixed point of f_d. (More directly, if the $\omega(A_{i+hk})$ were all zero, then each of these sectors would map homeomorphically onto a sector with strictly greater length, which is impossible.)

LEMMA 4. — *For each i between* 0 *and* $k-1$, *the q-fold sum*

$$\omega(A_i)+\omega(A_{i+k})+\ldots+\omega(A_{i+k(q-1)})$$

must be strictly positive. In other words, each of the arcs A_i *either contains a fixed point, or is mapped homeomorphically by an iterate of* f_d *onto an* A_j *that does contain a fixed point.* □

Remark. — In the sequel to this article, we will show that the weight $\omega(A_i)$ is equal to the number of critical points contained in an associated region of the dynamical plane of a polynomial map. (Compare Part II, § 2.)

An equivalent formulation of Lemma 4 in terms of deployment sequences is the following. Fix any $p/q\neq 0$.

LEMMA 5. — *A sequence* $0\leqq s_1\leqq s_2, \leqq\ldots, \leqq s_{d-1}=kq$ *is realized by a degree d rotation set if and only if every residue class modulo k is realized by at least one of the* s_i*'s.*

COROLLARY 6. — *We have* $k\leqq d-1$. *That is, a degree d rotation subset with rotation number p/q contains at most* $(d-1)q$ *points.* □

We summarize the results from this section as

THEOREM 7. — *A degree d rotation subset of the circle is uniquely determined by its rotation number and its deployment sequence. Conversely, a lowest terms fraction p/q and candidate deployment sequence*

$$0\leqq s_1\leqq s_2\leqq\ldots\leqq s_{d-1}=kq$$

determine a rotation subset of S^1 *only if every class modulo k is realized by at least one of the* s_i*'s.* □

COROLLARY 8. — *Quadratic rotation cycles are in one to one correspondence with the set of rational numbers modulo one.* □

3. Counting Rotation Cycles

Recall that the number of ways to deploy q indistinguishable balls in N labeled boxes is equal to the binomial coefficient $\binom{N+q-1}{q}$.

PROPOSITION 9. — *The map* f_d *has* $\binom{d+q-2}{q}$ *rotation cycles with rotation number* p/q.

Proof. — The conditions of Theorem 7 are satisfied for every candidate deployment sequence $(s_1, s_2, \ldots, s_{d-1}=q)$ corresponding to a rotation cycle. Consequently, the number of p/q-rotation cycles in degree d is precisely to the number of ways to deploy q indistinguishable balls in $d-1$ labeled boxes. □

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L. R. GOLDBERG,
Department of Mathematics,
Brooklyn College,
Brooklyn,
New York 11212
and
Graduate Center, CUNY,
33 West 42 Street,
New York,
New York 10036.