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The based ring of the lowest two-sided cell of an affine Weyl group. II


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THE BASED RING OF THE LOWEST TWO-SIDED CELL OF AN AFFINE WEYL GROUP, II

BY NANHUA XI

ABSTRACT. - We show that the lowest based ring of an affine Weyl group $W$ is very interesting to understand some simple representations of the corresponding Hecke algebra $H_{q_0}(q_0 \in \mathbb{C}^*)$ even when $q_0$ is a root of 1.

Let $H_{q_0}$ be the Hecke algebra (over $\mathbb{C}$) attached by Iwahori and Matsumoto [IM] to an affine Weyl group $W$ and to a parameter $q_0^2 \in \mathbb{C}^*$.

When $q_0$ is not a root of 1 or $q_0^2 = 1$, the simple $H_{q_0}$-modules have been classified (see [KL2]). However we know little about the simple $H_{q_0}$-modules when $q_0$ is a root of 1. In this paper we give some discussion to the representations of $H_{q_0}$ with $q_0$ a root of 1. Namely, let $J$ be the asymptotic Hecke algebra defined in [L3, III]. There exists a natural injection $\phi_{q_0} : H_{q_0} \rightarrow J$. Let $K(J)$ [resp. $K(H_{q_0})$] be the Grothendieck group of $J$-modules (resp. $H_{q_0}$-modules) of finite dimension over $\mathbb{C}$, then $\phi_{q_0}$ induces a surjective homomorphism $\phi_{q_0}^*: K(J) \rightarrow K(H_{q_0})$, when $q_0$ is not a root of 1 or $q_0^2 = 1$, $(\phi_{q_0})^*$ is an isomorphism (loc. cit.). For each two-sided cell $c$ of $W$, we can define the direct summand $K(J_c)$ [resp. $K(H_{q_0,c})$] of $K(J)$ [resp. $K(H_{q_0})$]. Thus $(\phi_{q_0})^*$ induces a homomorphism $(\phi_{q_0})_{*,c} : K(J_c) \rightarrow K(H_{q_0})$. The map $(\phi_{q_0})_{*,c}$ remains surjective and is an isomorphism if $q_0$ is not a root of 1 or $q_0^2 = 1$. In this paper we mainly discuss the map $(\phi_{q_0})_{*,c_0}$ where $c_0$ is the lowest two-sided cell of $W$.

1. Introduction

1.1. Let $G$ be a simply connected, almost simple complex algebraic group and $T$ a maximal torus. Let $P \subseteq X = \text{Hom}(T, \mathbb{C}^*)$ be the root lattice. The Weyl group $W_0 = N_G(T)/T$ of $G$ acts on $X$ in a natural way and this action is stable on $P$. Thus we can form the affine Weyl group $W_a = W_0 \ltimes P$, which is a normal subgroup of the extended affine Weyl group $W = W_0 \ltimes X$. There exists a finite abelian subgroup $\Omega$ of $W$ such that $W = \Omega \ltimes W_a$. Let $S = \{r_0, r_1, \ldots, r_n\}$ be the set of simple reflections of $W_a$ with $r_0 \notin W_0$. Then we have a standard length function $l$ on $W_a$ which can be extended

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to $W$ by defining $f(\omega w) = f(w)$ for any $\omega \in \Omega$, $w \in W_\alpha$. We keep the same notation for the extension of $f$.

1.2. For any $u = \omega_1 u_1$, $w = \omega_2 w_1$, $\omega_1$, $\omega_2 \in \Omega$, $u_1$, $w_1 \in W_\alpha$, we define $P_{u, w}$ to be $P_{u_1, w_1}$, as in [KL 1] if $\omega_1 = \omega_2$ and $P_{u, w}$ to be zero if $\omega_1 \neq \omega_2$. We say that $u \leq w$ or $u \leq w$ if $u_1 \leq w_1$ in the sense of [KL 1], we say that $u \leq w$ if $u^{-1} \leq w^{-1}$. These relations generate equivalence relations $\sim$, $\sim$, $\sim$ in $W$, respectively, and the corresponding equivalence classes are called two-sided cells, left cells, right cells of $W$, respectively. The relation $\leq$ (resp. $\leq$, $\leq$) in $W$ then induces a partial order $\leq$ (resp. $\leq$, $\leq$) in the set of two-sided (resp. left, right) cells of $W$. We extend the Bruhat order $\leq$ in $W_\alpha$ to $W$ by defining $u \leq w$ if and only if $\omega_1 = \omega_2$ and $u_1 \leq w_1$.

Let $q$ be an indeterminate and let $A = \mathbb{C}[q, q^{-1}]$. Let $H$ be the Hecke algebra of $W$ over $A$, that is a free $A$-module with basis $T_w (w \in W)$ and multiplication defined by

$$(T_r - q^2)(T_r + 1) = 0 \quad \text{if} \quad r \in S \quad \text{and} \quad T_w T_{w'} = T_{ww'} \quad \text{if} \quad l(ww') = l(w) + l(w').$$

For each $w \in W$, let

$$C_w = q^{-1(w)} \sum_{u \leq w} P_{u, w} (q^2) T_u \in H.$$ 

And we write

$$C_w C_u = \sum_z h_{w, u, z} C_z \in H, \quad h_{w, u, z} \in A.$$ 

For each $z \in W$, there is a well defined integer $a(z) \geq 0$ such that

$$q^{a(z)} h_{w, u, z} \in \mathbb{C} [q] \quad \text{for all} \quad w, u \in W,$$

$$q^{a(z)} h_{w, u, z} \notin \mathbb{C} [q] \quad \text{for some} \quad w, u \in W$$

(see [L 3, I, 7.3]). We have $a(z) \leq l(w_0)$, where $w_0$ is the longest element of $W_\alpha$. It is known that

$$C_0 = \{ w \in W \mid a(w) = l(w_0) \}$$

is a two-sided cell of $W$ (see [S, I]) which is the lowest one for the partial order $\leq$.

1.3. Let $\gamma_{w, u, z}$ be the constant term of $q^{a(z)} h_{w, u, z} \in \mathbb{C} [q]$. We have $\gamma_{w, u, z} \in \mathbb{N}$. Moreover (see [L 3, II])

$$(a) \quad \gamma_{w, u, z} \neq 0 \quad \Rightarrow \quad w \sim u^{-1}, \quad u \sim z, \quad w \sim z.$$ 

Let $J$ be the $\mathbb{C}$-vector space with basis $(t_w)_{w \in W}$. This is an associative $\mathbb{C}$-algebra with multiplication

$$l_w t_u = \sum_z \gamma_{w, u, z} t_z.$$
It has a unit element \( 1 = \sum t_d \), where \( \mathcal{D} = \{ d \in W \mid a(d) = l(d) - 2 \deg P, \} \) (\( e \) is the unit of \( W \)) (see [L3, II]).

For each two-sided cell \( c \) of \( W \), let \( J_c \) be the subspace of \( J \) spanned by \( t_{we} \), \( w \in c \), then \( J = \bigoplus J_c \), where the sum is over the set of all two-sided cells of \( W \). By (a) we see that \( J_c \) is a two-sided ideal of \( J \) and in fact is an associative \( C \)-algebra with unit \( \sum_{d \in \mathcal{D} \cap c} t_d \).

1.4. For each \( q_0 \in \mathbb{C}^* \), we denote \( H_{q_0} = H \otimes_A \mathbb{C} \), where \( C \) is an \( A \)-algebra with \( q \) acting as scalar multiplication by \( q_0 \). We shall denote \( T_w \otimes 1, C_w \otimes 1 \) in \( H_{q_0} \) again by \( T_w, C_w \). We also use the notation \( h_{w,u,z} \) for the specialization at \( q_0 \in \mathbb{C}^* \) of \( h_w,u,z \).

The \( A \)-linear map \( \phi : H \to J \otimes_C A \) defined by

\[
\phi(C_w) = \sum_{d \in \mathcal{D}} \sum_{z \in W} h_{w,d,z} t_z
\]

is a homomorphism of \( A \)-algebra with 1 (see [L3, II]). Let \( \phi_{q_0} : H_{q_0} \to J \) be the induced homomorphism for any \( q_0 \in \mathbb{C}^* \).

Any (left) \( J \)-module \( E \) gives rise, via \( \phi_{q_0} : H_{q_0} \to J \), to a (left) \( H_{q_0} \)-module \( E_{q_0} \). We denote by \( K(J) \) [resp. \( K(H_{q_0}) \)] the Grothendieck group of (left) \( J \)-modules (resp. \( H_{q_0} \)-modules) of finite dimension over \( \mathbb{C} \). The correspondence \( E \to E_{q_0} \) defines a homomorphism \( (\phi_{q_0})_* : K(J) \to K(H_{q_0}) \).

We similarly define \( K(J_c) \) for any two-sided cell \( c \) of \( W \). Then we have \( K(J) = \bigoplus_{c} K(J_c) \), where the sum is over the set of all two-sided cells of \( W \). Now we define \( K(H_{q_0}^c) \). For any simple \( H_{q_0} \)-module \( M \), we attach to \( M \) a two-sided cell \( c_M \) of \( W \) by the following two conditions:

\[
C_w M \neq 0 \text{ for some } w \in c_M
\]

\[
C_w M = 0 \text{ for any } w \text{ in a two-sided cell } c \text{ with } c \subseteq c_M, c \neq c_M.
\]

Then \( c_M \) is well defined since there are only a finite number of two-sided cells in \( W \). Let \( K(H_{q_0})_c \) be the subgroup of \( K(H_{q_0}) \) spanned by simple \( H_{q_0} \)-modules \( M \) with \( c_M = c \). Obviously we have \( K(H_{q_0}) = \bigoplus_{c} K(H_{q_0})_c \). Thus for a two-sided cell \( c \) of \( W \),

\[
(\phi_{q_0})_* \text{ induces a homomorphism } (\phi_{q_0})_{q_0,c} : K(J_c) \to K(H_{q_0})_c.
\]

The following result is due to Lusztig (see [L3, III, 1.9 and 3.4]).

**Proposition 1.5.** - The map \( (\phi_{q_0})_{q_0,c} \) is surjective for any \( q_0 \in \mathbb{C}^* \), moreover, \( (\phi_{q_0})_{q_0,c} \) is an isomorphism when \( q_0 \) is not a root of 1 or \( q_0^2 = 1 \).
Now we state a conjecture.

**Conjecture 1.6.** - The map $(\phi_{q_0})_{*,c}$ is injective if $(\phi_{q_0})_{*,c'}$ is injective for some two-sided cell $c'$ of $W$ with $c' \leq c$.

By proposition 1.6 one knows that $(\phi_{q_0})_{*,c}$ is injective is equivalent to that $(\phi_{q_0})_{*,c}$ is bijective.

We mainly discuss $(\phi_{q_0})_{*,c_0}$, where $c_0$ is the lowest two-sided cell of $W$. We prove that if $\sum_{w \in W_0} q_0^{2l(w)} \neq 0$, then $(\phi_{q_0})_{*,c_0}$ is injective (see Theorem 3.4) and show that $(\phi_{q_0})_{*,c_0}$ is likely not injective if $\sum_{w \in W_0} q_0^{2l(w)} = 0$ (see Theorem 3.6)

**1.7.** Let $H_{q_0}'$ be the subalgebra of $H_{q_0}$ spanned by $T_w$, $w \in W_0$. And let $J'$ be the subspace of $J$ spanned by $t_w$, $w \in W_0$. $J'$ is a $C$-algebra with unit $\sum_{d \in S \cap W_0} t_d$. Let $\phi_{q_0} : H_{q_0} \to J'$ be defined by

$$\phi_{q_0}(C_w) = \sum_{d \in S \cap W_0} h_{w,d,z}(q_0) t_z, \quad w \in W_0,$$

then $\phi_{q_0}$ is a $C$-algebra homomorphism preserving 1.

As in 1.4 we define $K(H_{q_0}, K(J'), K(H_{q_0})_{c'}, K(J')_{c'}, (\phi_{q_0})_{*,c'}, (\phi_{q_0})_{*,c'}, etc., where c' is a two-sided cell of $W_0$. We also have

**Proposition 1.8.** - $(\phi_{q_0})_{*,c'}$ is surjective for any $q_0 \in C^*$. Moreover $(\phi_{q_0})_{*,c'}$ is an isomorphism when $q_0$ is not a root of 1 or $q_0^2 = 1$.

**Conjecture 1.9.** - $(\phi_{q_0})_{*,c'}$ is injective if $(\phi_{q_0})_{*,c''}$ is injective for some two-sided cell $c''$ of $W_0$ with $c'' \leq c'$.

When $c'$ is the lowest two-sided cell of $W_0$, it is easy to see that $(\phi_{q_0})_{*,c'}$ is injective if and only if $\sum_{w \in W_0} q_0^{2l(w)} \neq 0$.

**2. The two-side cell $c_0$ and the ring $J_{c_0}$**

In this section we recall and prove some results on $c_0$ and $J_{c_0}$.

**2.1.** We denote by $w_0$ the longest element in $W_0$. Let

$$S = \{ w \in W | l(hw_0) = l(w) + l(w_0) \text{ and } wW_0 r \not\in c_0 \text{ for any } r \in S \cap W_0 \}.$$

Then $S = S \cap c_0 = \{ ww_0 w^{-1} | w \in S \}$ and $|S| = |W_0|$ (see [S, II]).

Let $X^+ = \{ w \in W \mid l(\rho x) > l(x) \text{ for any } \rho \in S' \}$, where $S' = S \cap W_0$. Let $x_i \in X^+ \ (i \in \{1, 2, \ldots, n\} = I_0)$ be the $i$-th basic dominant weight, then $x_i$ has the properties: $l(x_i\rho) < l(x_i)$, $x_i \rho = r_j x_i$, $l(x_i\rho) = l(x_i) + 1$ if $i \neq j \in I_0$. We have
\[
c_0 = \{ w \in W \mid w^{-1} x w, \ x \in X^+ \} \quad (\text{see [S, II]}).
\]
Moreover $l(w' w_0 x w^{-1}) = l(w') + l(w_0) + l(x) + l(w^{-1})$.

**Lemma 2.2.** Let $u \in C_0$, then $C_u = h C_{w_0} h'$ for some $h, h' \in H_{\Phi_0}$, i.e., the two-sided ideal $\bigoplus_{u \in C_0} C_u$ of $H_{\Phi_0}$ is generated by the element $C_{w_0}$.

**Proof.** Write $u = w' w_0 w$ for some $w', w \in W$ such that $l(u) = l(w') + l(w_0) + l(w)$. We use induction on $l(u)$ to prove that $C_u$ is in the two-sided ideal $N$ of $H_{\Phi_0}$ generated by $C_{w_0}$.

When $l(u) = l(w_0)$, then $C_u = C_{w_0} C_{w_0} C_{w'}$ for some $w$, $w' \in \Omega$. Now assume that $l(w') > 0$. Let $s \in S$ be such that $s \rho \leq w'$, then
\[
C_s, C_{s w} = C_u + \sum_{l(z) < l(u)} a_z C_z, \quad a_z \in \mathbb{N} \quad (\text{see [KL 1]}).
\]
By induction hypothesis we know that $C_u \in N$. Similarly we can prove that $C_u \in N$ if $l(w) > 0$. The lemma is proved.

**Corollary 2.3.** For a simple $H_{\Phi_0}$-module $M$, we have $c_M = c_0$ if and only if $C_{w_0} M \neq 0$.

For $w \in W$, set $L(w) = \{ r \in S \mid r w \leq w \}$ and $R(w) = \{ r \in S \mid w r \leq w \}$.

**Lemma 2.4.** (i) Let $w'$ be the longest element in the Weyl group generated by $L(w)$ (or $R(w)$), then $w = w' w''$ (or $w = w'' w'$) for some $w'' \in W$ and $l(w) = l(w') + l(w'')$.

(ii) Let $w'$ be the longest element in the Weyl group $W'$ generated by $S - L(w)$ (resp. $S - R(w)$), then $l(w') = l(w') + l(w)$ (resp. $l(w') = l(w') + l(w')$).

**Proof.** (i) follows from $T_w C_w = q^{l(w)} C_w$ or $C_w T_w = q^{l(w)} C_w$. (ii) follows from the fact that $w$ is the shortest element in $W' w$ or $w W'$.

Let $G_0$ be the left cell in $C_0$ containing $w_0$, then
\[
G_0 = \{ w w_0 x \mid x \in X^+, \ w \in W \} = \{ w \in W \mid R(w) = S' \}
\]

**Lemma 2.5.** Any element $u \in G_0$ has the form $w x w_0$, where $w \in W_0$, $x = \prod_{i=1}^{n} x_i \in X^+$. $w_j$ is the longest element in $W_j = \langle r_j | a_j = 0, j \in I_0 \rangle$, moreover $l(u) = l(w) + l(x) + l(w_j)$.

**Proof.** Choose $x = \prod_{i=1}^{n} x_i \in X^+$ such that $u \in G_0 \cap W_0 x W_0$.  

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Then the shortest element in $W_0 \times W_0$ is $x w_j w_0$ and the shortest element in $\Gamma_0 \cap W_0 \times W_0$ is $x w_j$ by lemma 2.4 (i), where $w_j$ is the longest element in $W_j = \langle r_j |_{a_j = 0}, j \in I_0 \rangle$. The lemma is proved.

**Lemma 2.6.** - (i) Let $I \subseteq K \subseteq I_0$, then in $H_q_0$, we have $C_{w_j} C_{w_k} = C_{w_j} w_k$, where $w_j$, $w_k$ are the longest element in $W_j = \langle r_j |_{j \in J} \rangle$, $W_k = \langle r_k |_{k \in K} \rangle$, respectively, $\eta_j = q_0^{-1}(w_j) \sum_{w \in W_j} q_0^1(w).

(ii) $C_{w_j} w = h C_{w_j}$, $C_{w_j} w' = C_{w_j} h'$ for some $h, h' \in H_q_0$ if $l(w w_j) = l(w) + l(w_j)$, $l(w_j w') = l(w_j) + l(w')$.

**Proof.** - First we prove (ii). We use induction on $l(w)$. Assume that $l(w) > 0$. Choose $r \in S$ such that $r w \leq w$, then

$$C_r C_{r w w_j} = C_{r w w_j} \sum_{z \in W} a_z C_z, \quad a_z \in \mathbb{N} \quad (\text{see [KL 1]})$$

Moreover $a_z \neq 0$ implies that $z \leq r w w_j$. So $R(z) \supseteq \{ r_j |_{j \in J} \}$ (see [KL 1]).

By Lemma 2.4 we see that $z = z' w_j$ for some $z' \in W$ and $l(z) = l(z') + l(w_j)$. By induction hypothesis we know that $C_{w_j} = h C_{w_j}$ for some $h \in H_q_0$. Similarly we have $C_{w_j} = h' C_{w_j}$ for some $h' \in H_q_0$.

(i) follows from $C_j C_j = \eta_j C_j$ and (ii).

**Corollary 2.7.** - Let $x, w_j$ be as in 2.5, then in $H_q_0$ we have

$$C_{w_0} C_{x w_j} = \eta_j \sum_{y \in X^+, y \subseteq w_0} a_{x, y} C_{w_0 y}, \quad a_{x, y} \in \mathbb{C} \quad \text{and} \quad a_{x, x} = 1.$$

**Proof.** - By 2.1 and 2.6 (ii) we see that $C_{x w_j} = C_{w x} = C_{w_j} h$, where

$$h = \sum_{w \in W} a_w T_w, \quad a_w \in \mathbb{C}, \quad a_x = q_0^{-1}(x).$$

By (2.6 (i)) we know that

$$(a) \quad C_{w_0} C_{x w_j} = C_{w_0} C_{x w_j} = C_{w_0} C_{w_j} h = \eta_j C_{w_0} h.$$

Note that $h_{w_0, x w_j, z} \neq 0$ implies that $z \sim x w_j, z \sim w_0$ (see [L 3, I]), we have $z \in \Gamma_0 \cap \Gamma_0^{-1} = \{ w_0, y | y \in X^+ \}$. So by (a) we get

$$C_{w_0} C_{x w_j} = \eta_j \sum_{y \in X^+} a_{x, y} C_{w_0 y}, \quad a_{x, y} \in \mathbb{C}.$$
Since \( a_y = q_0^{-1}(x) \) and \( l(w) < l(x) \) if \( a_w \neq 0, w \neq x \). We have \( a_{x, x} = 1 \) and \( a_{x, y} = 0 \) if \( l(y) > l(x) \) or \( l(y) = l(x) \) but \( x \neq y \). Let \( w \in W \) be such that \( a_w \neq 0 \). Consider the expression

\[ C_{w_0} \cdot T_w = \sum_{z^{-1} \in \Gamma_0} b_z C_z, \quad b_z \in \mathbb{C}. \]

Since \( w_0 w \leq w_0 x \), we have \( b_z \neq 0 \) implies that \( z \leq w_0 x \). Thus by (a) we know that \( a_{x, y} \neq 0 \) implies that \( w_0 y \leq w_0 x \). The Corollary is proved.

2.8. For any \( x \in X \), we choose \( x', x'' \in X^+ \) such that \( x = x' x''^{-1} \) and then define \( \bar{T}_x = q_0^{-1}(x) T_x (q_0^{-1}(x'') T_{x''})^{-1} \). \( \bar{T}_x \) is independent of the choices \( x' \) and \( x'' \). We denote the conjugacy class of \( x \in X \) in \( W \) by \( p_{x_0} \) and let \( z_x = \sum_{x' \in O_x} \bar{T}_{x'} \). \( z_x \) is in the center of \( H_{q_0} \). For \( x \in X^+ \), denote \( d(x', x) \) the dimension of the \( x' \)-weight space \( V(x)_x \) of \( V(x) \), where \( V(x) \) is the irreducible representation of \( G \) with highest weight \( x \). We set \( S_x = \sum_{x' \in X^+} d(x', x) z_x \), \( x \in X^+ \).

**Lemma 2.9.** (see [X]). In \( H_{q_0} \) we have \( C_{w'_0 w_0 w^{-1}} = S_x C_{w w_0 w^{-1}} = C_{w'_0 w_0 x w^{-1}} \) for any \( w', w \in \mathcal{O}, x \in X^+ \).

**Lemma 2.10.** Let \( u \in \Gamma_0 \), then

\[ C_u = \sum_{y \in X^+} h_{l, y} C_{x_0 w_0} S_y, \]

where \( h_{l, y} \in H_{q_0} = \sum_{w \in W_0} \mathbb{C} T_w = \sum_{w \in W_0} \mathbb{C} C_w \), \( x_0 = \prod_{l \in J} x_l \), \( J = \{ j \in \mathcal{I}_0 \mid a_j = 0 \} \).

**Proof.** By 2.5 we see that \( u = w x w_j \), where \( w \in W_0, x = \prod_{i=1}^n x_i^j \), \( J = \{ j \in \mathcal{I}_0 \mid a_j = 0 \} \).

We use induction on \( l(u) \), when \( w = e \), by 2.9 we see that \( C_u = C_{x_0 w_0} S_y \), where \( J' = \mathcal{I}_0 - J, y = \prod_{j \in J'} x_l^j \), i.e. the lemma is true. Now assume that \( l(w) > 0 \) and choose \( r \in S' \) such that \( rw \leq w \), then

\[ C_{r'} C_{rw x w_j} = C_{w x w_j} + \sum_{z \in \Gamma_0} a_z C_z, \quad a_z \in \mathbb{N}. \]

By induction hypothesis we know that there exists \( h_{l, y} \in H_{q_0} \) such that \( C_u = \sum_{y \in X^+} h_{l, y} C_{x_0 w_0} S_y \). The lemma is proved.

2.11. Let \( R_G \) be the ring of the rational representations ring of \( G \) tensor with \( \mathbb{C} \). Then \( R_G \) is a \( \mathbb{C} \)-algebra with a \( \mathbb{C} \)-basis \( V(x), x \in X^+ \). Let \( M_{\mathcal{O} \times \mathcal{O}}(R_G) \) be the \( \mathcal{O} \times \mathcal{O} \) matrix.
Then we have

**Theorem 2.12** (see [X]). There is a natural isomorphism $J_{c_0} \cong M_{\mathbb{R} \times \mathbb{R}}(R_G)$ such that

$$m_{w_1, w_2} = \begin{cases} V(x) & \text{if } w_1 = w', w_2 = w \\ 0 & \text{otherwise} \end{cases}$$

Hereafter we identify $J_{c_0}$ with $M_{\mathbb{R} \times \mathbb{R}}(R_G)$.

### 3. The homomorphism $(\phi_{q_0})^{*_{c_0}}$

**3.1.** For any semisimple conjugacy class $s$ in $G$, we have a simple representation $\psi_s$ of $J_{c_0} \cong M_{\mathbb{R} \times \mathbb{R}}(R_G)$:

$$\psi_s: \ M_{\mathbb{R} \times \mathbb{R}}(R_G) \to M_{\mathbb{R} \times \mathbb{R}}(\mathbb{C})$$

$$(m_{w, w'}) \to (\text{tr}(s, m_{w, w'})), w, w' \in \mathcal{S}.$$  

Any simple representation of $J_{c_0}$ is isomorphic to some $\psi_s$ (see [X]). Let $E_s$ be the simple $J_{c_0}$-module providing the representation $\psi_s$. $E_s$ gives rise, via

$$\phi_{q_0, c_0}: H_{q_0} \to J \to J_{c_0},$$

to an $H_{q_0}$-module $E_{s, q_0}$. Note that $\phi_{q_0, c_0}(S_x) = \sum_{w \in \mathcal{S}} t_{w w_0 x w^{-1}}$ for any $x \in X^+$, we see that $S_x$ acts on $E_{s, q_0}$ by scalar $\text{tr}(s, V(x))$.

**Proposition 3.2.** The set $\Lambda = \{ (\phi_{q_0})^{*_{c_0}}(E_s) | s \text{ semisimple conjugacy class of } G \} - \{0\}$ is a base of $K(H_{q_0})_{c_0}$.

**Proof.** It is easy to see that $(\phi_{q_0})^{*_{c_0}}(E_s) = \sum M_i$ where the sum is over the set of composition factors $M$ of $E_{s, q_0}$ with $c_M = c_0$ and $a_M$ is the multiplicity of $M$ in $E_{s, q_0}$.

Now let $F_i = (\phi_{q_0})^{*_{c_0}}(E_{s_i}) \in \Lambda, 1 \leq i \leq k$, and suppose that $\sum_{i=1}^{k} m_i F_i = 0, m_i \in \mathbb{Z}$. Let

$$F_i = \sum_{M_{ij}} a_{M_{ij}} M_{ij}$$

be simple $H_{q_0}$-module with $c_{M_{ij}} = c_0$. Since $S_x$ acts on $E_{s_i, q_0}$ by scalar $\text{tr}(s_i, V(x))$. $S_x$ acts on $M_{ij}$ by scalar $\text{tr}(s_i, V(x))$ if $a_{M_{ij}} \neq 0$. $F_i \in \Lambda$ implies that $a_{M_{ij}} \neq 0$ for some $M_{ij}$. Therefore $m_i = 0$. By 1.6 we know that $(\phi_{q_0})^{*_{c_0}}$ is surjective, hence $\Lambda$ is a base of $K(H_{q_0})_{c_0}$. The proposition is proved.

**Corollary 3.3.** $E_{s, q_0}$ has at most one composition factor to which the attached two-sided cell is $c_0$. Moreover the multiplicity $a_M$ is 1 if $E_{s, q_0}$ has such a composition factor $M$. 

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THEOREM 3.4. — If \( \sum_{w \in W_0} q_{21(w)}^0 = q_{11}^0 \eta_{I_0} \neq 0 \), then \((\phi_{\eta_{I_0}})_{* \cdot c_0}\) is injective, so \((\phi_{\eta_{I_0}})_{* \cdot c_0}\) is an isomorphism by 1.6.

Proof. — We have

\[
\phi_{\eta_{I_0} \cdot c_0} (C_{W_0}) = \sum_{\omega \in \mathfrak{F}} h_{w_0 \cdot w w_0 \omega w^{-1}, w_0 \omega x w^{-1}} I_{w_0 \omega x w^{-1}} \in J_{I_0}.
\]

We identify \( J_{I_0} \) with \( M_{\mathfrak{F} \times \mathfrak{G}} (R_G) \), then \( \phi_{\eta_{I_0} \cdot c_0} (C_{W_0}) = (m_{w', w}) \in M_{\mathfrak{F} \times \mathfrak{G}} (R_G) \) and

\[
m_{w', w} = \begin{cases} \sum_{x \in X^+} h_{w_0 \cdot w w_0 \omega w^{-1}, w_0 \omega x w^{-1}} V(x) & \text{if } w' = e, \\ 0 & \text{if } w' \neq e. \end{cases}
\]

Note that \( C_{W_0} \subseteq C_{W_0} = \eta_{I_0} \subseteq C_{W_0} \), we see that \( m_{e, e} = \eta_{I_0} \neq 0 \), where \( e \) is the unit in \( W \). So \( C_{W_0} E_{s_{q_0}} \neq 0 \) for any semisimple conjugacy class \( s \) of \( G \) since \( \psi_s \phi_{\eta_{I_0} \cdot c_0} (C_{W_0}) \neq 0 \). Now let \( 0 = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_i = E_{s_{q_0}} \) be a composition series of \( E_{s_{q_0}} \) and let \( i \) be the integer such that \( C_{W_0} F_i \neq 0 \) and \( C_{W_0} F_{i-1} = 0 \). Then \( C_{W_0} M \neq 0 \) where \( M = F_i / F_{i-1} \), otherwise, \( C_{W_0} F_i \subseteq F_{i-1} \), choose \( v \in E_{s_{q_0}} \) such that \( C_{W_0} v \neq 0 \), we have \( C_{W_0} F_i = \eta_{I_{f_0}} C_{W_0} v \neq 0 \). A contradiction, so \( C_{W_0} M = 0 \), i.e., \( c_M = c_0 \). That is to say \((\phi_{\eta_{I_0}})_{* \cdot c_0} (E_s) = 0 \). The theorem follows from proposition 3.2.

3.5. In the subsequent part of this section we assume that \( \eta_{I_0} = 0 \), i.e.,
\[ \sum_{w \in W_0} q_{21(w)}^0 = 0. \]

Let \( \Delta_{q_0} = \{ I \subseteq I_0 \mid \eta_I \neq 0 \text{ but } I_{r \cup \{i\}} = 0 \text{ for any } i \in I \} \). Here we use the convention that \( I' \) always denotes the complement of \( I \) in \( I_0 \), i.e., \( I' = I_0 - I \).

THEOREM 3.6. — Let \( s \) be a semisimple conjugacy class of \( G \), then \((\phi_{\eta_{I_0}})_{* \cdot c_0} (E_s) = 0 \) if and only if \( \alpha_I = 0 \) for all \( I \in \Delta_{q_0} \), where

\[
\alpha_I = \sum_{x \in X^+} h_{w_0 \cdot x \omega r \cdot w_0 x \omega} \text{tr} (s, V(x)) \text{ for any } I \subseteq I_0.
\]

We need two lemmas.

LEMMA 3.7. — The following conditions are equivalent.

(i) \( C_{W_0} E_{s_{q_0}} = 0 \).
(ii) \( \psi_s \phi_{\eta_{I_0} \cdot c_0} (C_{W_0}) = 0 \).
(iii) \( \sum_{x \in X^+} h_{w_0 \cdot w w_0 x w^{-1}, w_0 x w^{-1}} \text{tr} (s, V(x)) = 0 \) for all \( w \in \mathfrak{F} \).
(iv) \( \sum_{x \in X^+} h_{w_0 \cdot w w_0 x w^{-1}, w_0 x w^{-1}} \text{tr} (s, V(x)) = 0 \) for all \( w \in \mathfrak{F} \).
(v) \( \alpha_I = \sum_{x \in X^+} h_{w_0 \cdot x \omega r \cdot w_0 x \omega} \text{tr} (s, V(x)) = 0 \) for all \( I \subseteq I_0 \).
(vi) \( \alpha_I = \sum_{x \in X^+} h_{w_0 \cdot x \omega r \cdot w_0 x \omega} \text{tr} (s, V(x)) = 0 \) for all \( I \in \Delta_{q_0} \).
Proof. — (i) and (ii) are obviously equivalent.

Note that \( h_{w_0, w_0 w^{-1}, z} \neq 0 \) implies that \( z = w_0 x w^{-1} \) for some \( x \in X^+ \) and that
\[
\phi_{q_0, c_0}(C_{w_0}) = (m', w),
\]
\[
m'_{w', w} = \begin{cases}
\sum_{x \in X^+} h_{w_0, w_0 w^{-1}, w_0 x w^{-1}} V(x), & \text{if } w' = e \\
0, & \text{otherwise}
\end{cases}
\]
we see that (ii) \( \iff \) (iii).

By theorem 2.9 in [X] we have \( h_{w_0, w_0 w^{-1}, z} = h_{w_0, w_0 w^{-1}, w_0 x w^{-1}} \). So we have (iii) \( \iff \) (iv).

By Lemma 2.4 (i) we see that \( x_i w_r = w w_0 \) for some \( w \in W \). Using the method in [S] one knows that \( w \in \mathfrak{G} \). Thus we have (iv) \( \Rightarrow \) (v). Now we show that (v) \( \Rightarrow \) (iv). Let \( w \in \mathfrak{G} \), then \( w w_0 \in \Gamma_0 \), hence by 2.10
\[
C_{w w_0} = \sum_{y \in X^+} h_{y, y} C_{x_{y} w_r, S_y}, \quad h_{y, y} \in H'_{q_0}.
\]

Since \( C_{w_0} h_{y, y} = a_{y, y} C_{w_0} \) for some \( a_{y, y} \in \mathbb{C} \), we have
\[
\sum_{x \in X^+} h_{w_0, w_0 w^{-1}, w_0 x w^{-1}} tr(s, V(x)) = \sum_{x \in X^+} a_{y, y} \alpha_i tr(s, V(y)) = 0.
\]

Finally we prove that (v) and (vi) are equivalent.

One direction is obvious. Now assume that (vi) holds. Let \( J \subseteq \Gamma_0 \). We use induction on \( l(x_j) \) to prove that \( \alpha_j = 0 \). When \( \eta_{p_0} = 0 \) or \( J \in \Delta_{q_0} \) we have \( \alpha_j = 0 \) by 2.7 or by (vi). Suppose \( \eta_{p_0} \neq 0 \) and \( J \notin \Delta_{q_0} \). Choose \( j \in J \) such that \( \eta_{p_0, j} \neq 0 \). Let \( K = J - \{j\} \), then \( K' = J' \cup \{j\} \). We have
\[
C_{w_0} C_{x_j w_r} = \frac{1}{\eta_{K'}} C_{w_0} C_{w_k} C_{x_j w_r} \quad \text{(by 2.6)}
\]
\[
= \frac{\eta_{l_j}}{\eta_{K'}} C_{w_0} (C_{w_k x_j w_r} + \sum_{i \in \Gamma_0} h_{y, y} C_{x_j w_r, S_y}), \quad h_{y, y} \in H'_{q_0} \quad \text{(by 2.6, 2.10)}.
\]

Let \( C_{w_0} h_{y, y} = a_{y, y} C_{w_0} \), \( a_{y, y} \in \mathbb{C} \). By 2.7 we see that \( a_{y, y} \eta_{p_0} \neq 0 \) implies that \( l(x_j y) < l(x_j) \). Obviously \( l(x_K) < l(x_j) \). Using induction hypothesis we get
\[
\alpha_j = \frac{\eta_{l_j}}{\eta_{K'}} (\alpha_k tr(s, V(x_j)) + \sum_{i \in \Gamma_0} a_{y, y} \alpha_i tr(s, V(y))) = 0.
\]

The lemma is proved.

Lemma 3.8. — \( (\phi_{q_0, c_0} (E_0) = 0 \) if and only if \( C_{w_0} E_{x_0} = 0 \).

Proof. — The “if” part is obvious. The “only if” part need to do a little more.
Assume that $C_w E_{s, q_0} \neq 0$. By 3.7 we see that $\alpha_i \neq 0$ for some $I \subseteq I_0$. As in [LX] we define an automorphism $\alpha : W \to W$ by

$$\alpha(wx) = w_0 wx^{-1} w_0, \quad w \in W_0, \ x \in X.$$  

One verifies that $\alpha$ leaves stable $W_0, X, S, S'$. In particular, $\alpha$ induces a bijection $\alpha : I_0 \to I_0$ and an automorphism $\sigma : H_{q_0} \to H_{q_0}$ by defining $C_u \to C_{\alpha(u)}$, $u \in W$. Let $J = \alpha(I)$, we have $\alpha(x_j) = x_j$, $\alpha(w_j) = w_j$. Consider

$$\psi_s \phi_{q_0, c_0}(C_{s_j^{-1} w_j}) = (n_{w', w}) \in M_{X} \otimes \mathbb{C}.$$  

By 2.4 and 2.12, we know that $n_{w', w} = 0$ if $w' \neq e$ and

$$n_{e, w} = \sum_{x \in X^+} h_{s_j^{-1} w_j, w_{0x} w_{0x}^{-1}} tr(s, V(x)).$$

In particular,

$$n_{e, e} = \sum_{x \in X^+} h_{s_j^{-1} w_j, w_{0x} x_{0x}^{-1}} tr(s, V(x)).$$

We claim that $n_{e, e} = \alpha_i$. In fact, let $\iota$ be the antiautomorphism of $H_{q_0}$ defined by $C_u \to C_{-1} u$, $u \in W$. Apply $\iota$ to the equality

$$C_{w_0} C_{x_j w_0} = \sum_{x \in X^+} h_{w_{0x} x_{0x} w_0 x} C_{w_0 x}.$$

We get

$$C_{x_j^{-1} w_j} C_{w_0} = \sum_{x \in X^+} h_{w_{0x} x_{0x} w_0 x} C_{x^{-1} w_0}.$$

Apply $\sigma$ to the above identity we obtain

$$C_{x_j^{-1} w_j} C_{w_0} = \sum_{x \in X^+} h_{w_{0x} x_{0x} w_0 x} C_{w_0 x}.$$

Therefore $h_{x_j^{-1} w_j, w_{0x} x} = h_{w_{0x} x_{0x} w_0 x}$ and $n_{e, e} = \alpha_i \neq 0$. By this and $n_{w', w} = 0$ if $w' \neq e$ we see that $\alpha_i$ is an eigenvalue of $\psi_s \phi_{q_0, c_0}(C_{s_j^{-1} w_j})$. Let $0 \neq v \in E_{s, q_0}$ be such that $C_{s_j^{-1} w_j} v = \alpha v$. Let $F$ be the $H_{q_0}$-submodule of $E_{s, q_0}$ generated by $v$. Then $F$ has a maximal $H_{q_0}$-submodule $F_0$ which doesn't contain $v$. $F/F_0$ is an irreducible $H_{q_0}$-module. Moreover $C_{s_j^{-1} w_j} (F/F_0) \neq 0$ since $v \notin F_0$. We have proved that $(\phi_{q_0})_{s, c_0}(E_{s}) \neq 0$.

Theorem 3.6 follows from 3.7 and 3.8.

3.9. There are two special cases. One is that $\eta_{l_0} = 0$ but $\eta_i \neq 0$ for any proper subset $I$ of $I_0$. In this case we have $A_{q_0} = \{ \{ i \} | i \in I_0 \}$. Let $i' = I - \{ i \}$. By 2.7 we have $h_{w_0 x_{w_0 x}, w_{0x} x} = \eta_i a_i x$ for some $a_i x \in \mathbb{C}$. Moreover, $a_i x \neq 0$ implies that $w_0 x \leq w_0 x_i$ and $a_i x_i = 1$. By this we see that the equation system

$$\alpha_i = \eta_i \sum_{x \in X^+} a_i x \ tr(s, V(x)) = 0, \quad i \in I_0$$

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uniquely determines \( tr(s, V(x_i)), i \in I_0 \). In other words, there exists a unique semisimple conjugacy class \( s \) of \( G \) such that \( \alpha_{(i)} = 0 \) for all \( i \in I_0 \). By 3.6 we have got the following.

**Proposition.** — There exists a unique semisimple conjugacy class \( s \) of \( G \) such that 
\[
(\phi_{\eta_0})_{*, c_0} (E_s) = 0 \quad \text{when} \quad \eta_0 = 0 \quad \text{but} \quad \eta_1 \neq 0 \quad \text{for any proper subset} \quad I \quad \text{of} \quad I_0.
\]

When \( W \) is of type \( \Lambda_n \). We can determine the semisimple conjugacy class \( s \) in the proposition explicitly. We have \( a_{i,x} = 0 \) if \( x \neq x_i \) since \( x_i \) is a minimal dominant weight for any \( i \in I_0 \). So \( \alpha_{(i)} = \eta_i \cdot tr(s, V(x_i)) \). Let \( T \) be the diagonal subgroup of \( G = SL_{n+1}(\mathbb{C}) \). We may require that \( x_i \in \text{Hom}(T, \mathbb{C}^*) \) is defined by \( x_i(t) = t_1 t_2 \ldots t_i \) where \( t = \text{diag}(t_1, t_2, \ldots, t_{n+1}) \in T \). Thus, we have
\[
tr(s, V(x_i)) = \sum_{J = 1}^{n+1} t_{j_1} t_{j_2} \ldots t_{j_i}.
\]

where \( t = \text{diag}(t_1, t_2, \ldots, t_{n+1}) \in s \cap T \), \( s \) a semisimple conjugacy class of \( G \). Hence, \( tr(s, V(x_i)) = 0 \), \( 1 \leq i \leq n \) is equivalent to that \( t_i (1 \leq i \leq n+1) \) is the solution of the equation \( \lambda^{n+1} + (-1)^{n+1} = 0 \). So if \( \eta_0 = 0 \) but \( \eta_1 \neq 0 \) for any proper subset \( I \) of \( I_0 \),
\[
(\phi_{\eta_0})_{*, c_0} (E_s) = 0 \quad \text{if and only if the eigenpolynomial of} \quad s \quad \text{is} \quad \lambda^{n+1} + (-1)^{n+1}.
\]

Another special case is that \( q_0 + q_0^{-1} = 0 \). In this case \( \Delta_{\eta_0} = \{ I_0 \} \). So \( (\phi_{\eta_0})_{*, c_0} (E_s) = 0 \) if and only if \( \alpha_{(I_0)} = 0 \). If we identify the set \{ semisimple conjugacy classes of \( G \) \} with \( \mathbb{C}^n \) through the bijection
\[
s \rightarrow (tr(s, V(x_1)), tr(s, V(x_2)), \ldots, tr(s, V(x_n))),
\]
then \( \alpha_{(I_0)} = 0 \) defines a hypersurface in \( \mathbb{C}^n \). That is to say, the set \{ semisimple conjugacy class \( s \) of \( G \) \} \( (\phi_{\eta_0})_{*, c_0} (E_s) = 0 \} \) is a variety of dimension \( n-1 \).

When \( W_0 \) is of rank 2, if \( \eta_0 = 0 \), then either \( \eta_i \neq 0 \) for any proper subset \( I \subseteq I_0 \) or \( q_0 + q_0^{-1} = 0 \). The above discussion shows that \( (\phi_{\eta_0})_{*, c_0} \) is an isomorphism if and only if \( \eta_0 \neq 0 \).

**3.10.** In general it is difficult to compute \( C_{w_0} C_{x w_1} \) in \( H \). Now we compute it for the simplest case: \( x_1 \) is the highest short root.

When \( x_1 \in X^+ \) is the highest short root, \( x_1 w_1 = r_0 w_0 \), and \( w_0 x \leq w_0 x_1 \), \( x \in X^+ \) implies that \( x = e \) or \( x_1 \). So by 2.7, in \( H \) we have
\[
C_{w_0} C_{r_0 w_0} = C_{w_0} C_{x w_1} = \sigma_{Y} (C_{w_0 x_1} + a C_{w_0}),
\]
where \( \sigma_{Y} \in A = \mathbb{C}[q, q^{-1}] \) is determined by \( C_{w_1} C_{w_1} = \sigma_{Y} C_{w_1}, a \in A \). We need to determine the coefficient \( a \). Comparing the coefficient of \( T_e \) in both sides we get
\[
q^{-i(w_0)} \sigma_{I_0} = q^{-i(w_0 w)} \sigma_{Y} P_{w_0, w_0 w} (q^2) + a q^{-i(w_0)} \sigma_{I_1}.
\]

\( i.e. \)
\[
\sigma_{I_0} = q^{-i(x)} \sigma_{Y} P_{w_0, w_0 w} (q^2) + a q \sigma_{I_1}.
\]
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Using the formula 8.10 in [L 2] we get the following

**Proposition 3.11.** — If $x_i$ is the highest short weight, then

\[
P_{w_0, w_0^x i} = \begin{cases} 
\sum_{i=1}^{n} q^{i-1} & \text{for type } \tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \\
1 & \text{for type } C_n, G_2, \\
q^{2(n-1)} - 1 & \text{for type } \tilde{D}_n, \\
q^2 - 1 & \text{for type } \tilde{E}_6, \\
q^4 + 1 & \text{for type } \tilde{E}_7.
\end{cases}
\]

where $e_1, \ldots, e_n$ are the exponents of $W_0$.

By the proposition and 3.10(a) we obtain the following

**Proposition 3.12.** — In $H$ we have

\[
C_{w_0} C_{r_0 w_0} = C_{w_0} C_{x_{1wp}} = \sigma_{i} C_{w_0^x i} + \frac{\sigma_{10}}{[e_n + 1]} [e_n] C_{w_0},
\]

where $e_n$ is the largest exponent of $W_0$ and $[i] = (q^{i} - q^{-i})/(q - q^{-1})$ for any $i \in \mathbb{N}$.

3.13. When $W$ is of type $\tilde{A}_n$, the highest short weight is $x_1, x_n$.

\[
\eta_{10} = [2]_{q_0} [3]_{q_0} \cdots [n + 1]_{q_0},
\]

where $[i]_{q_0}$ is the specialization at $q_0 \in \mathbb{C}^*$ of $[i]$. By 3.12, in $H_{q_0}$ we have

\[
C_{w_0} C_{r_0 w_0} = [2]_{q_0} [3]_{q_0} \cdots [n - 1]_{q_0} (C_{w_0 x_1 x_n} + [n]_{q_0}^2 C_{w_0}).
\]

Now suppose $[n]_{q_0} = 0$ but $[i]_{q_0} \neq 0$ for $i, 1 \leq i \leq n - 1$, then $\Delta_{q_0} = \{ \{ 1, n \}, \{ 2 \}, \{ 3 \}, \ldots, \{ n - 1 \} \}$. By 3.9 and 3.12 we see that $\alpha_i = 1$ in $\Delta_{q_0}$ if and only if the eigenpolynomial of $s$ has the form

\[
\lambda^{n+1} - a \lambda^i + (-1)^i a^{-1} \lambda + (-1)^{i+1}, \quad a \in \mathbb{C}^*.
\]

In other words, if $[n]_{q_0} = 0$, $[i]_{q_0} \neq 0$, $1 \leq i \leq n - 1$, then $(\phi_{q_0})_{\ast, \tau_{c}}(E_3) = 0$ if and only if the eigenpolynomial of $s$ has the form

\[
\lambda^{n+1} - a \lambda^i + (-1)^i a^{-1} \lambda + (-1)^{i+1}, \quad a \in \mathbb{C}^*.
\]

**4. Examples**

4.1. Type $\tilde{A}_1$. In this case $G = SL_2(\mathbb{C})$, $S = \{ r_0, r_1 \}$, $x_1 = r_0 \omega$, $\Omega = \{ e, \omega \}$, $\eta_{10} = q_0 + q_0^{-1}$. $c_0 = \{ w \in W \mid \ell(w) > 0 \}$. Another two-sided cell $c$ of $W$ is $\Omega$.

$J_i$ has two irreducible modules $F_0, F_i$. Both have dimension 1 and $t_w$ acts on $F_i$ by scalar $(-1)^i$, $i = 0, 1$. Via, $\phi_{q_0} : H_{q_0} \rightarrow J \rightarrow J_c$, $F_i$ becomes $H_{q_0}$-module $F_i_{q_0}$. $T_w$ acts on $F_{i, q_0}$ by scalar $(-1)^i$ and $T_{r_i}$ acts on $F_{i, q_0}$ by scalar $-1$. $(\phi_{q_0})_{\ast, \tau_{c}}$ is an isomorphism for any $q_0 \in \mathbb{C}^*$.
For \( c_0 \), we have \( J_{c_0} = M_{2 \times 2}(R_G) \) and

\[
\begin{align*}
\phi_{q_0, c_0}(C_{r_1}) &= \begin{pmatrix} \eta_{l_0} & V(x_1) \\ 0 & 0 \end{pmatrix}, \\
\phi_{q_0, c_0}(C_{r_2}) &= \begin{pmatrix} 0 & 0 \\ V(x_1) & \eta_{l_0} \end{pmatrix}, \\
\phi_{q_0, c_0}(C_{c_0}) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{align*}
\]

Suppose that \( \eta_{l_0} \neq 0 \). Let \( s \) be the semisimple conjugacy class of \( G \) containing \( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in G \), then \( E_{s, q_0} \) is irreducible if and only if \( \eta_{l_0} \neq \pm (t + t^{-1}) \). When \( \eta_{l_0} = \pm (t + t^{-1}) \), \( E_{s, q_0}/F_{l_0, q_0} \simeq M_{s, q_0} \), where \( i = 0 \) if \( \eta_{l_0} = -(t + t^{-1}) \) and \( i = 1 \) if \( \eta_{l_0} = t + t^{-1} \). \( T_{r_0} \) acts on \( M_{s, q_0} \) by scalar \((-1)^{1-i} \) and \( T_{r_1} \) acts on \( M_{s, q_0} \) by scalar \( q_0^2 \). \( (\phi_{q_0})_{s, c}(E_{s, q_0}) = E_{s, q_0} \) if \( \eta_{l_0} \neq \pm (t + t^{-1}) \), \( (\phi_{q_0})_{s, c}(E_{s, q_0}) = M_{s, q_0} \) if \( \eta_{l_0} = \pm (t + t^{-1}) \). In particular, when \( \eta_{l_0} \neq 0 \), \((\phi_{q_0})_{s, c} \) is an isomorphism.

When \( \eta_{l_0} = 0 \), one verifies that \( E_{s, q_0} \) is irreducible if \( t + t^{-1} \neq 0 \) and \( E_{s, q_0} = F_{0, q_0} \oplus F_{1, q_0} \) if \( t + t^{-1} = 0 \). In particular rank ker \((\phi_{q_0})_{s, c} \) = 1.

4.2. Type \( \tilde{A}_2 \). In this case we have \( G = SL_3(C) \), \( S = \{r_0, r_1, r_2\} \), \( \Omega = \{1, \omega, \omega^2\} \) and \( \omega r_0 = r_1 \), \( \omega r_1 = r_2 \), \( \omega r_2 = r_0 \omega \), \( x_1 = r_0 r_2 \omega \), \( x_2 = r_0 r_1 \omega^2 \). \( W \) has three two-sided cells: \( c = \Omega \), \( c_0 \), \( c' = W \cap c_0 \). \( c' \) is the two-sided cell of \( W \) containing \( r_0, r_1, r_2 \).

It is obviously \((\phi_{q_0})_{s, c} \) is an isomorphism.

Now consider \( J_\omega \). Any element in \( c' \) has one of the following forms: \( \omega^i r_1 x_1^a \omega^j \), \( \omega^{i+1} x_1^a \omega^j \), \( \omega^{i+2} r_2 x_2^j \omega^{i+1} \), \( \omega^{i+1} x_2^j \omega^{i+1} \), \( i, j = 0, 1, 2 \). We define a \( C \)-linear map \( \theta: J_\omega \to M_{3 \times 3}(A) \), \( A = C[q, q^{-1}] \), by \((\theta(w))_{(M_{2x2})} \in M_{3 \times 3}(A) \), \( w \in c' \). Assume that \( w \) is of one of the above forms, then \( m_{ab} = 0 \) except \((a, b) = (i+1, j+1) \) and

\[
m_{i+1,j+1} = \begin{cases}
q^{2a} & \text{if } w = \omega^i r_1 x_1^a \omega^j \\
q^{2a-1} & \text{if } w = \omega^{i+1} x_1^a \omega^j \\
q^{-2a} & \text{if } w = \omega^{i+2} r_2 x_2^j \omega^{i+1} \\
q^{-2a+1} & \text{if } w = \omega^{i+1} x_2^j \omega^{i+1}.
\end{cases}
\]

By [L1, 3.8] we know that \( \theta \) is a \( C \)-algebra isomorphism. We have

\[
\begin{align*}
\theta \phi_{q_0, c}(C_{r_1}) &= \begin{pmatrix} 2q & q^{-1} & q \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\theta \phi_{q_0, c}(C_{c_0}) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\end{align*}
\]

Specialize \( q \) to \( a \in C^* \), we get a simple representation \( \psi_a \) of \( J_\omega = M_{3 \times 3}(A) \) and any simple representation of \( J_\omega \) is isomorphic to some \( \psi_a \), \( a \in C^* \). Let \( E_a \) be a simple \( J_\omega \)-module providing \( \psi_a \).
A little surprisingly, the homomorphism \((\phi_{q_0})^*: K(J_c) \to K(H_{q_0})^c\) is an isomorphism for any \(q_0 \in \mathbb{C}^*\). In fact, via \(\phi_{q_0}: H_{q_0} \to J \to J_c\), \(E_a\) gives rise to an \(H_{q_0}\)-module \(E_{a,q_0}\). One verifies that \(E_{a,q_0}\) has a unique quotient \(M_{a,q_0}\) such that the attached two-sided cell is \(c^c\) and \((\phi_{q_0})^*: E = M_{a,q_0}\); moreover, \(M_{a,q_0}\) is not isomorphic to \(M_{b,q_0}\) whenever \(a \neq b\).

When \(\eta_0 = [2]_{q_0} [3]_{q_0} \neq 0\), \((\phi_{q_0})^*: E = M_{a,q_0}\) is an isomorphism by 3.4. So \((\phi_{q_0})^*\) is an isomorphism. When \([3]_{q_0} = 0\), by 3.9 we see that \((\phi_{q_0})^*: E = 0\) if and only if

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^2
\end{pmatrix} \in s,
\]

here we regard \(\omega\) as a 3-th primitive root of 1 in \(\mathbb{C}\). When \([2]_{q_0} = q_0 + q_0^{-1} = 0\), by 3.13 we see that \((\phi_{q_0})^*: E = 0\) if and only if the eigenpolynomial of \(s\) has the form \(\lambda^3 - a_1 \lambda^2 + a_1^{-1} \lambda - 1\), \(a \in \mathbb{C}^*\).

REFERENCES


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ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE