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A LIMITING CASE FOR VELOCITY AVERAGING

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ABSTRACT. – We complete the theory of velocity averaging lemmas for transport equations by studying the limiting case of a full space derivative in the source term. Although the compactness of averages does not hold any longer, a specific estimate remains, which shows compactness of averages in more general situations than those previously known. Our method is based on Calderon-Zygmund theory. © Elsevier, Paris

RÉSUMÉ. – Nous complétons les lemmes de moyenne pour les équations de transport en étudiant le cas limite d'une dérivée en espace dans le terme source. La compacité des moyennes ne peut être obtenue, mais nous démontrons une estimation spécifique qui permet de montrer la compacité en moyenne dans les situations les plus générales connues actuellement. Notre méthode s'appuie sur la théorie de Calderon-Zygmund. © Elsevier, Paris

I. Introduction

We consider the regularity properties of averaged quantities like

$$\rho(t,x) = \int f(t,x,v)\varphi(v)dv, \tag{I.1}$$

where φ is a given function and $f : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ solves the transport equation

$$f_t + v \cdot \nabla_x f = \sum_{j=1}^{d} \partial_{x_j} \partial_v^k g_j, \tag{I.2}$$

and k stands for the multiindex $(k_1, k_2, \ldots, k_d)$ of length $|k| = k_1 + k_2 + ... + k_d$, and $\partial_v^k = \partial_{v_1}^{k_1} \cdots \partial_{v_d}^{k_d}$.

Our goal is to study what remains from the velocity-averaging results i.e. improved regularity of ρ compared to f, for the equation (I.2), where the source terms contains a full space derivative, and $f, g_j \in L^p(\mathbb{R}^{2d+1})$ with $p \in (1,\infty)$.

The classical averaging results say quantities like ρ in (I.1) satisfy better estimates, and hence are more regular, than $f(t,x,v)$ itself. This phenomenon was discovered by V. I. Agoshkov [1] and F. Golse, B. Perthame and R. Sentis [11] where applications to the analysis of transport equations were pointed out. The final form of such results was established by F. Golse, P.-L. Lions, B. Perthame and R. Sentis [12] when there are no derivatives in the source term g. Two important steps are subsequently due to R. J. DiPerna and P.-L. Lions [6], [7] (velocity derivatives on g in an L^2 framework), and R. J. DiPerna, P.-L. Lions and Y. Meyer [8] (L^p framework for general sources as in (I.2), but with less than a full derivative in space). The last result was developed in the context of Sobolev spaces by M. Bézard [3] and was proved to be sharp by P.-L. Lions [13]. Applications of the averaging method to the analysis of nonlinear transport equations are numerous; see for example C. Bardos, F. Golse, B. Perthame and R. Sentis [2], R. J. DiPerna and P.-L. Lions [6], [7]. Applications to hyperbolic equations are due to P.-L. Lions, B. Perthame and E. Tadmor [14]. Extensions of averaging compactness to discrete times have been used by L. Desvillettes and S. Mischler [5], and generalized by F. Bouchut and L. Desvillettes [4].

A limiting case of the averaging lemmas was obtained by P. Gérard [9], [10], who showed that if the sequence $\{g_n\}_{n\geq 1}$ is compact in $L^2(\mathbb{R}^{2d+1})$, then the corresponding averages supply a family ρ_n which is compact in $L^2(\mathbb{R}^{d+1})$. Here, we develop a new approach based on a combination of Calderon-Zygmund theory and the classical averaging method in L^2, which explains what remains true when g is merely bounded in L^p.

We denote $C_0^k(B_R)$ the space of C^k functions supported in B_R where k is an integer and B_R is the ball of radius R and center 0 in $\mathbb{R}^d$. Also, p' denotes the conjugate exponent to p. Our result is

THEOREM 1. – *For all* $1 < p < +\infty$, $0 \leq \alpha < \min(\frac{1}{p}, \frac{1}{p'})$ *and* $\varphi \in C_0^{|k|}(B_R)$, *we have,*

$$\|\rho\|_{L^p(\mathbb{R}^{d+1})} \leq C(d,\alpha,p,R,\|\varphi\|_{k,\infty})\|f\|_{L^p(Q_R)}^{1-\frac{\alpha}{|k|+1}}\|\, g\|_{L^p(Q_R)}^{\frac{\alpha}{|k|+1}}, \tag{I.3}$$

with $Q_R = \mathbb{R}_x^d \times \mathbb{R}_t \times B_R$, $\|\varphi\|_{k,\infty} = \|\varphi\|_{L^\infty} + \|D^k\varphi\|_{L^\infty}$ *and* $\|g\|_{L^p(Q_R)} = \sum_{j=1}^d \|g_j\|_{L^p(Q_R)}$.

Remark. – 1. The constant C blows up for $p \to 1$ $(or + \infty)$ and for $\alpha \to 1/p'$ or $(1/p)$ except for $p = 2$, $\alpha = \frac{1}{2}$.

2. The inequality (I.3) is interesting for 'g small', then it improves the obvious inequality

$$\|\rho\|_{L^p(\mathbb{R}^{d+1})} \leq \|\varphi\|_{L^{p'}(\mathbb{R}^d)}\|f\|_{L^p(Q_R)}.$$

3. When $p = 1$, one cannot hope anything better than the trivial inequality $\|\rho\|_{L^1} \leq \|f\|_{L^1}$. But for $p > 1$, the value $\alpha = 1/p'$ is certainly allowed. A possible direction could be the method of [3].

4. A derivative $\partial_t\partial_v^k g_0$ is also possible in the source terms of (I.1), the only difference being that the constants will depend on R with a different homogeneity.

A direct application of the Theorem 1 gives the following compactness theorem,

THEOREM 2. – *Let* f^n *be bounded in* $L^p(Q_R)$ *with* $1 < p < +\infty$, *and* g^n *be relatively compact in* $[L^p(Q_R)]^d$. *If* f^n, g^n *satisfy the equation (I.1) and* $\varphi \in L^{p'}(B_R)$, *then* ρ^n *is relatively compact in* $L^p(\mathbb{R}^{d+1})$.

The outline of the paper is as follows: Theorem 1 is proved in Section II. In Section III, we present some extensions, where $v \cdot \nabla_x$ in (I.1) is replaced by $a(v) \cdot \nabla_x$. This turns out to be a natural structure in several applications (see [14]).

II. The Proofs of Theorems 1 and 2

Our approach differs somewhat of that initiated in [11] and [12] and used afterwards. Here, we do not split the integral (I.2) in two parts depending on the Fourier frequencies. Instead we perturb directly the equation (I.1) to make it elliptic. The perturbation parameter is not only related to Fourier frequencies, as it is done usually, but also to the L^p-norms of f and g.

We divide this section in three subsections. First, we present the method together with technical lemmas. Then, we prove these lemmas in subsections 2 and 3.

II.1. Method of proof

Denoting $\widehat{f}(\tau, \xi, v)$ the Fourier transform of f in the (t, x) variables, the equation (I.1) yields

$$(\tau + v \cdot \xi)\widehat{f} = \sum_{j=1}^{d} \xi_j \partial_v^k \widehat{g}_j,$$

which can be rewritten for $\beta > 0$ as

$$\widehat{f}\,[(\tau + v \cdot \xi)^2 + \beta^2|\xi|^2] = \beta^2|\xi|^2\widehat{f} + \sum_{j=1}^{d} \xi_j(\tau + v \cdot \xi)\partial_v^k \widehat{g}_j.$$

In other words

$$f = f_0 + \sum_{j=1}^{d} f_j \tag{II.1}$$

with

$$\widehat{f_0} = \frac{\beta^2|\xi|^2}{(\tau + v \cdot \xi)^2 + \beta^2|\xi|^2}\widehat{g}_0, \quad \text{where } g_0 = f, \tag{II.2}$$

and, for $1 \leq j \leq d$,

$$\widehat{f_j} = \frac{\xi_j(\tau + v \cdot \xi)}{(\tau + v \cdot \xi)^2 + \beta^2|\xi|^2}\partial_v^k \widehat{g}_j. \tag{II.3}$$

We study separately the operators $(T_j)_{0 \leq j \leq d}$ which are defined by

$$\begin{aligned} \rho_j(t, x) &= \int \varphi(v) f_j(t, x, v)dv \\ &:= T_j g_j(x, t). \end{aligned} \tag{II.4}$$

We need the following two lemmas, which we state below and then prove in Subsections II.2 and II.3.

LEMMA 3. – *Let* $\beta > 0$ *and* $p \in (1, \infty)$. *Then:*

(i) $$\|f_0(\cdot, v)\|_{L^p(\mathbb{R}^{d+1})} \leq C(d,p)\|g_0(\cdot,\cdot,v)\|_{L^p(\mathbb{R}^{d+1})} \quad \textit{for all } v \in \mathbb{R}^d,$$

(ii) $$\|\rho_0\|_{L^2(\mathbb{R}^{d+1})} \leq C(d, R, \|\varphi\|_\infty)\beta^{1/2}\|f\|_{L^2(Q_R)},$$

and thus, for all $0 \leq \alpha < \min\left(\frac{1}{p}, \frac{1}{p'}\right)$,

(iii) $$\|\rho_0\|_{L^p(\mathbb{R}^{d+1})} \leq C(d, \alpha, p, R, \|\varphi\|_\infty)\beta^\alpha\|f\|_{L^p(Q_R)} .$$

LEMMA 4. – *Let* $0 < \beta \leq 1$ *and* $p \in (1, \infty)$. *Then, for all* $1 \leq j \leq d$,

(i) $$\|\rho_j\|_{L^p(\mathbb{R}^{d+1})} \leq C(d, p, R, \|\varphi\|_{k,\infty})\beta^{-|k|-1}\|g_j\|_{L^p(Q_R)},$$

(ii) $$\|\rho_j\|_{L^2(\mathbb{R}^{d+1})} \leq C(d, R, \|\varphi\|_{k,\infty})\beta^{-|k|-\frac{1}{2}}\|g_j\|_{L^2(Q_R)},$$

and thus, for all $0 \leq \alpha < \min\left(\frac{1}{p}, \frac{1}{p'}\right)$,

(iii) $$\|\rho_j\|_{L^p(\mathbb{R}^{d+1})} \leq C(d, \alpha, p, R, \|\varphi\|_{k,\infty})\beta^{\alpha-|k|-1}\|g_j\|_{L^p(Q_R)}.$$

We now continue with the proof of Theorem 1. It is obtained combining these two lemmas. The average in (I.1) is exactly, with the notations (II.4),

$$\rho = \sum_{j=0} \rho_j,$$

and it is upper bounded by (we only use (iii) in the above lemmas)

$$\|\rho\|_{L^p(\mathbb{R}^{d+1})} \leq C(d, \alpha, p, R, \|\varphi\|_{k,\infty})\Big[\beta^\alpha\|g_0\|_{L^p(Q_R)} + \frac{\beta^\alpha}{\beta^{|k|+1}}\|g\|_{L^p(Q_R)}\Big]. \tag{II.5}$$

Next, choosing β as

$$\beta^{|k|+1} = \|g\|_{L^p}/\|f\|_{L^p}, \tag{II.6}$$

we obtain Theorem 1 for $\|g\|_{L^p}/\|f\|_{L^p} \leq 1$. If $\beta \geq 1$, we just use the Remark 2 after Theorem 1. This completes the proof.

II.2. The Proof of Lemma 3

1. We fix $v \in \mathbb{R}^d$, change the variables and define

$$G_0(y, s) = g_0\big(y + v\frac{s}{\beta}, \frac{s}{\beta}\big) \text{ and } F_0(y, s) = f_0\big(y + v\frac{s}{\beta}, \frac{s}{\beta}\big).$$

We have

$$\beta\frac{d}{ds}F_0 = \partial_t f_0 + \xi \cdot \nabla_x f_0,$$

therefore the equation (II.2) for f_0 writes

$$\frac{d^2}{ds^2}F_0 + \Delta_y F_0 = \Delta_y G_0.$$

Hence, denoting $\widehat{G}_0$ and $\widehat{F}_0$ the Fourier transforms in (y, s) of G_0 and F_0, we have

$$\widehat{F}_0(\xi, \sigma) = \frac{|\xi|^2}{|\xi|^2 + \sigma^2} \widehat{G}_0(\xi, \sigma).$$

It then follows from the Calderon-Zygmund Theorem (see E. Stein [15] for instance) that $\|F_0\|_{L^p} \leq C(d, p)\|G_0\|_{L^p}$. This estimate rewritten in variables (x, t) yields the first statement. We also deduce, for later purposes, that for all $q \in (1, +\infty)$,

$$\text{(II.7)} \qquad \begin{aligned} \|T_0 g_0\|_{L^q(\mathbb{R}^{d+1})} &\leq C(d, q) \int \|f_0(x, v, t)\|_{L^q(\mathbb{R}^{d+1})} \varphi(v) dv \\ &\leq C(d, q, \|\varphi\|_\infty) R^{\frac{d}{q'}} \|g_0\|_{L^q(Q_R)}. \end{aligned}$$

2. Using the averaging technique we obtain from (II.2) with $\alpha = \frac{\tau}{\beta|\xi|}$ and $v_1 = \frac{v \cdot \xi}{|\xi|}$,

$$\begin{aligned} |\widehat{\rho}_0(\xi, \tau)|^2 &\leq \int |\widehat{g}_0|^2 dv \int \varphi^2(v) \frac{\beta^4 |\xi|^4}{[(\tau + v \cdot \xi)^2 + \beta^2 |\xi|^2]^2} dv \\ &\leq C(d) \|\varphi\|_\infty^2 R^{d-1} \int \frac{dv_1}{[(\alpha + \frac{v_1}{\beta})^2 + 1]^2} \int |\widehat{g}_0|^2 dv \end{aligned}$$

Since the above integral in v_1 is proportional to β, we obtain (ii), i.e.

$$\text{(II.8)} \qquad \|T_0 g_0\|_{L^2(\mathbb{R}^{d+1})} \leq C(R) \beta^{1/2} \|g_0\|_{L^2(Q_R)}.$$

3. Interpolating (II.7) and (II.8) with

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1 - \theta}{q}, \quad \text{and} \quad \alpha = \frac{\theta}{2},$$

we obtain (iii).

4. Note that $\alpha = 0$ corresponds to $q = p$, $\alpha = \frac{1}{p'}$ corresponds to the (forbidden) cases $q = 1$ and $p < 2$ and $\alpha = \frac{1}{p}$ corresponds to the case $q = +\infty$, $p > 2$. ■

II.3. The Proof of Lemma 4

We begin by the proof of the case $k = 0$. It follows the lines of that of Lemma 3. We change variables and obtain

$$\widehat{F}_j(\xi, \sigma) = \frac{1}{\beta} \frac{\xi_j \sigma}{|\xi|^2 + \sigma^2} \widehat{G}_j(\xi, \sigma),$$

which gives (i) in place of (II.7) because T_j simply scales like $\beta^{-1} T_0$. For the estimate (ii), we have

$$\widehat{\rho}_j(\tau, \xi) = \int \varphi \frac{\xi_j \xi_1}{(\tau + v \cdot \xi)^2 + \beta^2 |\xi|^2} \widehat{g}_j dv,$$

and thus

$$|\widehat{\rho}_j(\tau,\xi)|^2 \leq \frac{C(R)}{\beta^2} \int |\widehat{g}_j|^2 dv \int \frac{|\alpha + v_1/\beta|^2}{[|\alpha + v_1/\beta|^2 + 1]^2} dv_1,$$

and (ii) follows. We obtain (iii) by optimizing the value of β.

2. Next, we prove (i) for $|k| = 1$, for instance $k = (1, 0, \ldots)$. We need a preliminary step. Using Green's formula, we have

$$\widehat{\rho}_j(\tau,\xi) = -\int \frac{\partial \varphi}{\partial v_1} \frac{\xi_j(\tau + v\cdot\xi)}{(\tau + v\cdot\xi)^2 + \beta^2|\xi|^2} \widehat{g}_j dv$$

$$-\int \varphi \frac{\xi_j \xi_1}{(\tau + v\cdot\xi)^2 + \beta^2|\xi|^2} \widehat{g}_j dv + 2\int \varphi \frac{\xi_j(\tau + v\cdot\xi)^2 \xi_1}{[(\tau + v\cdot\xi)^2 + \beta^2|\xi|^2]^2} \widehat{g}_j dv.$$

This defines three operators, S_1, S_2, S_3 for which we may apply the same proof as for (II.7), but with different powers of β. The only scaling factors in β play a role to estimate the different operators S_k. The operator S_1 scales as T_j for $k = 0$ and, as in (II.7), its L^p-norm is $C(R)\beta^{-1}$. The operator S_2 scales like $\beta^{-2}T_0$ and, in place of (II.7), its L^p-norm is $C(R)\beta^{-2}$. Finally S_3 scales like S_2 and has a L^p-norm $C(R)\beta^{-2}$. This gives the point (i) for $|k| = 1$.

3. For other values of k, we always obtain an operator of L^p-norm of order β^{-1} corresponding to the term containing $\frac{\partial^k \varphi}{\partial v^k}$, and a term of L^p-norm of order $\beta^{-|k|-1}$, corresponding to the term containing

$$\frac{\xi_j(\tau + v\cdot\xi)^{|k|+1}\xi^k}{[\cdots]^{|k|+1}}.$$

This last term corresponds to the general cases of (i).

4. When considering the L^2 norms by averaging, we obtain as before that S_1, like T_j for $k = 0$, has L^2-norm of order $C(R)\beta^{-1/2}$ (this gives the dominant term for $\beta > 1$), while the L^2-norms of S_2 and S_3 are of order $C(R)\beta^{-3/2}$ and give the dominant terms for $\beta < 1$. Again, this generalizes to other values of k as indicated above, thus proving (ii). The end of the proof is the same as for Lemma 3.

III. Extensions

In this section we extend the main result to another situation typical to kinetic equations. Using the same notations as before, we consider a solution f to the equation

$$\text{(III.1)} \qquad \frac{\partial f}{\partial t} + a(v)\cdot\nabla_x f = \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \frac{\partial^k g_j}{\partial v^k},$$

where $v \in \mathbb{R}^n$ and $a : \mathbb{R}^n \to \mathbb{R}^d$ is continuous. We also need a nondegeneracy assumption (see the remark below) which we express in the way it comes in the proof.For all $R > 0$, there exists a constant $C_R > 0$ and a $\delta > 0$ such that

$$\text{(III.2)} \qquad \begin{cases} \forall \alpha \in \mathbb{R},\ \forall \chi \in S^{d-1},\ \forall \beta \in (0,1), \\ \int_{|v|\leq R} \left[1 + \left(\alpha + \frac{a(v)\cdot\chi}{\beta}\right)^2\right]^{-1} dv \leq C_R \beta^\delta. \end{cases}$$

Notice that, for $a(v) = v$ and $n = d$, we recover the case of Theorem 1 and (III.2) holds with $\delta = 1$.

We have:

THEOREM 5. – *Assume the assumptions and notations of Theorem 1, and (III.2). Then*

$$\|\rho\|_{L^p(\mathbb{R}^{d+1})} \leq C(d,\alpha,p,\varphi)\|f\|_{L^p(Q_R)}^{1-\frac{\alpha\delta}{\|k\|+1}} \|g\|_{L^p(Q_R)}^{\frac{\alpha\delta}{\|k\|+1}}. \tag{III.3}$$

THEOREM 6. – *Assume, instead of (III.2) that*

$$\sup_{\chi\in S^{\alpha-1},\alpha\in\mathbb{R}} \int_{|v|\leq R} \left[1+\left(\alpha+\frac{a(v)\cdot\chi}{\beta}\right)^2\right]^{-1} dv \underset{\beta\to 0}{\longrightarrow} 0,$$

and let f^n be bounded in $L^p(Q_R)$ for some $p\in(1,+\infty)$ and g^n be relatively compact in $[L^p(Q_R)]^d$. If f^n, g^n satisfy the equation (III.1), then ρ^n is relatively compact in $L^p(\mathbb{R}^{d+1})$.

Remark. – The non-degeneracy assumptions on $a(v)$, could also be expressed more classically in other terms. Arguing as in [12], it is possible to prove that the condition (III.2) is equivalent to

$$\forall\alpha\in\mathbb{R},\ \forall\beta\in(0,1),\ \forall\chi\in S^{d-1},\ \text{meas}\{v;\ |v|\leq R,\ |\alpha+a(v).\chi|\leq\beta\}\leq C_R\beta^\delta$$

for the same value δ as in (III.2) (and such a δ has to be less than 2). While the condition in Theorem 6 is equivalent to

$$\forall\alpha\in\mathbb{R},\ \forall\chi\in S^{d-1},\ \alpha+a(v).\chi\neq 0 \text{ for a.e. } v\in\mathbb{R}^d.$$

We only indicate the main steps of the proof of these theorems. To prove the Theorem 5, we use the same notations as in section II. Again, we only need to treat the case $\|g\|_{L^p}\leq\|f\|_{L^p}$ $(\beta\leq 1)$, the result in the other case being obvious (see the remark 2 after the Theorem 1). First, the estimates (II.7) and (i) of Lemma 4 remain unchanged, since we only need to replace v by $a(v)$. Next, the averaging case is modified to yield

$$\|\rho_0\|_{L^2(\mathbb{R}^{d+1})}\leq C\beta^{\delta/2}\|f\|_{L^2(Q_R)}, \tag{III.4}$$

and

$$\|\rho_j\|_{L^2(\mathbb{R}^{d+1})}\leq C\beta^{\frac{\delta}{2}-|k|-1}\|g_j\|_{L^2(Q_R)}. \tag{III.5}$$

Here, the largest integrals of the type (III.2) are those arising in estimating T_j, $j\geq 1$, for $k=0$ or S_1 for $|k|>0$. Finally, interpolating as before, we obtain

$$\|\rho_0\|_{L^p(\mathbb{R}^{d+1})}\leq C\beta^{\delta\alpha}\|f\|_{L^p(Q_R)}$$

and

$$\|\rho_j\|_{L^p(\mathbb{R}^{d+1})}\leq C\beta^{\delta\alpha-(|k|+1)}\|g\|_{L^p(Q_R)}.$$

The Theorem 5 follows by choosing β as in (II.6).

To prove the Theorem 6, we first reduce it to the case when $g_n\to 0$ in $[L^p(Q_R)]^d$ by substracting the limits to the equation. Then, by choosing β small so that $\|\rho_0^n\|_{L^p(\mathbb{R}^{d+1})}$ is small thanks to the assumption of Theorem 6 and to the inequality corresponding to (III.4). Then, by a choosing n large enough the terms $\|\rho_j^n\|_{L^p(\mathbb{R}^{d+1})}$ can also be made as small as we want (see (III.5)), and the result follows.

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