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On moduli of $G$-bundles of a curve for exceptional $G$

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ON MODULI OF G-BUNDLES
ON A CURVE FOR EXCEPTIONAL G

BY CHRISTOPH SORGER

ABSTRACT. - Let X be a complex, smooth, projective and connected curve, G be a simple and simply connected complex algebraic group and \( \mathcal{M}_{G,X} \) be the stack of G-bundles on X. We show, using the decomposition formulas of Tsuchiya-Ueno-Yamada [T-U-Y] and Faltings [F], the existence of certain line bundles on \( \mathcal{M}_{G,X} \) conjectured in [L-S]. The result is then applied to the question of local factoriality of the coarse moduli space of semi-stable G-bundles. © Elsevier, Paris


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1. Introduction

Let G be a simple and simply connected complex algebraic group. Let \( \mathcal{M}_{G,X} \) be the stack of G-bundles on the smooth connected and projective complex curve X of genus g. If \( \rho : G \to \text{SL}_r \) is a representation of G, denote by \( \mathcal{D}_\rho \) the pullback of the determinant...
bundle [D-N] under the morphism \( \mathcal{M}_{G,X} \to \mathcal{M}_{SL,X} \) defined by extension of the structure group. Associate to \( G \) the number \( d(G) \) and the representation \( \rho(G) \) as follows:

<table>
<thead>
<tr>
<th>Type of ( G )</th>
<th>( A_r )</th>
<th>( B_r ) (( r \geq 3 ))</th>
<th>( C_r )</th>
<th>( D_r ) (( r \geq 4 ))</th>
<th>( E_6 )</th>
<th>( E_7 )</th>
<th>( E_8 )</th>
<th>( F_4 )</th>
<th>( G_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d(G) )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>12</td>
<td>60</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>( \rho(G) )</td>
<td>( \omega_1 )</td>
<td>( \omega_1 )</td>
<td>( \omega_1 )</td>
<td>( \omega_1 )</td>
<td>( \omega_6 )</td>
<td>( \omega_7 )</td>
<td>( \omega_8 )</td>
<td>( \omega_4 )</td>
<td>( \omega_1 )</td>
</tr>
</tbody>
</table>

**THEOREM 1.1.** There is a line bundle \( \mathcal{L} \) on \( \mathcal{M}_{G,X} \) such that \( \text{Pic}(\mathcal{M}_{G,X}) \cong 2\mathcal{L} \). Moreover we may choose \( \mathcal{L} \) in such a way that \( \mathcal{L}^{2 \cdot d(G)} = D_{\rho(G)} \).

The above theorem is proved, for classical \( G \) and \( G_2 \), in [L-S] where it is also proved that the space of sections \( H^0(\mathcal{M}_{G,X}, \mathcal{L}^\ell) \) may be identified to the space of conformal blocks \( B_{G,X}(\ell; p; 0) \) (see (2.2.1) for its definition). Now, once the generator of the Picard group is known in the exceptional cases, this identification is also valid in general, as well as what happens when we additionally consider parabolic structures as we did in [L-S] (Theorems 1.1 and 1.2). The general case had been conjectured by Laszlo and the author [L-S] and figures as a question in Fallings [F] (5.(c)).

There is a topological approach to Theorem 1.1: as suggested in [L-S], in order to prove Theorem 1.1 it is sufficient to show that the group of algebraic morphisms from \( X - \{ p \} \) to \( G \) is simply connected, which would follow from the fact that this group is homotopy equivalent to the group of smooth morphisms from \( X - \{ p \} \) to \( G \). A proof of the last statement, hence of Theorem 1.1, is discussed in [T]. Our proof however, avoids this question and is purely algebraic in nature: the basic idea is not only to identify the space of conformal blocks \( B_{G,X}(\ell; p; 0) \) with sections of \( \mathcal{L}^\ell \) provided that \( \mathcal{L} \) exists, but also to use the space of conformal blocks and its properties as the decomposition formulas of [T-U-Y] and [F] to prove the existence of \( \mathcal{L} \).

Suppose \( g(X) \geq 2 \). For the coarse moduli spaces \( M_{G,X} \) of semi-stable \( G \)-bundles, we will see that the roots of the determinant bundle of Theorem 1.1 do only exist on the open subset of regularly stable \( G \)-bundles. This will allow us to complete the following result of [B-L-S], which was proved there for classical \( G \) and \( G_2 \).

**THEOREM 1.2.** Let \( G \) be semi-simple. Then \( M_{G,X} \) is locally factorial if and only if \( G \) is special in the sense of Serre.

I would like to thank V. Drinfeld for a helpful question on a previous version of this paper and P. Polo for useful discussions on (4.1.1).

## 2. Conformal Blocks

### 2.1. Affine Lie algebras

Let \( \mathfrak{g} \) be a simple finite dimensional Lie algebra of rank \( r \) over \( \mathbb{C} \). Let \( P \) be the weight lattice, \( P_+ \) be the subset of dominant weights and \( (\omega_i)_{i=1,\ldots,r} \) be the fundamental weights. Given a dominant weight \( \lambda \), we denote \( L(\lambda) \) the associated simple \( \mathfrak{g} \)-module with highest weight \( \lambda \). Finally \( (\cdot, \cdot) \) will be the Cartan-Killing form normalized so that for the highest
root \theta we have \((\theta, \theta) = 2\). Let \(Lg = g \otimes_C C((z))\) be the loop algebra of \(g\) and \(\widehat{Lg}\) be the central extension of \(Lg\)

\[(2.1.1) \quad 0 \rightarrow C \rightarrow \widehat{Lg} \rightarrow Lg \rightarrow 0\]

defined by the 2-cocycle \((X \otimes f, Y \otimes g) \mapsto (X, Y) \text{Res}_0(gdf)\).

Fix an integer \(\ell\). Call a representation of \(\widehat{Lg}\) of level \(\ell\) if the center acts by multiplication by \(\ell\). The theory of affine Lie algebras affirms that the irreducible and integrable representations of \(\widehat{Lg}\) are classified by the dominant weights belonging to \(P_+ = \{\lambda \in P_+ | (\lambda, \theta) \leq \ell\}\). For \(\lambda \in P_+\), denote \(\mathcal{H}_\ell(\lambda)\) the associated representation.

### 2.2. Definition of conformal blocks

Fix an integer (the level) \(\ell \geq 0\). Let \((X, p)\) be an \(n\)-pointed smooth and projective curve (we set \(p = (p_1, \ldots, p_n)\)) and suppose that the points are labeled by \(\lambda = (\lambda_1, \ldots, \lambda_n) \in P_+^n\) respectively. Choose an additional point \(q \in X\) and a local coordinate \(z\) at \(p\). Let \(X^* = X \setminus \{p\}\) and \(L_{X^*} g\) be the Lie algebra \(g \otimes O(X^*)\). We have a morphism of Lie algebras \(L_{X^*} g \rightarrow Lg\) by associating to \(X \otimes f\) the element \(X \otimes \hat{f}\), where \(\hat{f}\) is the Laurent development of \(f\) at \(p\). By the residue theorem, the restriction to \(L_{X^*} g\) of the central extension (2.1.1) splits and we may see \(L_{X^*} g\) as a Lie subalgebra of \(Lg\). In particular, the \(\widehat{Lg}\)-module \(\mathcal{H}_\ell(0)\) may be seen as a \(L_{X^*} g\)-module. Moreover, we may consider the \(g\)-modules \(L(\lambda_i)\) as a \(L_{X^*} g\)-modules by evaluation at \(p_i\). The vector space of conformal blocks is defined as follows:

\[(2.2.1) \quad B_{G, X}(\ell; p; \lambda) = [\mathcal{H}_\ell(0) \otimes_C L(\lambda_1) \otimes_C \ldots \otimes_C L(\lambda_n)]_{L_{X^*} g}\]

where \([\cdot]_{L_{X^*} g}\) means that we take co-invariants. It is known ([T-U-Y] or [S], 2.3.5) that the definition of \(B_{G, X}(\ell; p; \lambda)\) may be extended to \(n\)-pointed stable \((X, p)\) and that these vector spaces are finite-dimensional ([T-U-Y] or [S], 2.5.1 for a simple proof). Important properties are as follows:

a) \(\dim B_{G, p_1}(\ell; p_1; 0) = 1\).

b) Upon adding a non-singular point \(q \in X\), spaces \(B_{G, X}(\ell; p; \lambda)\) and \(B_{G, X}(\ell; p; q; \lambda; 0)\) are canonically isomorphic.

c) Suppose \(X\) is singular in \(c\) and let \(\tilde{X} \rightarrow X\) be a partial desingularization of \(c\). Let \(a\) and \(b\) be the points of \(\tilde{X}\) over \(c\). Then there is a canonical isomorphism

\[\bigoplus_{\mu \in P_+} B_{G, \tilde{X}}(\ell; p; a, \lambda, \mu, \mu^*) \cong B_{G, X}(\ell; p; \lambda)\]

### Remark.

- If \(\tilde{X}\) becomes disconnected and \(a \in \tilde{X}'\) and \(b \in \tilde{X}''\) are its connected components, then \(B_{G, \tilde{X}}(\ell; p; a, \lambda, \mu, \mu^*)\) should be understood as the tensor product \(B_{G, \tilde{X}'}(\ell; p'; a, \lambda', \mu) \otimes_{C} B_{G, \tilde{X}''}(\ell; p'', b, \lambda'', \mu^*)\) where \(p'\) and \(p''\) are the points lying on \(\tilde{X}'\) and \(\tilde{X}''\) respectively.

d) The dimension of \(B_{G, X}(\ell; p; \lambda)\) does not change when \((X, p)\) varies in the stack of \(n\)-pointed stable curves \(M_{g, n}\) ([T-U-Y]).
2.3. Application

Let $X$ be a smooth connected curve with one marked point $p \in X$. Using $d)$ and $c)$ and then $b)$ and $a)$ it follows that $B_{G,X}(1;p;0)$ is non trivial, which will be crucial in the proof of Theorem 1.1.

3. The Picard group of $\mathcal{M}_{G,X}$

3.1. The uniformization theorem

Let us recall the description of $\text{Pic}(\mathcal{M}_{G,X})$ of [L-S], which uses as main tool the uniformization theorem which we now recall: let $LG$ be the loop group $G(C((z)))$, seen as an ind-scheme over $\mathbb{C}$, $L^+G$ the sub-group scheme $G(C[[z]])$ and $Q_G = LG/L^+G$ be the infinite Grassmannian, which is a direct limit of projective integral varieties (loc. cit.). Finally let $L_XG$ be the sub-ind group $G(O(X^*))$ of $LG$. The uniformization theorem states that there is a canonical isomorphism of stacks $L_XG \backslash Q_G \cong \mathcal{M}_{G,X}$ and moreover that $Q_G \to \mathcal{M}_{G,X}$ is a $L_XG$-bundle ([L-S], 1.3).

Let $\text{Pic}^{L_XG}(Q_G)$ be the group of $L_XG$-linearized line bundles on $Q_G$. Recall that a $L_XG$-linearization of the line bundle $L$ on $Q_G$ is an isomorphism $m^*L \cong \text{pr}_2^*L$, where $m : L_XG \times Q_G \to Q_G$ is the action of $L_XG$ on $Q_G$, satisfying the usual cocycle condition. It follows from the uniformization theorem that

$$\text{Pic}(\mathcal{M}_{G,X}) \cong \text{Pic}^{L_XG}(Q_G);$$

hence, in order to understand $\text{Pic}(\mathcal{M}_{G,X})$, it suffices to understand $\text{Pic}^{L_XG}(Q_G)$. The Picard group of $Q_G$ itself is infinite cyclic; let us recall how its positive generator may be defined in terms of central extensions of $LG$.

3.2. The Picard group of the infinite Grassmannian

If $\mathcal{H}$ is an (infinite) dimensional vector space over $\mathbb{C}$, we define the $\mathbb{C}$-space $\text{End}(\mathcal{H})$ by $R \mapsto \text{End}(\mathcal{H} \otimes_{\mathbb{C}} R)$, the $\mathbb{C}$-group $\text{GL}(\mathcal{H})$ as the group of its units and $\text{PGL}(\mathcal{H})$ by $\text{GL}(\mathcal{H})/\text{Gm}$. The $\mathbb{C}$-group $LG$ acts on $L\mathfrak{g}$ by the adjoint action which is extended to $\hat{L}\mathfrak{g}$ by the following formula:

$$\text{Ad}(\gamma).(\alpha', s) = (\text{Ad}(\gamma).\alpha', s + \text{Res}_{z=0}(\gamma^{-1}\frac{d}{dz}\gamma, \alpha'))$$

where $\gamma \in LG(R)$, $\alpha = (\alpha', s) \in \hat{L}\mathfrak{g}(R)$ and $(\cdot, \cdot)$ is the $R((z))$-bilinear extension of the Cartan-Killing form. The main tool we use, due to Faltings, is that if $\varphi : \hat{L}\mathfrak{g} \to \text{End}(\mathcal{H})$ is an integral highest weight representation, then for $R$ a $\mathbb{C}$-algebra and $\gamma \in LG(R)$ there is, locally over $\text{Spec}(R)$, an automorphism $u_\gamma$ of $\mathcal{H}_R = \mathcal{H} \otimes_{\mathbb{C}} R$, unique up to $R^*$, such that

$$\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\varphi(\alpha)} & \mathcal{H} \\
\downarrow{u_\gamma} & & \downarrow{u_\gamma} \\
\mathcal{H} & \xrightarrow{\varphi(\text{Ad}(\gamma).\alpha)} & \mathcal{H}
\end{array}$$

is commutative for any $\alpha \in \hat{L}\mathfrak{g}(R)$ ([L-S], Prop. 4.3).
By the above, the representation \( \pi \) may be “integrated” to a (unique) algebraic projective representation of \( LG \), i.e. there is a morphism of \( \mathbb{C} \)-groups \( \pi : LG \to PGL(\mathcal{H}) \) whose derivative coincides with \( \pi \) up to homothety. Indeed, thanks to the unicity property, the automorphisms \( u \) associated locally to \( \gamma \) glue together to define an element \( \pi(\gamma) \in PGL(\mathcal{H})(R) \) and, again because of the unicity property, \( \pi \) defines a morphism of \( \mathbb{C} \)-groups. The assertion on the derivative is consequence of (3.2.1). We apply this to the basic representation \( \mathcal{H}_1(0) \) of \( LG \). Consider the central extension

\[(3.2.2) \quad 1 \longrightarrow G_m \longrightarrow GL(\mathcal{H}_1(0)) \longrightarrow PGL(\mathcal{H}_1(0)) \longrightarrow 1.\]

The pull back of (3.2.2) to \( LG \) defines a central extension to which we refer as the canonical central extension of \( LG \):

\[(3.2.3) \quad 1 \longrightarrow G_m \longrightarrow \widehat{LG} \longrightarrow LG \longrightarrow 1.\]

A basic fact is that the extension (3.2.3) splits canonically over \( L^+G \) ([L-S], 4.9), hence we may define a line bundle on the homogeneous space \( Q_G = \widehat{LG}/L^+G \) via the character \( G_m \times L^+G \to G_m \) defined by the first projection. Then this line bundle generates \( \text{Pic}(Q_G) \) ([L-S], 4.11); we denote its dual by \( \mathcal{O}_{Q_G}(1) \).

### 3.3. Restriction of the canonical central extension to \( LXG \)

By ([L-S], 6.2) the forgetful morphism \( \text{Pic}_{LXG}(Q_G) \to \text{Pic}(Q_G) \) is injective. Recall the Kumar-Narasimhan-Ramanathan lemma ([L-S], 6.8): if \( \rho : G \to \text{SL}_r \) is a representation, then the pullback of the determinant bundle \( D \) to \( Q_G \) under \( Q_G \to \mathcal{M}_{\text{SL}_r,X} \) is \( \mathcal{O}_{Q_G}(d_{\rho}) \), where \( d_{\rho} \) is the Dynkin index of \( \rho \) [D]. As \( \gcd(d_{\rho}) \) is \( d(g) \) when \( \rho \) runs over all (finite dimensional) representations of \( g \) ([L-S], 2.6), proving theorem 1.1 is equivalent to showing that there is a \( LXG \)-linearization of \( \mathcal{O}_{Q_G}(1) \). This in turn is equivalent ([L-S], 6.4) to the splitting of the central extension (3.2.3) when restricted to \( LXG \) and so, the proof is complete once we know the following.

**Proposition.** – The restriction of the central extension (3.2.3) to \( LXG \) splits.

**Proof.** – Let \( B = B_{G,X}(\ell'; p; 0) \). We know from (2.3) that \( B \neq 0 \). Remark that the commutativity of (3.2.1) implies that for \( \gamma \in LXG(R) \) the associated automorphism \( u_\gamma \) of \( \mathcal{H} \) maps coinvariants to coinvariants. We get a morphism of \( \mathbb{C} \)-groups \( \pi : LXG \to PGL(B) \) and so we may consider the diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & G_m & \longrightarrow & \widehat{LXG} & \longrightarrow & LXG & \longrightarrow & 1 \\
\| & & & & & & & & \\
1 & \longrightarrow & G_m & \longrightarrow & GL(B) & \longrightarrow & PGL(B) & \longrightarrow & 1
\end{array}
\]

By construction, the central extension of \( LXG \) above coincides with the central extension obtained by restriction of (3.2.3) to \( LXG \). By definition of \( B \), the derivative of \( \pi \) is trivial. As \( LXG \) is an integral ind-group ([L-S], 5.1) it follows that \( \pi \) has to be the constant map with value the identity. Indeed, write \( LXG \) as the direct limit of integral schemes \( V_n \) and remark that \( \pi \) has to be constant on \( V_n \); for large \( n \), as \( V_n \) contains 1, this constant is \( \pi(1) = 1 \). So \( \pi \) being the identity, \( \pi \) factors through \( G_m \) which gives the desired splitting. 

\[\Box\]
4. Proof of theorem 1.2

According to ([B-L-S], §13) it remains to prove that the coarse moduli space \( M_{G,X} \) of semi-stable \( G \)-bundles is not locally factorial for \( G = F_4, E_6, E_7 \) or \( E_8 \). Let \( M_{G,X}^{\text{reg}} \) be the open subset of \( M_{G,X} \) corresponding to regularly stable \( G \)-bundles \( E \) (i.e. \( E \) is stable and \( \text{Aut}(E) = Z(G) \)). Denote by \( \text{Cl} \) the group of Weil divisor classes. We have a commutative diagram, with \( r_* \) the restriction, \( c \) and \( c_{\text{reg}} \) the canonical, and \( f \) and \( f_{\text{reg}} \) the forgetful morphisms:

\[
\begin{array}{cccc}
\text{Pic}(M_{G,X}) & r_1 & \text{Pic}(M_{G,X}^{\text{reg}}) & f_* \text{Pic}(M_{G,X}^{\text{reg}}) \\
\downarrow & & \downarrow r_2 & \downarrow r_3 \\
\text{Pic}(M_{G,X}^{\text{reg}}) & c_{\text{reg}} \text{Pic}(M_{G,X}^{\text{reg}}) & \text{Cl}(M_{G,X}) & \end{array}
\]

It is known (see [L-S], 9.2 and 9.3) that \( r_3 \) is injective (normality of \( M_{G,X} \)), that \( c_{\text{reg}} \) is an isomorphism (smoothness of \( M_{G,X}^{\text{reg}} \)) and that \( r_4 \) is an isomorphism (the complement of \( M_{G,X}^{\text{reg}} \) in \( M_{G,X} \) is of codimension \( \geq 2 \)). So in order to prove that \( M_{G,X} \) is not locally factorial, it is sufficient to show that \( r_3 \) is not surjective. In order to see this, we will consider the generator \( L \) of theorem 1.1. Indeed, there is an element \( L' \) of \( \text{Pic}(M_{G,X}^{\text{reg}}) \) such that \( f_{\text{reg}}(L') = r_2 \circ r_1(L) \); as the center \( Z(G) \) of \( G \) acts trivially on \( H_1(0) \), the restriction of \( O_G \) to \( Q_{G}^{\text{reg}} \) is \( L \times G/Z(G) \)-linearized, hence descends to a line bundle \( L' \) to \( M_{G,X}^{\text{reg}} \) (use that \( Q_{G}^{\text{reg}} \rightarrow M_{G,X}^{\text{reg}} \) is a \( L \times G/Z(G) \)-bundle). On the other hand, \( L' \) cannot be in the image of \( r_3 \). Let us suppose the contrary. Then there is a line bundle \( L'' \) such that \( f^*(L'') = r_1(L) \). Now consider the well known tower of inclusions

\[
\text{Spin}_8 \stackrel{\alpha}{\rightarrow} F_4 \stackrel{\beta}{\rightarrow} E_6 \stackrel{\gamma}{\rightarrow} E_7 \stackrel{\delta}{\rightarrow} E_8.
\]

An easy calculation (using for example [Sl], tables 77-128) shows that the restriction of the representation \( L(\omega_8) \) of \( E_8 \) to \( \text{Spin}_8 \) is \( 28L(0) \oplus 8L(\omega_1) \oplus L(\omega_2) \oplus 8L(\omega_3) \oplus 8L(\omega_4) \), hence has Dynkin index 60, since \( d(\omega_i) = 2 \) for \( i = 1,3,4 \) and \( d(\omega_2) = 12 \) ([L-S], 2.6). It follows from Theorem 1.1 (and the discussion in 3.3) that \( L \) pulls back to the (positive) generator \( P \) of \( \text{Pic}(M_{\text{Spin}_8}) \), which is the pfaffian line bundle of [L-S]. The pullback of \( L'' \) then defines a line bundle \( P'' \) on \( M_{\text{Spin}_8} \) such that \( f^*(P'') = P \). But this is a contradiction, as the pfaffian line bundle does not descend to the coarse moduli space \( M_{\text{Spin}_8} \) ([B-L-S], 8.2).

\[\square\]

REFERENCES


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