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**NOTE RECTIFICATIVE D'Y. GUIVARCH  
ET Y. LE JAN CONCERNANT L'ARTICLE**

**ASYMPTOTIC WINDING OF THE GEODESIC FLOW  
ON MODULAR SURFACES  
AND CONTINUOUS FRACTIONS**

(paru dans le tome 26, fascicule 1, 1993, pp. 23-50)

The object of this note is to complete the proof of theorem 2.1 in [G-L]. The main steps of the proof are valid, but the replacement of the discrete coding time used in Proposition 4-3 by the time of the flow should have been done more carefully. We use the notations of [G-L].

We recall that the geodesic flow  $U_t$  on  $\Gamma/G$  has been presented as a special flow over  $\hat{S} = [0, 1]^2 \times \mathbb{Z}/2\mathbb{Z} \times \Gamma_0/\Gamma$ . The transformation  $\hat{\theta}$  on  $\hat{S}$  is an extension of the continuous fraction transformation and the height function  $\Phi$  is explicit. We denote by  $\hat{g}$  the projection of  $g \in G$  on  $\Gamma/G$  and by  $t_{m_k}(\hat{g}) = t_{m_k}(g)$  the sequence of return times to the section  $\hat{S}$ . The following points are essential in the proof.

a) Set  $N_t(g) = \text{Inf}\{k \in \mathbb{N}; t_{m_k}(\hat{g}) > t\}$ ,  $A(\hat{g}) = \int_{\hat{\gamma}_{t_{m_1}}(g)} \omega$ .

One has  $\int_{\hat{\gamma}^t(g)} \omega - \sum_1^{N_t(g)-1} \psi[\hat{\theta}^n \hat{p}(g)] = A(\hat{g}) - A(\hat{g}U_t)$ .

Because the Liouville measure is  $U_t$ -invariant, the laws of  $A(\hat{g})$  and  $A(\hat{g}U_t)$  are the same; hence  $\frac{A(\hat{g})}{t}$  and  $\frac{A(\hat{g}U_t)}{t}$  converge to zero in probability. The case of  $\omega^c$  can be treated in the same way.

b) To reduce Theorem 2.1 to Proposition 4-3, we will use, in addition to Lemma 4.1 (Lemma 4.2 is incorrect) the tightness of  $\frac{N_t(g) - \pi^2 t / 6 \text{Log } 2}{\sqrt{t}}$ , as  $t \uparrow \infty$ .

To show this result, set  $\ell = \pi^2 / 6 \text{Log } 2$  and note that:

$$N_t(g) - [\ell^{-1} t] \geq A\sqrt{t} \quad \text{iff} \quad \tau_{[\ell^{-1} t] + [A\sqrt{t}]}(g) + T(g) \leq t$$

and that:

$$N_t(g) - [\ell^{-1} t] \leq -A\sqrt{t} \quad \text{iff} \quad \tau_{[\ell^{-1} t] - [A\sqrt{t}]}(g) + T(g) \geq t.$$

Hence the result will follow from the tightness of the sequence  $\frac{\tau_n - n\ell}{\sqrt{n}}$ . Set  $(\chi_-^{(n)}, -\chi_+^{(n)}) = \theta^n(\chi_-, \chi_+)$  (N.B.  $\theta(\chi_-, \chi_+) = ((\chi_- + [\chi_+^{-1}])^{-1}, \chi_+^{-1} - [\chi_+^{-1}])$ : a parenthesis was dropped in [G-L].) From the expression of  $\Phi$ , it is enough to prove a central limit theorem for the sequences  $\sum_1^n \text{Log}(\chi_-^{(m)})$  and  $\sum_1^n \text{Log}(\chi_+^{(m)})$  under  $\nu$ .

The law of  $\{\chi_-^{(m)}, m \in \mathbb{N}\}$  is the law of the stationary Markov chain  $x_n$  studied in §5. The law of  $(\chi_+^{(n)}, \dots, \chi_+^{(1)}, \chi_+)$  given that  $\chi_+^{(n)} = x$  coincides with the law of  $(\chi_-, \chi_-^{(1)}, \dots, \chi_-^{(n)})$  given that  $\chi_- = x$ .

Henceforth, it is sufficient to prove the central limit theorem for  $\sum_1^n \text{Log}(x_m)$ . This can be done easily by following the steps given in §5, using the perturbed transfer operator  $Q'_\lambda$  defined by  $Q'_\lambda u(x) = \sum_1^\infty p(x, k) e^{i\lambda \text{Log}(k \cdot x)} u(k \cdot x)$  if we take into account the following remarks (see also [G-H]):

The function  $\text{Log} x$  on  $[0, 1]$  is not Lipchitz as in [G-H] but the key properties of the operators are the same, namely.

- 1)  $Q'_\lambda$  acts on the space of Lipchitz functions on  $[0, 1]$  and satisfies the basic inequality

$$\|Q'_\lambda u\| \leq \frac{3}{8} \|u\| + C|u|_\infty$$

for some constant  $C > 0$ .

- 2)  $Q'_\lambda$  is an analytic family of operators ( $|\lambda| < 1$ ). In particular, if we set

$$D' u(x) = \sum_1^\infty p(x, k) e^{i\lambda \text{Log}(k \cdot x)} u(k \cdot x)$$

there exist a constant  $C' > 0$  such that  $\|Q'_\lambda - Q' - i\lambda D'\| \leq C'|\lambda|^2$ .

In order to get the required inequality for  $\|Q'_\lambda u\|$  one needs to control the Lipchitz coefficient. One has:

$$\begin{aligned} |Q'_\lambda u(x) - Q'_\lambda u(x')| &\leq \sum_1^\infty p(x, k) \left| \frac{1}{(k+x)^{i\lambda}} - \frac{1}{(k+x')^{i\lambda}} \right| |u|_\infty \\ &+ \sum_1^\infty |p(x, k) - p(x', k)| |u(k \cdot x)| + \sum_1^\infty p(x, k) |u(k \cdot x) - u(k \cdot x')|. \end{aligned}$$

Clearly  $\left| \frac{1}{(k+x)^{i\lambda}} - \frac{1}{(k+x')^{i\lambda}} \right| \leq |\lambda| |x - x'| \frac{1}{k}$  hence the first term is bounded by  $|\lambda| |x - x'| |u|_\infty \sum_1^\infty p(x, k) \frac{1}{k} \leq C_1 |\lambda| |u|_\infty |x - x'|$ .

The other terms are bounded by  $[C_2 |u|_\infty + \frac{3}{8} \|u\|] |x - x'|$ .

The analyticity of the family  $Q'_\lambda$  follows from the Cauchy formula on a loop  $\ell$  in the unit disk:

$$\int_\ell Q'_z u dz = \int_\ell \sum_1^\infty p(x, k) \frac{1}{(x+k)^z} u(k \cdot x) dz.$$

If  $|z| < 1 - \varepsilon$ , one has  $|p(x, k) \frac{1}{(x+k)^z}| \leq p(x, k) (k+1)^{1-\varepsilon}$ .

Hence, summation and integration can be exchanged and the result follows from the relation  $\int_{\ell} \frac{1}{(x+k)^z} dz = 0$ .

c) To show that  $\frac{1}{t} \left( \sum_1^{N_t} \psi \circ \hat{\theta}^n - \sum_1^{[\ell^{-1}t]} \psi \circ \hat{\theta}^n \right)$  converges to zero in probability, it is enough to show the convergence to 0 in probability of  $\frac{1}{n} \sum_1^{[A\sqrt{n}]} |\psi \circ \hat{\theta}^m|$  as  $n \uparrow \infty$ . From the expressions of  $\psi$  and  $\hat{\nu}$ , it is clear that  $\int |\psi|^{1-\varepsilon} d\hat{\nu}$  is finite for any  $\varepsilon$  in  $]0, 1[$ . Moreover the inequality  $\left( \sum_1^k |a_i| \right)^\varepsilon \leq \sum_1^k |a_i|^\varepsilon$  ( $0 < \varepsilon < 1$ ) implies:

$$\left( n^{-1} \sum_1^{[A\sqrt{n}]} |\psi \circ \hat{\theta}^m| \right)^{1-\varepsilon} \leq n^{\varepsilon-1} \sum_1^{[A\sqrt{n}]} |\psi \circ \hat{\theta}^m|^{1-\varepsilon}.$$

Hence, for any  $p$  such that  $1 < p < (1-\varepsilon)^{-1}$ , from the triangular inequality in  $L^p$

$$\left\| \left( n^{-1} \sum_1^{[A\sqrt{n}]} |\psi \circ \hat{\theta}^m| \right)^{1-\varepsilon} \right\|_{L^p(\hat{\nu})} \leq [A\sqrt{n}] n^{\varepsilon-1} \left( \int |\psi|^{p(1-\varepsilon)} d\hat{\nu} \right)^{1/p}$$

which converges to zero as  $n \uparrow \infty$  for  $\varepsilon < \frac{1}{2}$ .

This completes the proof of 2.1 a) (A different approach was also given in [LJ], and developed in [E] for a) and b) using Brownian motion instead of the coding.)

d) To show that  $\frac{1}{\sqrt{t}} \left( \sum_1^{N_t} \eta \circ \hat{\theta}^n - \sum_1^{[\ell^{-1}t]} \eta \circ \hat{\theta}^n \right)$  converges to zero in probability, it is enough to show the convergence to 0 in probability of  $\frac{1}{\sqrt{n}} S_{A\sqrt{n}}^*$  with  $S_n = \sum_1^n \eta \circ \hat{\theta}^m$ ,  $S_n^* = \sup_{k \leq n} |S_k|$ .

Since  $\eta$  is bounded and of integral zero, an extension of the classical Kolmogorov inequality is required here, in the case of the Markov chain on  $[0, 1] \times \mathbb{Z}/2\mathbb{Z} \times \Gamma_0/\Gamma$  defined by  $Q$ , and the functional  $S_n = \sum_1^n \eta \circ \hat{\theta}^m$ : there exists a constant  $C > 0$  such that:

$$\tilde{m} \{ S_n^* > b \} \leq C \frac{n}{b^2}.$$

This implies  $\tilde{m} \{ \frac{1}{\sqrt{n}} S_{[A\sqrt{n}]}^* > b \} \leq C \frac{A}{b^2 \sqrt{n}}$ , hence the convergence to zero in probability of  $\frac{1}{\sqrt{n}} S_{[A\sqrt{n}]}^*$ .

Such an inequality is proved in [G] p. 451 Cor. 1, on the basis of the Berry-Essen estimate for  $S_n$ . In turn, this estimate is obtained in [G-H] p. 80 Th. 2. The operator  $Q$  which occurs here satisfies the properties required in [G-H] as follows from Prop. 5.2, p. 36. Such estimates are studied in full detail in [Br] in the context of expanding transformations of the interval, a situation close to the situation considered here.

The proof of 2.1 b) and c) is now completed.

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