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ERGODICITY AND RIGIDITY FOR CERTAIN SUBGROUPS OF $\mathrm{Diff}^{\omega}(S^1)$

BY JULIO C. REBELO*

ABSTRACT. – We consider the non solvable subgroups of the group of real analytic diffeomorphisms of the circle which admit a finite generating set whose elements belong to an appropriate and fixed neighborhood of the identity. If G is such a group, we prove that there are non trivial local analytic vector fields which are a sort of "limit" of some local diffeomorphisms in G. Finally we apply these vector fields to prove, in particular, that either the group G is ergodic or it has a finite orbit. These vector fields also enable us to show that the dynamics of G is topologically rigid. © Elsevier, Paris

RÉSUMÉ. – Nous considérons les sous-groupes non résolubles du groupe des difféomorphismes analytiques réels du cercle qui admettent un ensemble fini de générateurs dont les éléments appartiennent à un voisinage adéquat fixé de l'identité. Si G est un tel groupe, nous montrons que localement il existe des champs de vecteurs analytiques réels, non triviaux, qui sont une sorte de "limite" de certains difféomorphismes locaux de G. Finalement nous utilisons ces champs de vecteurs pour démontrer, en particulier, que le groupe G est ergodique sauf s'il a une orbite finie. Ces champs de vecteurs nous permettent aussi de montrer que la dynamique de G est topologiquement rigide. © Elsevier, Paris

1. Introduction

Let $\mathrm{Diff}^{\omega}(S^1)$ be the group of orientation preserving real analytic diffeomorphisms of the circle. Among the subgroups of $\mathrm{Diff}^{\omega}(S^1)$ two kinds of examples have been more intensively studied: discrete subgroups of $\mathrm{PSL}(2, \mathbf{R})$ acting on S^1 (identified to the real projective line) and subgroups generated by diffeomorphisms which are close to the identity. The former examples arise mostly from problems related to Riemannian Geometry (especially from the Hyperbolic one) and they have inspired a very well developed theory. Indeed a clear picture of the dynamics associated to such actions is known today (see [Su] for an account). Furthermore, these actions have interesting rigidity properties concerning the degree of differentiability for conjugacies (cf. [Gh-2] and references therein). On the other hand, groups generated by diffeomorphisms close to the identity are intimately linked to many examples of codimension 1 foliations (cf. [Bo]). Nevertheless the dynamics of these subgroups still have many points that need to be elucidated. Even though an unpublished theorem of Duminy asserts that either every orbit for such a group is everywhere dense or the group has a finite orbit, which is obviously a remarkable topological feature, one does

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not know the structure of these dynamics in a measure-theoretic level as well as whether it is possible to construct deformations with a fixed topological or differentiable type.

In this paper we set up a theory tackling both questions in the real analytic context. In what follows we shall deal with orientation preserving diffeomorphisms; however, the generalization of the results presented here to the more general case is completely straightforward. One of the main results in our work is an analogous, in the measure theory viewpoint, of Duminy's theorem whose analytic version can be found in [Gh-1]. This result is stated as:

Theorem A: *There exists an open neighborhood $\mathcal{U}$ of the identity in the real analytic topology (cf. section 2) such that, if $G \subseteq \mathrm{Diff}^{\omega}(S^1)$ is a non solvable group admitting a (finite) set of generators whose elements belong to $\mathcal{U}$, then one of two possibilities holds:*

$\imath$) G acts in a ergodic way with respect to the Lebesgue measure of S^1.

$\imath\imath$) G has a finite orbit. Besides there is a finite set $P \subset S^1$ such that the (restriction of the) action to the union of the images by G of each connected component in the complement of P is ergodic with respect to the restriction of the Lebesgue measure.

Alternatively, item $\imath\imath$) of Theorem A can be stated as:

$\imath\imath'$) G has a finite orbit. Besides there is a finite set $P \subset S^1$ with the following property: to each connected component of $S^1 \setminus P$, it corresponds a subgroup of G which fixes this component and whose action on it is ergodic with respect to the restriction of the Lebesgue measure.

The technical device on which is based the proof of Theorem A is the existence of some locally defined vector fields. These vector fields may be seen as a generalization of those constructed by Nakai in his paper [Na-1]. To the best of my knowledge the idea of associating (in a non trivial way) vector fields to the dynamics of certain groups is due to A. Scherbakov who considered subgroups of $\mathrm{Diff}(\mathbf{C}, 0)$ (the group of germs of holomorphic diffeomorphisms fixing $0 \in \mathbf{C}$, see [Sch]). Some of these results were later rediscovered and accurately stated by Nakai. We present below a simplified version of our main theorem concerning existence of vector fields having "special" features.

Theorem B: *There is an open neighborhood $\mathcal{U}$ of identity in $\mathrm{Diff}^{\omega}(S^1)$ with the following property: if G is a non solvable subgroup of $\mathrm{Diff}^{\omega}(S^1)$ and G admits a finite set of generators contained in $\mathcal{U}$, then there exists a (local and nowhere zero) vector field $\mathcal{X}$ defined in a neighborhood of any point $p \in S^1$, provided that p is not a periodic point for G. Yet this vector field $\mathcal{X}$ defines a local flow which is contained in the C^{∞}-closure of the restriction of elements in G to this neighborhood.*

It should be observed that if G is as above and p is a periodic point for G then the *stabilizer* of p is non solvable and thus Nakai's theorem can be applied to describe the local dynamics of G around p. On the other hand, for a "generic" G as above, the stabilizer of any point $p \in S^1$ is solvable (in particular such a G has no periodic points) so that Nakai's theorem says nothing about its dynamics.

The statement of Theorem B needs further comments to explain in a little rough way its meaning. Let $I \subseteq S^1$ be a connected open interval endowed with a real analytic vector field $\mathcal{X}$. One denotes by ϕ^t the local flow generated by $\mathcal{X}$ on I. We say that $\mathcal{X}$ is (a local) vector field in the C^{∞}-closure of the restriction of elements in G to I if, and only if, for every open (relatively compact) interval $I_0 \subset I$ and $t_0 \in \mathbf{R}$ for which the map ϕ^{t_0} is defined on I_0, this map is a C^{∞}-limit (of the restriction to I_0) of diffeomorphisms in G.

This definition may seem complicated and technical at the first glance, however it is very natural and useful (as testified by Theorems A and C below). In section 4, we shall give more accurate definitions about these vector fields (using the notion of pseudogroup) as well as a stronger version of the statement related to its existence and regularity.

As it happens in Nakai's paper, this type of vector fields, in addition to enabling us to prove Theorem A, becomes a useful tool for studying topological conjugacies between two such groups. The last question is settled by Theorem C which implies in particular the topological invariance of Godbillon-Vey class for some type of foliated S^1-bundles (see Corollary (5.4)):

Theorem C: *Let $\mathcal{U}$ be a neighborhood of the identity in the real analytic topology as in the statement of Theorem B. Let G_1 and G_2 be non solvable subgroups of* $\mathrm{Diff}^{\omega}(S^1)$ *respectively generated by $\{f_{11}, \ldots, f_{1k}\}$ and $\{f_{21}, \ldots, f_{2k}\}$. Assume that all the f_{ij} ($i = 1, 2$ and $j = 1, \ldots, k$) belong to $\mathcal{U}$. If u is a homeomorphism which conjugates the actions of G_1 and G_2 (i.e. $f_{2j} = u^{-1} \circ f_{1j} \circ u$) then u is in fact a real analytic diffeomorphism of S^1.*

Let $\pi_1(\Sigma_g)$ be the fundamental group of the oriented surface with genus $g \geq 2$. It is known that any two cocompact embedding of $\pi_1(\Sigma_g)$ in $\mathrm{PSL}(2, \mathbf{R})$ are topologically conjugated. Moreover they are C^1-conjugated if and only if they are conjugated in $\mathrm{PSL}(2, \mathbf{R})$ (s.t. if they define the same point in the Teichmüller space, see the references in [Gh-2]).

The examples above show that the main assumption of Theorem C, namely the fact that our groups admit a finite set of generators close to the identity, cannot be dropped.

In an incoming paper, we shall extend our main construction of vector fields in order to study measurable conjugacies between two such groups.

It is my pleasant duty to thank Etienne Ghys who motivated me to study this problem and whose work [Gh-1] is at the origin of the present work. I am also grateful to R.S. Mol who read a first draft of this article.

2. Some background on circle diffeomorphisms

In this section we will gather some foundation material which will be used in the remaining sections.

2.1. Diffeomorphisms close to the identity

First, let us introduce the (real) analytic topology. We consider a real analytic embedding of S^1 into some euclidean space $\mathbf{R}^N$. A *complexification* of S^1 is a *non compact* Riemann surface $\tilde{S} \subseteq \mathbf{C}^N$ such that any real analytic function defined on S^1 can be extended to a unique germ of holomorphic function defined in a neighborhood of S^1 in $\tilde{S}$. Two complexifications of S^1 agree in a neighborhood of S^1, so we fix hereafter one of them.

For $\tau > 0$, one denotes $\tilde{S}_\tau$ the set of points in $\tilde{S}$ whose distance (the euclidean distance of $\mathbf{C}^N$) to S^1 is smaller than τ. We consider a real analytic function $u : S^1 \to \mathbf{R}^l$ from S^1 into $\mathbf{R}^l$. If there exists a holomorphic extension $\tilde{u}_\tau : \tilde{S}_\tau \to \mathbf{C}^l$ of u, we denote by

$\| u \|_\tau$ the supremun of the norm $\| \tilde{u}(z) \|$ with $z \in \tilde{S}_\tau$. If there is no such extension, we put $\| u \|_\tau = \infty$.

For $\tau > 0$ and $\varepsilon > 0$, we define $\mathcal{U}_\tau^\varepsilon$ as the set given by real analytic diffeomorphisms $f : S^1 \to S^1 \subset \mathbf{R}^N$ verifying $\| f - id \|_\tau < \varepsilon$. Now we are able to introduce a topology on $\text{Diff}^\omega(S^1)$ called the *real analytic topology* (or only analytic topology) simply declaring that:

$\imath$) the sets $\mathcal{U}_\tau^\varepsilon$, $\tau > 0$ and $\varepsilon > 0$, define a base of neighborhoods for the identity.

$\imath\imath$) a base of neighborhoods for an arbitrary element $f \in \text{Diff}^\omega(S^1)$ is obtained by a translation of the identity's one.

It turns out that this topology does not depend on the embedding chosen at the beginning. Therefore it is a well defined and *intrinsic* topology on $\text{Diff}^\omega(S^1)$. When endowed with its analytic topology, $\text{Diff}^\omega(S^1)$ supports a unique structure of topological group. This topological group shares an important property with finite dimensional Lie groups namely, the convergence of commutators. Before discussing this question it is worth getting some knowledge on solvable subgroups of $\text{Diff}^\omega(S^1)$.

Let $\mathcal{G}$ be an abstract group. For $a, b \in \mathcal{G}$, the symbol $[a, b]$ denotes the *commutator* of a and b defined by $[a, b] = aba^{-1}b^{-1}$. In the same way, an element $c \in \mathcal{G}$ is called *a commutator* if there are elements a and b in $\mathcal{G}$ such that $c = [a, b]$. The smallest subgroup of $\mathcal{G}$ containing *all commutators* in $\mathcal{G}$ is denoted $D^1\mathcal{G}$ and is called the *first derived group* of $\mathcal{G}$. One associates to $\mathcal{G}$ a sequence $D^k\mathcal{G}$ $(k = 1, 2, \ldots)$ called the (descending) *derived series* of $\mathcal{G}$, by means of the following recurrence relation: $D^k\mathcal{G} = D^1(D^{k-1}\mathcal{G})$ (provided that $k \geq 2$).

The group $\mathcal{G}$ is called *solvable* if there exists $k \in \mathbf{N}^*$ for which $D^k\mathcal{G} = \{id\}$.

Going back to $\text{Diff}^\omega(S^1)$, solvable subgroups of $\text{Diff}^\omega(S^1)$ are studied in particular in [Gh-1]. In this article the author shows that it is possible to decide whether a given subgroup of $\text{Diff}^\omega(S^1)$ is solvable from the commutators of a set of generators for this subgroup. Let us clarify the precise meaning of this claim and also deduce some consequences.

We assume that $G \subseteq \text{Diff}^\omega(S^1)$ is a subgroup of $\text{Diff}^\omega(S^1)$ generated by the set $S = \{f_1, \ldots, f_m\} \subset \text{Diff}^\omega(S^1)$. According to [Gh-1], let us associate to S a sequence of sets $S(k) \subseteq G$ as follows:

$\imath$) $S(0) = S$,

$\imath\imath$) $S(k+1)$ is the set whose elements are the commutators written under the form $[g^{\pm 1}, h^{\pm 1}]$ where $g \in S(k)$ and $h \in S(k) \cup S(k-1)$ ($h \in S(0)$ if $k = 0$).

The group G is said *pseudo-solvable* if it is possible to find a (finite) set of generators S for which $S(k)$ is reduced to the identity for some $k \in \mathbf{N}^*$. The proposition below asserts that the notion of pseudo-solvability coincides with the ordinary notion of solvability in $\text{Diff}^\omega(S^1)$.

Proposition 2.1 ([Gh-1]). – *If G is a subgroup of* $\text{Diff}^\omega(S^1)$, *then the following are equivalent:*

a) G is pseudo-solvable;

b) G is solvable;

c) G is metabelian (i.e. $D^2G = \{id\}$).

Now we return to the convergence of commutators. One of the main theorems in [Gh-1] claims the existence of a neighborhood $\mathcal{U}$ for the identity (in the analytic topology) where iterations of commutators converge to the identity. It means that if S is a subset of $G \cap \mathcal{U}$ then the elements of the sequence $S(k)$ converge to the identity when k goes to infinity. Comparing this theorem with the proposition above, one deduces the following theorem (explicitly stated in [Gh-1]) which is adapted to our interests:

THEOREM 2.2 (E. Ghys [Gh1]). – *We consider S^1 as the unit circle contained in $\mathbf{C}$. For every real number τ verifying $0 < \tau < 1/2$, there exists a neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}^{\omega}(S^1)$ such that, if G is generated by a (finite) set S of diffeomorphisms in $\mathcal{U}$, one has the alternatives:*

ı) G is solvable;

ıı) There exists a sequence $\{h_i\}_{i\in\mathbf{N}}$ ($h_i \neq id$ for all i) of diffeomorphisms h_i in G such that each h_i has a holomorphic extension to the annulus $\{z \in \mathbf{C}\,;\, 1-\tau <\mid z \mid< 1+\tau\}$, moreover these extensions converge uniformly to the identity on this annulus. Finally, the h_i are obtained as elements of $S(k)$.

We complete this paragraph with Proposition (2.3) below that is a particular case of a theorem in [Gh-1].

PROPOSITION 2.3. – *Let G be a non solvable subgroup of $\mathrm{Diff}^{\omega}(S^1)$ generated by a set $S = \{f_1, \ldots, f_l\}$. Assume that S is contained in the neighborhood $\mathcal{U}$ of identity in $\mathrm{Diff}^{\omega}(S^1)$ mentioned above. Then the union of all finite orbits of G is a finite set denoted by P. Moreover, if $p \in S^1$ does not belong to P, then its orbit is everywhere dense in the connected component of $S^1 \setminus P$ containing p.*

It results from Proposition (2.3) that such a group cannot exhibit a minimal set which is a Cantor set. Furthermore, if the group has no finite orbit, then every orbit is dense in S^1.

2.2. Germs of biholomorphisms in $(\mathbf{C}, 0)$

Let $\mathrm{Diff}(\mathbf{C}, 0)$ be the group of all germs of holomorphic diffeomorphisms (biholomorphisms) fixing $0 \in \mathbf{C}$. If a diffeomorphism $f \in \mathrm{Diff}^{\omega}(S^1)$ has a fixed point, it projects (in a neighborhood of this fixed point) onto an element of $\mathrm{Diff}(\mathbf{C}, 0)$ (well defined up to conjugacies). Actually this projection takes place onto $\mathrm{Diff}_{\mathbf{R}}(\mathbf{C}, 0)$, the subgroup of $\mathrm{Diff}(\mathbf{C}, 0)$ defined by germs of holomorphic diffeomorphisms which preserve the real line (equivalently the germs of holomorphic diffeomorphisms for which the respective Taylor's series have real coefficients). This observation leads us to consider some facts about $\mathrm{Diff}_{\mathbf{R}}(\mathbf{C}, 0)$ and its subgroups.

The foremost goal of this paragraph is to state a result obtained by I. Nakai in [Na-1]. We will also give a very rough outline of the corresponding proof since some similar ideas are going to be used in the sequel (fortunately the hardest technical part of Nakai's proof will not be considered in our discussion, concerning this part just the statement of his theorem will be enough for our applications).

Let us consider a germ of holomorphic diffeomorphism f which can be written under the form $f(z) = z + cz^n + \ldots$ ($c \neq 0$). The topological picture described by the local dynamics of f is called a "flower" ([Ca]): there are points $z \in \mathbf{C}$ such that $f^n(z)$ is defined for every $n \in \mathbf{Z}$ and furthermore $f^n(z)$ converges toward the origin when n (or $-n$) goes to infinity. These points form the "petals" of the "flower". Assuming that f has real coefficients and $c < 0$ (so that $f(x) < x$ for small $x \in \mathbf{R}_+^*$), the following well known lemma holds (cf. [Ca]):

LEMMA 2.4. – *Every sufficiently small positive real number a, is contained in an open neighborhood $V \subset \mathbf{C}$ verifying:*

1. *$f^n(z)$ is defined for every $z \in V$ and $n \in \mathbf{N}$. Moreover $f^n(z)$ converges to the origin when n converges to infinity for any $z \in V$.*
2. *The mapping $f^n : V \to f^n(V)$ is a holomorphic diffeomorphism of V onto its image $f^n(V) \subset \mathbf{C}$.*

Now we turn our attention to non solvable subgroups of $\mathrm{Diff}(\mathbf{C}, 0)$. Indeed it is more convenient to work with pseudogroups. So we let Γ be the pseudogroup generated by diffeomorphisms f and g (both defined in a neighborhood of $0 \in \mathbf{C}$). We assume Γ is non solvable (it means that the germ of Γ is non solvable), so Γ contains elements (also denoted by f and g) that can be written in Taylor's expansion as

$$f(z) = z + az^{i+1} + \cdots \quad , \quad g(z) = z + bz^{j+1} + \cdots \quad (a, b \neq 0 \,\mathrm{and}\, i < j).$$

Furthermore, one has

$$[f, g](z) = f \circ g \circ f^{-1} \circ g^{-1}(z) = z + cz^{l+1} \cdots \quad (c \neq 0 \,\mathrm{and}\, j < l)\,.$$

For $h \in \mathrm{Diff}(\mathbf{C}, 0)$, $\mathcal{B}_h$ denotes the *basin* of h, that is the set of the points z in the domain of h for which $h^n(z)$ converges to $0 \in \mathbf{C}$ when n goes to infinity. We remark for any $h \in \mathrm{Diff}(\mathbf{C}, 0)$, $\mathcal{B}_h \cup \mathcal{B}_{h^{-1}}$ is a neighborhood of $0 \in \mathbf{C}$.

In his paper [Na-1], Nakai showed the existence of some holomorphic vector field $\mathcal{X} = \mathcal{X}(f, g)$ defined on $\mathcal{B}_f \setminus \{0\}$ which is a (uniform) limit of sequence $\lambda_n(f^{-n} g f^n - id)$ for a suitable sequence of real numbers λ_n. In the same way, it can be defined, on $\mathcal{B}_{f^{-1}}$, another vector field $\mathcal{X}_- = \mathcal{X}(f^{-1}, g)$ as the limit of the sequence $\lambda'_n(f^n g f^{-n} - id)$ (for suitable λ'_n). Finally replacing f by g and g by $[f, g]$, we still have vector fields $\mathcal{Z}$ and $\mathcal{Z}_-$, defined respectively on $\mathcal{B}_g \setminus \{0\}$ and $\mathcal{B}_{g^{-1}} \setminus \{0\}$ verifying analogous properties. The vector fields $\mathcal{X}$, X_-, $\mathcal{Z}$ and $\mathcal{Z}_-$ will be referred to for short as Nakai's vector fields. He proved in particular the following theorem:

THEOREM 2.5 (I. Nakai [Na-1]). – *Let V be a relatively compact subset of $\mathcal{B}_f \setminus \{0\}$ (resp. $\mathcal{B}_g \setminus \{0\}$). If $t_0 \in \mathbf{R}$ and* $\exp t\mathcal{X}$ *(resp.* $\exp t\mathcal{Z}$*) is defined on V for all $0 \leq t \leq t_0$, then the local diffeomorphisms* $\exp t_0\mathcal{X}$ *(resp.* $\exp t_0\mathcal{Z}$*) (from V onto its image) is a uniform limit in V of maps $f^{-n} g^{l_n} f^n$ (resp. $g^{-n}[f, g]^{l'_n} g^n$) for an appropriate sequence of integers l_n (resp. l'_n) in the pseudogroup generated by f and g. In fact these sequences of integers are given by l_n such that $l_n n^{(i-j)/i} \to t_0$ and $l'_n n^{(j-l)/j} \to t_0$.*

3. A construction of a local vector field

In the sequel, we are going to carry out the fundamental construction of a local vector field associated to the dynamics of a special kind of pseudogroups. This vector field corresponds of course to the one announced in the introduction. Actually, we shall make a number of assumptions concerning a pseudogroup in order to settle its existence. In the next section, we will see that a group as in the statement of Theorems A and C always contains (in a natural way) pseudogroups verifying these assumptions. In any case, I believe this construction for pseudogroups is interesting in itself and it may be helpful in other problems related to the theory of real analytic codimension 1 foliations.

So we set B the unit ball centered at $0 \in \mathbf{C}$. Let Γ be a pseudogroup of holomorphic mappings from open subsets of B onto its image in $\mathbf{C}$ which preserves the real line (i.e. for any $g \in \Gamma$, $g(U \cap \mathbf{R}) \subseteq \mathbf{R}$, where U is the domain of g). We make the following assumptions:

1. There exists $f \in \Gamma$, whose domain of definition contains B, verifying $f(0) = 0$ and $\mid f'(0) \mid \neq 1$.
2. There exists a sequence $\mathcal{H} = \{h_i\}_{i \in \mathbf{N}}$ of elements in Γ ($h_i \neq id$ for all i) such that each h_i is defined on B and the sequence converges uniformly on B to the identity map.
3. For all $h_i \in \mathcal{H}$, one has $h_i(0) > 0$.

Let us put $f'(0) = \lambda$. Replacing f by f^{-1} if necessary, we may suppose $0 < \lambda < 1$. Thus Poincare's Linearization Theorem guarantees that f is conjugated to the homothety $z \mapsto \lambda z$ in some small disc around $0 \in \mathbf{C}$. Since we can perform a change of coordinates followed by a rescaling, we are able to suppose $f(z) = \lambda z$ in the whole B ($0 < \lambda < 1$).

We denote by D_λ (resp. $D_{\lambda/2}$) the disc centered at $(\lambda + \lambda^2)/2$ whose radius is $(\lambda - \lambda^2)/2$ (resp. $(\lambda - \lambda^2)/4$). The main result of the present section is the proposition below:

PROPOSITION 3.1. – *Under assumptions 1, 2 and 3 above and provided that $f(z) = \lambda z$ on B, there exists a nowhere zero real analytic vector field $\mathcal{X}$ on $D_{\lambda/2}$ verifying:*

1. *If Φ^t designates the local real flow arised from $\mathcal{X}$, then, for any open set $U \subset D_{\lambda/2}$ and real number t_0 for which $\Phi^t(U) \subset D_{\lambda/2}$ whenever $0 \leq t < t_0$, the map $\Phi^{t_0} : U \to \Phi^{t_0}(U)$ is a uniform limit (in U) of maps $g_i : U \to g_i(U)$ belonging to Γ.*
2. *These maps g_i can be written as $g_i(z) = \lambda^{-k_i}.h_i^{s_i(t_0)}(\lambda^{k_i} z)$ for appropriate exponents $s_i(t_0)$ and k_i in $\mathbf{N}$.*

The first property of the vector field described by the proposition above motivates the following definition:

DEFINITION 3.2. – *Let V be an open subset of $\mathbf{C}$ and let Γ be a pseudogroup of mappings from open subsets of V into $\mathbf{C}$. Assume finally that $\mathcal{X}$ is an analytic vector field defined on V. We denote by Φ^t the local (real) flow of $\mathcal{X}$. We say that $\mathcal{X}$ is in the closure of Γ (relative to V) if and only if Φ^t verifies the following condition: for any open set $U \subset V$ and real number t_0 for which $\Phi^t(U) \subset V$ whenever $0 \leq t < t_0$, the mapping $\Phi^{t_0} : U \to \Phi^{t_0}(U)$ is holomorphic and is a uniform limit (in U) of the restriction to U of mappings g_i which belong to Γ*

REMARK 3.3. – We stress that if Γ is a non solvable pseudogroup generated by some local diffeomorphisms in $\mathrm{Diff}(\mathbf{C}, 0)$ and U is an open subset of $\mathbf{C}$ where a Nakai vector field $\mathcal{X}$ is defined, then $\mathcal{X}$ is in the closure of Γ relative to U.

We start our approach to the proposition with a simple lemma related to iterations of diffeomorphisms C^1-close to the identity. For an open (relatively compact) set $U \subseteq \mathbf{C}$ ($\simeq \mathbf{R}^2$) and a C^1-mapping h from U onto its image in $\mathbf{C}$ ($\simeq \mathbf{R}^2$), one denotes $\| h \|_{1,U} = \sup\{\| h(x) \| + \| D_x h \|, x \in U\}$ (where $D_x h$ is the differential of h at x). Sometimes we denote by $d(.\,,.)$ the natural euclidean distance on $\mathbf{C}$ ($\simeq \mathbf{R}^2$), in other cases we use the symbol $\| \quad \|$ for the modulus of some vector. Finally, for maps f_1 and f_2 defined on a domain of $\mathbf{C}$ the notation $(f_1 - f_2)(x) = f_1(x) - f_2(x)$ designates the vector with origin at $f_2(x)$ and end point at $f_1(x)$.

LEMMA 3.4. – *Let U be an open set and consider an relatively compact open subset $W \subset U$. Suppose we are given numbers $n \in \mathbf{N}$ and $\epsilon \in \mathbf{R}_{+}^{*}$. There exists $\delta > 0$ such that, as long as $\| h - id \|_{1,U} < \delta$, one has*

$$\| (h^n - id)(x) - n(h - id)(x) \| < \epsilon \| (h - id)(x) \| , \tag{1}$$

for every $x \in W$, where h^n stands for the n–th iterate $h \circ \cdots \circ h$.

Proof. – First, we choose δ_1 so small that if x belongs to W and $d(x,y) < \delta_1$ (for some $y \in \mathbf{C}$) then y belongs to U. We assume $\| h \|_{1,U} < \delta_1/n$. Thus for all $0 \leq j \leq n$ and $x \in V$ one has

$$\| (h^j - id)(x) \| \leq \sum_{i=1}^{j} \| (h^i - h^{i-1})(x) \| \leq j\delta_1/n \leq \delta_1$$

(where $h^0 = id$). In particular, $h^j(x)$ belongs to U.

For the proof, let us use an induction argument. Since our lemma is obviously true for $n = 1$, let us suppose that the result holds for $n - 1$. Therefore there exists $\delta_2 > 0$ such that

$$\| (h^{n-1} - id)(x) - (n-1)(h - id)(x) \| < \frac{\epsilon}{2} \| (h - id)(x) \| , \tag{2}$$

for any h satisfying $\| (h - id) \|_{1,U} < \delta_2$.

Now we choose δ smaller than $\inf\{\delta_1/n, \delta_2, \epsilon/2(n - 1 + \epsilon/2)\}$. We obviously have

$$(h^n - h^{n-1})(x) - (h - id)(x) = (h - id)(h^{n-1}(x)) - (h - id)(x).$$

Thus the Mean Value Theorem gives us that

$$\| (h^n - h^{n-1})(x) - (h - id)(x) \| \leq (\sup_{z \in U} \| D_z(h - id) \|) d(h^{n-1}(x), x).$$

Recalling that $\sup_{z \in U} \| D_z(h - id) \| \leq \epsilon/2(n - 1 + \epsilon/2)$, it follows from (2) that

$$\| (h^n - h^{n-1})(x) - (h - id)(x) \| \leq \frac{\epsilon(n - 1 + \epsilon/2)}{2(n - 1 + \epsilon/2)} \| (h - id)(x) \| = \frac{\epsilon}{2} \| (h - id)(x) \| . \tag{3}$$

On the other hand we write $(h^n - id)(x) - n(h - id)(x)$ as

$$[(h^n - h^{n-1})(x) - (h - id)(x)] + [(h^{n-1} - id)(x) - (n-1)(h - id)(x)] .$$

Therefore inequalities (2) and (3) show that

$$\begin{aligned} \| (h^n - id)(x) - n(h - id)(x) \| &\leq \frac{\epsilon}{2} \| (h - id)(x) \| + \frac{\epsilon}{2} \| (h - id)(x) \| \\ &= \epsilon \| (h - id)(x) \| . \end{aligned}$$

This completes the proof. □

LEMMA 3.5. – *Let W be an open (and relatively compact) subset of U. Let n be a natural number fixed and $Const_1, Const_2$ given real numbers. There exists $\delta > 0$ such that, for $x, y \in V$, one has:*

$$\| (h - id)(y) - (h - id)(x) \| \leq \frac{1}{4Const_2} \| (h - id)(x) \| ,$$

provided that $\| (h - id) \|_{1,U} < \delta$ and $d(y, x) < Const_1 \| (h^n - id)(x) \|$.

Proof. – According to the Mean Value Theorem, we know that

$$\begin{aligned} \| (h - id)(y) - (h - id)(x) \| &\leq (\sup_{z \in U} \| D_z(h - id) \|) d(y, x) \\ &\leq Const_1 . (\sup_{z \in U} \| D_z(h - id) \|) . \| (h^n - id)(x) \| . \end{aligned}$$

Let $\delta_1 > 0$ be such that Lemma (3.4) holds for W, n and $\epsilon = 1/2$. It results that

$$\| (h - id)(y) - (h - id)(x) \| \leq Const_1 . (\sup_{z \in U} \| D_z(h - id) \|)(n + 1/2) \| (h - id)(x) \| .$$

Thus the lemma is proved by taking $\delta < \inf\{\delta_1, 1/4(n + 1/2)Const_2\}/Const_1$. □

Next we shall prove a kind of "selection lemma" which gives us useful additional estimates.

LEMMA 3.6. – *Let n be a (sufficiently great) fixed integer. There exists $n_0 \in \mathbf{N}$ for which, whenever $N \geq n_0$, the diffeomorphism $h_N \in \mathcal{H}$ satisfies*

$$\lambda^{k_N}(1 - \lambda) < n \| (h_N - id)(\lambda^{k_N}) \|$$

and

$$(n - 1) \| (h_N - id)(\lambda^{k_N}) \| \leq \lambda^{k_N - 1}(1 - \lambda) ,$$

for a suitable $k_N \in \mathbf{N}$. Moreover for any $y \in B$ such that $d(y, \lambda^{k_N}) \leq \lambda^{k_N - 1}(1 - \lambda)$, one has

$$\| (h_N - id)(y) - (h_N - id)(\lambda^{k_N}) \| \leq \| (h_N - id)(\lambda^{k_N}) \| /4. \tag{4}$$

Proof. – We recall that the sequence $\mathcal{H} = \{h_i\}_{i \in \mathbf{N}}$ converges uniformly to the identity on B. Thus Cauchy's Integral Formula shows that it converges C^1 (C^∞) to the identity on every relatively compact subset of B. Let $B_{(1+\lambda)/2}$ be the disc centered at $0 \in \mathbf{C}$ with radius $(1 + \lambda)/2$ (< 1).

We define $n_1 = [2n/\lambda] + 1$ (where $[.]$ denotes the integral part). Replacing in the statement of Lemma (3.5) $Const_1$ by λ^{-1} and $Const_2$ by n, it results the existence of $\delta > 0$ (with $2n\delta < \lambda^2(1 + \lambda)$) such that h_i verifies

$$\| (h_i - id)(y) - (h_i - id)(x) \| \leq \frac{1}{4n} \| (h_i - id)(x) \| , \tag{5}$$

for $x, y \in B_{(1+\lambda)/2}$ as long as $d(x, y) < \lambda^{-1} \| (h_i^{n_1} - id)(x) \|$ and $\| (h_i - id) \|_{1,B'} < \delta$ (where B' denotes a disc with center at $0 \in \mathbf{C}$ and radius slightly greater than $(1 + \lambda)/2$).

Furthermore, setting $\epsilon = 1/4$ in Lemma (3.4), reducing modulo δ, we are able to assume in addition that

$$|| (h_i^j - id)(x) - j(h_i - id)(x) || \leq \frac{1}{4} || (h_i - id)(x) ||, \tag{6}$$

for every $0 \leq j \leq n_1 = [2n/\lambda] + 1$ and $x \in B_{(1+\lambda)/2}$.

Finally we choose $n_0 \in \mathbf{N}$ so large that $|| (h_N - id)(x) ||_{1,B'} < \delta$ for every $N \geq n_0$ (hence inequalities (5) and (6) hold for h_N). We are going to prove that to each $N \geq n_0$ one can assign a positive integer k_N so that the conclusions of our lemma are true.

We consider the points $\lambda, \lambda^2, \ldots, \lambda^k, \ldots$ and the corresponding sequence of fundamental domains:

$$[\lambda^2, \lambda], [\lambda^3, \lambda^2], \ldots, [\lambda^{k+1}, \lambda^k] \ldots.$$

Because $h_N(0) > 0$ and $|| \ (h_N - id)(x) ||_{1,B'} < \delta_1 < \lambda^2(1-\lambda)/2n$, we can consider the *smallest* positive integer k_N verifying

$$n \, || (h_N - id)(\lambda^{k_N}) || \geq \lambda^{k_N}(1 - \lambda). \tag{7}$$

Without loss of generality we can suppose that $h_N(\lambda^{k_N}) > \lambda^{k_N}$. Now inequalities (6) and (7) imply that

$$d(\lambda^{k_N}, \lambda^{k_N+1}) < d(\lambda^{k_N}, \lambda^{k_N-1}) \leq \lambda^{-1} \, || (h_N^{n+1} - id)(\lambda^{k_N}) || \ .$$

Since $n + 1 < n_1$, inequality (5) shows that

$$|| (h_N - id)(\lambda^{k_N}) - (h_N - id)(\lambda^{k_N \pm 1}) || \leq \frac{1}{4n} || (h_N - id)(\lambda^{k_N}) || \tag{8}$$

(where $k_N \pm 1$ means that the estimate holds for both $k_N + 1$ and $k_N - 1$). In particular we obtain that $h_N(\lambda^{k_N+1}) > \lambda^{k_N+1}$ and $h_N(\lambda^{k_N-1}) > \lambda^{k_N-1}$ (recall that $h_N(\lambda^{k_N}) > \lambda^{k_N}$).

We now claim that

$$(n-1) \, || (h_N - id)(\lambda^{K_N}) || < \lambda^{k_N-1}(1 - \lambda). \tag{9}$$

To check (9), observe that inequality (8) gives us

$$n \, || (h_N - id)(\lambda^{K_N-1}) || \geq \left(n - \frac{1}{4}\right) || (h_N - id)(\lambda^{K_N}) || > (n-1) \, || (h_N - id)(\lambda^{K_N}) || \ .$$

Hence if (9) were false we would obtain $n \, || (h_N - id)(\lambda^{K_N-1}) || \geq \lambda^{K_N-1}(1 - \lambda)$ which contradicts the minimality of K_N.

It remains to prove only the second part of the statement, namely the estimate (4). Replacing x by λ^{K_N} in (5), this estimate is reduced to see that $\lambda^{K_N-1}(1 - \lambda) < \lambda^{-1} \, || (h_N^{n_1} - id)(\lambda^{K_N}) ||$. In fact if it is so, then (4) follows at once from applying estimate (5). However it results from (6) and (7) that $|| (h_N^{n_1} - id)(\lambda^{K_N}) || > || (h_N^{n+1} - id)(\lambda^{K_N}) || > n \, || (h_N - id)(\lambda^{K_N}) || > \lambda^{K_N}(1 - \lambda)$ which in turn establishes the required estimate. The lemma is proved. □

We shall build up the vector field $\mathcal{X}$ as a limit of a sequence of vector fields. Let us begin by fixing a sequence $\{n_i\}$ converging to infinity. To each n_i we assign $\delta_i > 0$ so small that estimates (5) and (6) (with $n = n_i$) hold for any diffeomorphism h verifying $\| (h - id) \|_{1,B'} < \delta_i$. Next we fix $h_{N_i} \in \mathcal{H}$ such that $\| (h_{N_i} - id) \|_{1,B'} < \delta_i$. According to Lemma (3.6), we can choose N_i large enough to assure the existence of $k_{N_i} \in \mathbf{N}$ for which one has the inequalities

$$\lambda^{k_{N_i}}(1-\lambda) < n_i \| (h_{N_i} - id)(\lambda^{k_{N_i}}) \| \tag{10}$$

$$(n_i - 1) \| (h_{N_i} - id)(\lambda^{k_{N_i}}) \| \leq \lambda^{k_{N_i}-1}(1-\lambda) \tag{11}$$

and

$$\| (h_{N_i} - id)(y) - (h_{N_i} - id)(\lambda^{k_{N_i}}) \| \leq \| (h_{N_i} - id)(\lambda^{k_{N_i}}) \| /4n\,, \tag{12}$$

provided that $d(y, \lambda^{k_{N_i}}) \leq \lambda^{k_{N_i}-1}(1-\lambda)$ (cf. Lemma (3.6)).

Recall that it has been assumed (without loss of generality) that $h_{N_i}^{j_{N_i}}(\lambda^{k_{N_i}}) \geq \lambda^{k_{N_i}}$. We now define $j_{N_i} \in \mathbf{N}$ to be the *smallest* positive integer for which $h_{N_i}^{j_{N_i}}(\lambda^{k_{N_i}}) \geq \lambda^{k_{N_i}-1}$. The lemma below compares j_{N_i} and n_i.

LEMMA 3.7. – *There are uniform constants $c_2 \geq c_1 > 0$ such that*

$$c_1 n_i < j_{N_i} \leq c_2 n_i\,.$$

Proof. – Thanks to Lemma (3.6) one has $(n_i - 1) \| (h_{N_i} - id)(\lambda^{k_{N_i}}) \| \leq \lambda^{k_{N_i}-1}(1-\lambda)$. On the other hand, the estimate (6) allows us to conclude that

$$\| (h_{N_i}^{n_i-2} - id)(\lambda^{k_{N_i}}) \| \leq \left(n_i - 2 + \frac{1}{4}\right) \| (h_{N_i} - id)(\lambda^{k_{N_i}}) \| < (n_i - 1) \| (h_{N_i} - id)(\lambda^{k_{N_i}}) \|\;.$$

Thus $\| (h_{N_i}^{n_i-2} - id)(\lambda^{k_{N_i}}) \| < \lambda^{k_{N_i}-1}(1-\lambda)$ and hence $j_{N_i} \geq n_i - 2$. The left side of the desired inequality follows.

To finish the proof of the lemma it is enough to show that $j_{N_i} \leq [2n_i/\lambda] + 1$. Since (6) holds for every $0 \leq j \leq [2n_i/\lambda] + 1$, it results that

$$\begin{aligned} \| (h_{N_i}^{[2n_i/\lambda]} - id)(\lambda^{k_{N_i}}) \| &\geq \left([2n_i/\lambda] + 1 - \frac{1}{4}\right) \| (h_{N_i} - id)(\lambda^{k_{N_i}}) \| \\ &\geq \lambda^{-1} n_i \| (h_{N_i} - id)(\lambda^{k_{N_i}}) \|\,. \end{aligned}$$

However (10) implies that $\lambda^{-1} n_i \| (h_{N_i} - id)(\lambda^{k_{N_i}}) \| \geq \lambda^{k_{N_i}-1}(1-\lambda)$. Hence $h_{N_i}^{[2n_i/\lambda]}(\lambda^{k_{N_i}}) > \lambda^{k_{N_i}-1}$. Because j_{N_i} is the smallest positive integer having this property, we get $j_{N_i} \leq [2n_i/\lambda]$ as required. □

On D_λ, we associate to each $i \in \mathbf{N}$ a vector field $\mathcal{X}_i$ defined by the formula

$$\mathcal{X}_i(z) = j_{N_i}.\lambda^{-k_{N_i}+2}.(h_{N_i} - id)(\lambda^{k_{N_i}-2} z) \tag{13}$$

LEMMA 3.8. – *There exist uniform constants $Const_1 \geq Const_2 > 0$ such that*

$$Const_2 \leq \inf_{D_\lambda} \| \mathcal{X}_i(z) \| \leq \sup_{D_\lambda} \| \mathcal{X}_i(z) \| \leq Const_1\,. \tag{14}$$

Proof. – Inequality (12) shows that

$$\frac{3}{4} \| \mathcal{X}_i(\lambda^2) \| \leq \| \mathcal{X}_i(z) \| \leq \frac{5}{4} \| \mathcal{X}_i(\lambda^2) \|\,,$$

for any $z \in D_\lambda$ (cf. formula (13)). Thus it suffices to see that the sequence $\{\| \mathcal{X}_i(\lambda^2) \| \}_{i\in\mathbf{N}}$ has an upper bound C_1 and a lower bound $C_2 > 0$. According to inequalities (10) and (11), this is equivalent to find upper and lower bounds for the sequence $\{j_{N_i}/n_i\}_{i\in\mathbf{N}}$, which is assured by Lemma (3.7). This proves the lemma. □

We go back to the sequence $\{n_i\}$ and the vector fields $\mathcal{X}_i$ defined on D_λ. The vector field $\mathcal{X}_i$ can be interpreted as a mapping from D_λ into $\mathbf{C}$, moreover it is a holomorphic mapping (see formula (13)). On the other hand the mappings $\mathcal{X}_i : D_\lambda \to \mathbf{C}$ are uniformly bounded after Lemma (3.8). Therefore (maybe choosing a subsequence), we can apply Montel's theorem to find a limit for this sequence of mappings restricted to $D_{\lambda/2}$. It means that we can define a vector field $\mathcal{X}$ on $D_{\lambda/2}$ by taking

$$\mathcal{X} = \lim_{i\to\infty} \mathcal{X}_i \,. \tag{15}$$

Proof of Proposition (3.1). – At this point the proof is rather standard. We follow essentially [Na-1], although we point out that the argument is equivalent to the one given (in a different context) in [Ka]. Naturally we are going to prove that the vector field $\mathcal{X}$ defined just above fulfils the conditions of Proposition (3.1). First of all, we observe $\mathcal{X}$ is a nowhere *zero* vector field since $0 < Const_2 \leq \| \mathcal{X}_i(z) \|$ for all $z \in D_{\lambda/2}$ and $i = 1, 2, \dots$ (Lemma (3.8)).

We consider Φ^t the local flow generated by $\mathcal{X}$ and let U be an open (relatively compact) subset of $D_{\lambda/2}$. Let $t_0 \in \mathbf{R}$ be so small that $\Phi^t(U) \subset D_{\lambda/2}$ whenever $0 \leq t < t_0$. We know that there is a sequence $\mathcal{X}_i$ of vector fields converging uniformly on $D_{\lambda/2}$ to $\mathcal{X}$; moreover, each $\mathcal{X}_i$ is given by the expression (13) (for appropriate integers j_{N_i}, k_{N_i} and a diffeomorphism $h_{N_i} \in \mathcal{H}$). The proof of the proposition is hence accomplished by the claim below:

Claim: The mapping $\Phi^{t_0} : U \to \Phi^{t_0}(U) \subseteq D_{\lambda/2}$ is a uniform limit on $D_{\lambda/2}$ of the sequence $\lambda^{-k_{N_i}}.h_{N_i}^{p_i(t_0)}(\lambda^{k_{N_i}} z)$ where $p_i(t_0) = [j_{N_i} t_0]$ (the brackets [ ] stands for the integral part).

Proof of the Claim. – Given $\varepsilon > 0$, we search for $i_0 \in \mathbf{N}$ such that, if $i > i_0$, one has $\| \Phi^{t_0}(z) - \lambda^{-k_{N_i}}.h_{N_i}^{p_i(t_0)}(\lambda^{k_{N_i}} z) \| < \varepsilon$ for any $z \in U$. Let Φ_i^t be the local real flow associated to $\mathcal{X}_i$. Because $\mathcal{X}_i$ and $\mathcal{X}$ are holomorphic and $\{\mathcal{X}_i\}$ converges uniformly to $\mathcal{X}$ on $D_{\lambda/2}$, the real trajectories $\Phi_i^t(z)$ (with $0 \leq t < t_0$) are arbitrarily approximated by sequences of the form:

$$z_{i,l+1} = z_{i,l} + t_l \mathcal{X}_i(z_{i,l}) \;\; ; \;\; z_{i,0} = z,$$

where $l = 0, 1, \dots, m-1$ and with sufficiently small $0 < t_l$, $\sum_{l=0}^{m-1} t_l = t_0$. This proves that $\| \Phi_i^{s_l}(z) - z_{i,l} \|$ (where $s_l = t_1 + \cdots + t_{l-1}$) has a uniform upper bound $C(\delta)$ depending only on $\delta = \max\{t_l\}$. Besides, using the relative compactness of U on $D_{\lambda/2}$, it is easy to see that all these constants can be chosen uniformly on U. Now we set $t_l = 1/j_{N_i}$ (consequently $s_l = l/j_{N_i}$). It results that

$$z_{i,l+1} = z_{i,l} + \frac{1}{j_{N_i}} \mathcal{X}_i(z_{i,l}) = \lambda^{-k_{N_i}}.h_{N_i}(\lambda^{k_{N_i}} z_{i,l}) = \lambda^{-k_{N_i}}.h_{N_i}^{l+1}(\lambda^{k_{N_i}} z_0)\,,$$

for $l = 0, 1, \dots$. Therefore the estimate above implies $\| \Phi_i^{s_l}(z) - \lambda^{-k_{N_i}}.h_{N_i}^{l}(\lambda^{k_{N_i}} z) \| < C(1/j_{N_i})$ (where $0 \leq s_l < t_0$). Nevertheless, we know $\{\Phi_i^t\}$ converges uniformly to Φ^t

$(0 \le t < t_0)$ and t_0 is approximated (from below) by sequences $s_{l_i} = [t_0 j_{N_i}]/j_{N_i}$, so that $\lambda^{-k_{N_i}}.h_{N_i}^{[t_0 j_{N_i}]}(\lambda^{k_{N_i}} z)$ converges to $\Phi_i^{s_{l_i}}(z)$ which by its turn converges to $\Phi^t(z)$.

Since all the constants involved regardless the point $z \in U$ chosen, we finally conclude the claim. □

The proof of the proposition is over. □

4. Local vector fields in $\mathrm{Diff}^{\omega}(S^1)$

Throughout this section G will be a non solvable subgroup of $\mathrm{Diff}^{\omega}(S^1)$. We assume that G admits a (finite) set of generators $\{g_1, \ldots, g_l\}$ such that g_i (for each $i = 1, \ldots, l$) belongs to the neighborhood of the identity described in Theorem (2.2) (we recall that S^1 is regarded as the unit circle embedded in $\mathbf{C}$).

For a point $p \in S^1$, $\mathrm{Stab}_G(p)$ denotes the *stabilizer* of p in G, namely the subgroup of G consisting of diffeomorphisms $f \in G$ for which $f(p) = p$. Clearly for any $p \in S^1$, $\mathrm{Stab}_G(p)$ defines in a natural way (and unequivocally up to conjugacies) a subgroup of $\mathrm{Diff}(\mathbf{C}, 0)$ that will be designated by $\mathrm{Stab}_G(p, 0)$. Furthermore $\mathrm{Stab}_G(p, 0)$ is cyclic (solvable) if and only if $\mathrm{Stab}_G(p)$ is so.

We turn our attention to diffeomorphisms f which belong to G. Since f is real analytic, f has a holomorphic extension to some neighborhood of S^1 in $\mathbf{C}$. Hence f defines in a natural way a pseudo-dynamics (in the sense of pseudogroups) on this neighborhood (however f is not necessarily injective on this neighborhood). Now let U be an open subset of $\mathbf{C}$ $(U \cap S^1 \neq \emptyset)$ and let us define a pseudogroup, denoted by $\Gamma_{G,U}$, of mappings from open subsets of U into $\mathbf{C}$ declaring that this pseudogroup is generated by mappings f_U which are holomorphic, injective (as mapping from U into $\mathbf{C}$) and such that they agree with the restriction to U of the (maximal) holomorphic extension of some diffeomorphism $f \in G$.

In this section we will prove a different version of the Theorem B announced in the introduction.

THEOREM 4.1. – *There exists an open neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}^{\omega}(S^1)$ with the following property: Let G be a non solvable subgroup of $\mathrm{Diff}^{\omega}(S^1)$ and assume that G has a set of generators $\{g_1, \ldots, g_l\}$ contained in $\mathcal{U}$. Suppose that $p \in S^1$ is not a periodic point for G. Then there exists an open neighborhood U of p in $\mathbf{C}$ endowed with a nowhere zero analytic vector field $\mathcal{X}$ such that $\mathcal{X}$ is in the closure (in the sense of definition (3.2)) of $\Gamma_{G,U}$.*

REMARK 4.2. – Later we shall observe that the stabilizer of a periodic point $p \in S^1$ is necessarily non solvable. Therefore around these points we can use Nakai's vector fields do study the local dynamics.

The proof of Theorem (4.1) above requires some further information on non solvable subgroups of $\mathrm{Diff}^{\omega}(S^1)$ generated by diffeomorphisms which are close to the identity. We recall that Hölder's theorem (cf. [H-H]) claims, in particular, the existence of points in S^1 with non trivial stabilizer for every non abelian subgroup of $\mathrm{Diff}^{\omega}(S^1)$. More precisely, if a non abelian subgroup of $\mathrm{Diff}^{\omega}(S^1)$ has all orbits dense, then a result by Ghys (cf. [E-T]) based on Sacksteder's Theorem guarantees the existence of hyperbolic fixed points for certain elements of the group. On the other hand, we will need of the following proposition:

PROPOSITION 4.3. – *Let G be a non solvable subgroup of $\mathrm{Diff}^{\omega}(S^1)$ generated by a finite set of diffeomorphisms in $\mathcal{U}$ (the neighborhood of the identity fixed in the beginning of the* *section). Assume G has no finite orbit. Then to each open (non empty) interval $I \subseteq S^1$, it corresponds a finite set $\{f_1, \dots, f_l\}$ of elements in G so that the union $\bigcup_{i=1}^{l} f_i(I)$ covers S^1.*

Proof. – Thanks to the compacity of S^1, it is enough to show that $\bigcup_{f \in G} f(I)$ covers S^1. However this follows at once from the density of all orbits of G (cf. Proposition (2.3)). The proposition is proved. □

We still have one last proposition corresponding to an analogous of Proposition (4.3) in the presence of finite orbits for G.

PROPOSITION 4.4. – *Assume $G \subseteq \mathrm{Diff}^{\omega}(S^1)$ is a group as in Proposition (4.3) except that G displays at least one finite orbit. So there is a finite subset $P \subset S^1$ with the following properties:*

1. *P is the union of all finite orbits of G.*
2. *Let I_{p_i} be a connected component of $S^1 \setminus P$. Let K_I be a connected open relatively compact subset of I_{p_i} and I_0 a (non empty) open interval of I_{p_i}. Then there exists a finite set $\{f_1, \dots, f_l\}$ of elements in G such that the union $\bigcup_{i=1}^{l} f_i(I_0)$ covers K_I.*

Proof. – First, we *define* P as the union of all finite orbits of G. According to Proposition (2.3) P is finite.

Now we fix I_{p_i}, a connected component of $S^1 \setminus P$, and a relatively compact subset K_I of I_{p_i}. Recall that the orbit by G of any point $p \in I_{p_i}$ is dense in I_{p_i}. Since the closure of K_I is compact, the proposition results from the same argument employed in Proposition (4.3). The proposition is proved. □

Proof of Theorem (4.1). – The proof is naturally divided in two cases according to the existence of finite orbits in G. First we assume that G has no finite orbits and thus every point in S^1 has a dense orbit.

Let $\{g_1, \dots g_s\}$ be a finite generating set of G which is contained in a small neighborhood of identity in $\mathrm{Diff}^{\omega}(S^1)$ and let $S(k)$ be the respective sequence of subsets $S(k) \subset G$ defined in paragraph 2.1. After Ghys' result (Theorem (2.2)), one can assume that the elements of $S(k)$ converge uniformly to the identity on the annulus $\tilde{S}_{\tau} = \{z \in \mathbf{C}\, ; 1 - \tau \mid z \mid < 1 + \tau\}$ for some $\tau > 0$.

For k large enough we consider the set $H_k \subset S^1$ defined as the intersection of the fixed points of diffeomorphisms in $S(k) \cup S(k-1)$, i.e. $H_k = \bigcap_{h \in S(k) \cup S(k-1)} \mathrm{Fix}(h)$ (where $\mathrm{Fix}(h)$ stands for the set of fixed points by h). We designate by G_k the subgroup of G generated by $S(k) \cup S(k-1)$. We observe G_k is non solvable for every k, since otherwise the sequence $S(k)$ would degenerate in $\{id\}$ (for some appropriate k). Let us consider two subcases: in accordance with H_k being empty (for every k).

1) Suppose H_k is non empty for some k.

Let p be a point in H_k. The point p is fixed by all diffeomorphisms in G_k which in turn is a non solvable group. Because G_k defines (by natural projection) a non solvable subgroup of $\mathrm{Diff}_{\mathbf{R}}(\mathbf{C}, 0)$ around p, there exists an open (non empty) neighborhood $U \subset \mathbf{C}$ of some point $q \in S^1$ ("close" to p) endowed with a (nowhere *zero*) Nakai vector field $\mathcal{X}$. We put $I = U \cap S^1$. After Proposition (4.3) there is a finite set $\{f_1, \dots, f_r\}$ of diffeomorphisms in G such that the union $\bigcup_{i=1}^{r} f_i(I)$ covers S^1. Hence the union $\bigcup_{i=1}^{r} f_i(U)$ is a neighborhood of S^1 in $\mathbf{C}$ and, modulo choosing U very "narrow" (i.e. U enclosed in a narrow annulus containing S^1), we can assume that f_i is injective on U (for $i = 1, \dots, r$). Therefore each

$f_i(U)$ is naturally equipped with the direct image of $\mathcal{X}$ by f_i which is a vector field in the closure of the pseudogroup $\Gamma_{G,f_i(U)}$ as required (see remark (3.3)).

2) Suppose H_k is empty for every $k \in \mathbf{N}$.

Using Ghys' result, we choose a point $p \in S^1$ and a diffeomorphism $f \in G$ such that $f(p) = p$ and $f'(p) < 1$. Since H_k is empty for every k, we can find a sequence $\{h_k\} \in G$ ($h_k \in S(k)$ and $h_k \neq id$ for all k) such that $h_k(p) \neq p$. Besides, maybe replacing h_k by its inverse, it can be supposed that $h_k(p) > p$.

Now Proposition (3.1) assures the existence of some open (non empty) set $R \subset \mathbf{C}$ endowed with a nowhere *zero* vector field $\mathcal{X}$ contained in the closure of $\Gamma_{G,R}$. Setting $I = R \cap S^1$, Proposition (4.3) assigns a finite set $\{f_1, \dots, f_r\}$ of elements in G such that $\bigcup_{i=1}^{r} f_i(I)$ covers S^1. Thus $f_i(R)$ is naturally endowed with a (nowhere *zero*) vector field contained in the closure of $\Gamma_{G,f_i(R)}$ (choosing U very "narrow"). Moreover the union $\bigcup_{i=1}^{r} f_i(U)$ is a neighborhood of S^1 in $\mathbf{C}$. This accomplishes the proof in the case G has no finite orbits.

Let us now suppose that G has a finite orbit. We consider the finite set P formed by all finite orbits of G. Since the diffeomorphisms in G_k converges to the identity when k increases, for k sufficiently large they must fix the points of P. Thus the stabilizer of any point p in P is non solvable. So, in every connected component I_i of $S^1 \setminus P$ there is a point q_i and an open neighborhood $U_i \subset \mathbf{C}$ of q_i endowed with a Nakai vector field $\mathcal{X}$. Therefore we can complete the proof applying Proposition (4.4) in the same manner we have applied Proposition (4.3) in the other case. The proof of Theorem (4.1) is over. □

5. Topological rigidity

In this section we are going to apply Theorem (4.1) to get the proof of Theorem C in the introduction (topological rigidity). Actually, topological rigidity is a consequence of two main results of rigidity for certain types of pseudogroups, one of them due to Nakai ([Na-2]). The other one, concerning pseudogroups like those considered in section 3, corresponds to Proposition (5.1). Let us resume some notation.

Let $B \subset \mathbf{C}$ be the unit ball. Let Γ_1, Γ_2 be pseudogroups of holomorphic diffeomorphisms from open subsets of B into $\mathbf{C}$ which preserve the real line. Suppose in addition that

1. There is $f_1 \in \Gamma_1$ (resp. $f_2 \in \Gamma_2$) whose domain of definition contains B and such that $f_1(0) = 0$ (resp. $f_2(0) = 0$). Moreover $f_1(z) = \lambda z$ (for certain $0 < \lambda < 1$).
2. There exists a sequence $\mathcal{H}_1 = \{h_{1,i}\}_{i \in \mathbf{N}}$ (resp. $\mathcal{H}_2 = \{h_{2,i}\}_{i \in \mathbf{N}}$) of elements in Γ_1 (resp. Γ_2), $h_{1,i} \neq id$ for all i (resp. $h_{2,i} \neq id$ for all i) such that each $h_{1,i}$ (resp. $h_{2,i}$) is defined on B and the sequence converges uniformly on B to the identity. Furthermore $h_{1,i}(0) > 0$ (resp. $h_{2,i}(0) > 0$) for all i.

In this section we shall assume that *all the homeomorphisms considered* preserve the orientation of S^1. We leave the analogous considerations related to orientation-reversing homeomorphisms to the reader. In the proof of topological rigidity the main role is played by Proposition (5.1) below.

PROPOSITION 5.1. – *Let Γ_1, Γ_2 be pseudogroups as above and let u be a homeomorphism defined on $B \cap \mathbf{R}$ (and taking values in $\mathbf{R}$). Assume that the following equations hold (as long as both members are defined):*

a) $f_2 = u(\lambda u^{-1})$,

b) $h_{2,i} = u \circ h_{1,i} \circ u^{-1}$ *(for every* $i = 1, 2, \ldots$*).*

Then there exists an open (non empty) interval such that the restriction of u *to this interval is a real analytic diffeomorphism.*

It is important to point out the fact that u is defined only in $B \cap \mathbf{R}$ and not in the whole of B. Besides, it is not known *a priori* whether 0 is a hyperbolic fixed point for f_2. Hence we should be able to handle with the possibility that $| f_2'(0) |= 1$. By iterating f_2 twice, one sees that it is enough to deal with the case $f_2'(0) = 1$.

The proof of Proposition (5.1) is deeply influenced by the very nature of the construction carried out in section 3 (which culminates in the proof of Proposition (3.1)). Indeed it will result from a sequence of observations concerning this construction and its natural behavior under topological conjugacies.

Let us consider the fundamental domain for $f_1 = \lambda z$ given by $I_1 = [\lambda^2, \lambda]$. Because u conjugates λz to f_2, it follows that $u(\lambda^2) = f_2 \circ u(\lambda)$, hence $u(I_1) = [u(\lambda^2), u(\lambda)]$ is a fundamental domain for f_2.

Suppose we are given $\varepsilon > 0$. Let R_ε be the rectangle of vertices (λ^2, ε), (λ, ε), $(\lambda, -\varepsilon)$ and $(\lambda^2, -\varepsilon)$. Analogously, $R_{u\varepsilon}$ will be the rectangle whose vertices are $(f_2 \circ u(\lambda), \varepsilon)$, $(u(\lambda), \varepsilon)$, $(u(\lambda), -\varepsilon)$ and $(f_2 \circ u(\lambda), -\varepsilon)$. We now observe that $f_2(x) < x$ for small $x \in \mathbf{R}_+^*$. Hence modulo a rescaling and taking ε sufficiently small, it can be supposed that $R_{u\varepsilon}$ is enclosed in a strictly larger domain V which has the properties stated in Lemma (2.4) (remark that this lemma is trivial if $| f_2'(0) |< 1$). This fact will be assumed throughout this section.

The new ingredient related to f_2 is the following lemma:

LEMMA 5.2. – *Given* $\varepsilon > 0$ *sufficiently small, there exists a uniform constant* $C_1 > 0$ *such that, whenever* $z \in R_{u\varepsilon}$, *one has*

$$\| f_2^n(z) - f_2^n \circ u(\lambda) \| \leq C_1 \| f_2^n \circ u(\lambda) - f_1^{n+1} \circ u(\lambda) \| \quad \textit{and} \tag{16}$$

$$C_1^{-1} \| d_{f_2^n(z)}(f_2^{-n}) \| \leq \| d_{f_2^n(u(\lambda))}(f_2^{-n}) \| \leq C_1 \| d_{f_2^n(z)}(f_2^{-n}) \| \; . \tag{17}$$

Proof. – The existence of such a constant is clear if f_2 is hyperbolic, for in this case we can thing of f_2 as being a homothety. Thus we assume that $f_2'(0) = 1$. Recall that $R_{u\varepsilon}$ is enclosed in V which satisfies the conclusions of Lemma (2.4).

We put $\tilde{\lambda} = (u(\lambda)+u(\lambda^2))/2$ and $\tilde{\lambda}_n = (d_{f_2^n(u(\lambda))}(f_2^{-n}))^{-1}$. In order to prove the lemma, let us consider the mappings $T_n : V \rightarrow \mathbf{C}$ given by $z \mapsto \tilde{\lambda}_n f_2^n(z) + (\tilde{\lambda} - \tilde{\lambda}_n f_2^n(\tilde{\lambda}))$. Clearly T_n is injective on V. Besides $T_n(\tilde{\lambda}) = \tilde{\lambda}$ and $T_n'(\tilde{\lambda}) = 1$. Thanks to the homogeneity of (16) and (17), it suffices to show that these estimates hold when f_2^n is replaced by T_n. Since there is a Riemann mapping taking $\tilde{\lambda}$ into the center of the unit ball, these lattest inequalities are easy consequences of Köebe's Bounded Distortion Theorem (and its version for derivatives, cf. [Ca]). The lemma is proved. □

Let us proceed to the construction of vector fields on R_ε and $R_{u\varepsilon}$. Choose a sequence of positive integers $\{n_i\}$ going to infinity. Fix $n_i \in \mathbf{N}$ and a ball $B' \subset B$ of radius greater than $(1 + \lambda)/2$. Take $C = \sup\{C_1, \lambda^{-1}\}$ and let $n_i^* = [2n_iC] + 1$. After Lemmas (3.4) and (3.5), there is $\delta_i > 0$ for which any diffeomorphism h fulfilling $\| (h - id) \|_{1,B'} < \delta_i$

also satisfies the inequality

$$\| (h-id)(y)-(h-id)(x) \| \leq \frac{1}{4n} \| (h-id)(x) \| , \tag{18}$$

as long as $d(y,x) \leq C \| (h^{n_i^*}-id)(x) \|$. Furthermore there is also

$$\| (h^j-id)(x)-j(h-id)(x) \| \leq \frac{1}{4} \| (h-id)(x) \| , \tag{19}$$

for any $0 \leq j \leq n_i^* = [2n_iC]+1$. Next we choose $N_i \in \mathbf{N}$ so large that h_{1,N_i}, h_{2,N_i} satisfies (18) and (19). According to Lemma (3.6) we can find $k_{N_i} \in \mathbf{N}$ such that estimates in the statement of this lemma hold (with $h_N = h_{1,N_i}$).

Assuming without loss of generality that $h_{1,N_i}(\lambda^{k_{N_i}}) > \lambda^{k_{N_i}}$, we finally define j_{N_i} as the smallest positive integer for which $h_{1,N_i}^{j_{N_i}}(\lambda^{k_{N_i}}) \geq \lambda^{k_{N_i}-1}$. Observe that j_{N_i} is also the smallest integer for which $h_{2,N_i}^{j_{N_i}} \circ f_2^{k_{N_i}-1}(u(\lambda)) \geq f_2^{k_{N_i}-2}(u(\lambda))$.

So to each i we associate a vector field $\mathcal{X}_i$ on R_ε by formula (13). Yet we define another vector field $\mathcal{X}_{ui}$ on $R_{u\varepsilon}$ by the expression

$$\mathcal{X}_{ui}(z) = j_{N_i}.d_{f_2^{k_{N_i}-2}(z)}(f_2^{-k_{N_i}+2})(h_{2,N_i}-id)(f_2^{k_{N_i}-2}) . \tag{20}$$

The next lemma is fundamental for the proof of Proposition (5.1).

LEMMA 5.3. – *There are uniform constants $C_2 \geq C_3 > 0$ such that*

$$C_3 \leq \inf_{R_{u\varepsilon}} \| \mathcal{X}_{ui}(z) \| \leq \sup_{R_{u\varepsilon}} \| \mathcal{X}_{ui}(z) \| \leq C_2 .$$

Proof. – It is known that

$$\| (h_{2,N_i}-id) \circ f_2^{k_{N_i}-1} \circ u(\lambda) \| \geq \| f_2^{k_{N_i}-2} \circ u(\lambda) - f_2^{k_{N_i}-1} \circ u(\lambda) \| . \tag{21}$$

Because Lemma (3.7) asserts that $j_{N_i} \leq n_i^*$, this estimate combined with the first inequality of Lemma (5.2) establishes that $d(f_2^{k_{N_i}-2}(z), f_2^{k_{N_i}-1} \circ u(\lambda)) \leq C \| (h_{2,N_i}^{n_i^*}-id) \circ f_2^{k_{N_i}-1} \circ u(\lambda) \|$ for every $z \in R_{u\varepsilon}$. Hence estimate (18) yields

$$\begin{aligned}\frac{3}{4} \| (h_{2,N_i}-id) \circ f_2^{k_{N_i}-1} \circ u(\lambda) \| &\leq \| (h_{2,N_i}-id) \circ f_2^{k_{N_i}-1}(z) \| \\ &\leq \frac{5}{4} \| (h_{2,N_i}-id) \circ f_2^{k_{N_i}-1} \circ u(\lambda) \| .\end{aligned}$$

Using again Lemma (5.2) (the second inequality), it results the existence of a constant $Const$ such that

$$Const^{-1} \| \mathcal{X} \circ f_2 \circ u(\lambda) \| \leq \inf_{R_{u\varepsilon}} \| \mathcal{X}_{ui}(z) \| \leq \sup_{R_{u\varepsilon}} \| \mathcal{X}_{ui}(z) \| \leq Const \| \mathcal{X} \circ f_2 \circ u(\lambda) \| . \tag{22}$$

The estimate (22) above reduces the proof of the lemma to find constants C_2', C_3' verifying $C_3' \leq \| \mathcal{X} \circ f_2 \circ u(\lambda) \| \leq C_2'$. In order to get that, observe that Lemma (5.2) proves in particular that

$$\begin{aligned}\frac{C_1^{-1} \| u(\lambda) - f_2 \circ u(\lambda) \|}{\| f_2^{k_{N_i}-1}(u(\lambda)) - f_2^{k_{N_i}-2}(u(\lambda)) \|} &\leq \| d_{f_2^{k_{N_i}-1}(\lambda)}(f_2^{-k_{N_i}+1}) \| \\ &\leq \frac{C_1 \| u(\lambda) - f_2 \circ u(\lambda) \|}{\| f_2^{k_{N_i}-1}(u(\lambda)) - f_2^{k_{N_i}-2}(u(\lambda)) \|} .\end{aligned}$$

In view of that, the desired double inequality results from the fact that j_{N_i} is the smallest integer verifying (21) together with inequality (18). The lemma is proved. $\square$

Proof of Proposition (5.1). – We consider on R_ε the sequence of vector fields $\{\mathcal{X}_i\}$ and on $R_{u\varepsilon}$ the sequence $\{\mathcal{X}_{ui}\}$. After Lemmas (3.8) and (5.3), choosing subsequences and reducing their domain of definition if necessary, it can be supposed that $\{\mathcal{X}_i\}$ converges uniformly toward some vector field $\mathcal{X}_1$ and $\{\mathcal{X}_{ui}\}$ converges uniformly toward a holomorphic vector field $\mathcal{X}_2$. Applying Proposition (3.1) it results that $\mathcal{X}_1$ is a uniform limit of dynamics like $f_1^{-k_{N_i}} \circ h_{1,N_i}^l \circ f_1^{k_{N_i}}$. Similarly one can verify that $\mathcal{X}_2$ is a uniform limit of dynamics like $f_2^{-k_{N_i}} \circ h_{2,N_i}^l \circ f_2^{k_{N_i}}$. We consider the restrictions of these dynamics to $B \cap \mathbf{R}_+^*$. Using the fact that u conjugates (the restrictions to $B \cap \mathbf{R}_+^*$ of) f_1 with f_2 and h_{1,N_i} with h_{2,N_i}, one obtains $u \circ f_1^{-k_{N_i}} \circ h_{1,N_i}^l \circ f_1^{k_{N_i}} \circ u^{-1} = f_2^{-k_{N_i}} \circ h_{2,N_i}^l \circ f_2^{k_{N_i}}$. Hence it conjugates the restrictions to $B \cap \mathbf{R}_+^*$ of $\mathcal{X}_1$ and $\mathcal{X}_2$ in a time-preserving way, that is, if ϕ_1^t and ϕ_2^t designate respectively the real flows associated to the restriction of $\mathcal{X}_1$ and $\mathcal{X}_2$ to $B \cap \mathbf{R}_+^*$, one has $u \circ \phi_1^t \circ u^{-1}(x) = \phi_2^t(x)$ (for any t and x such that both members be defined). Thus u is real analytic in this domain and the proposition is proved. $\square$

We are now able to prove the Theorem C.

Proof of Theorem C. – Let G_1 and G_2 be subgroups of $\mathrm{Diff}^\omega(S^1)$ as in the statement of this theorem. Let us first suppose G_1 (and thus G_2) has finite orbits. In this case there is a finite set P_1 (resp. P_2) formed of all the finite orbits of G_1 (resp. G_2) and such that G_1 (resp. G_2) acts in a minimal way on each connected component of $S^1 \setminus P_1$ (resp. $S^1 \setminus P_2$).

For $p_1 \in P_1$ and $u(p_1) = p_2 \in P_2$, we consider the respective stabilizers $\mathrm{Stab}_{G_1}(p_1)$ and $\mathrm{Stab}_{G_2}(p_2)$ which project onto non solvable subgroups of $\mathrm{Diff}(\mathbf{C}, 0)$. Moreover u conjugates the restriction to the real line of these subgroups. Under these assumptions a theorem of Nakai [Na-2] assures that u is real analytic in a neighborhood of p_1. Since p_1 is an arbitrary point in P_1, one concludes that u is real analytic in a neighborhood of any point in P_1. Now Proposition (4.4) allows us to conclude the analycity of u in the entire S^1.

Let us now assume G_1, G_2 have no finite orbit. Therefore these groups verify the conclusions of Proposition (4.3). As in the proof of Theorem (4.1) we define the sets $S_1(k) \cup S_1(k-1) \subset G_1$ (resp. $S_2(k) \cup S_2(k-1) \subset G_2$) and let $G_{1,k}$ (resp. $G_{2,k}$) be the subgroup generated by $S_1(k) \cup S_1(k-1)$ (resp. $S_2(k) \cup S_2(k-1)$). Moreover $H_{1,k}$ (resp. $H_{2,k}$ is the intersection of the fixed points for the diffeomorphisms in $S_1(k) \cup S_1(k-1)$ (resp. $S_2(k) \cup S_2(k-1)$).

Assume that $H_{1,k}$ (and consequently $H_{2,k}$) is non empty for some k. Then the stabilizer of a point $p_1 \in H_{1,k}$ (resp. $p_2 = u(p_1) \in H_{2,k}$) is non solvable since it contains $G_{1,k}$ (resp. $G_{2,k}$). Therefore Nakai's theorem applies to ensure the analycity of u around p_1. Finally Proposition (4.3) allows us to conclude the analycity of u in the entire S^1.

On the other hand, assume that $H_{1,k}, H_{2,k}$ are empty for every k. There is $p_1 \in S^1$ and $f_1 \in G_1$ such that $f_1(p_1) = p_1$ and $| f_1'(p_1) | < 1$. Let $f_2 = u \circ f_1 \circ u^{-1} \in G_2$ and $p_2 = u(p_1)$. Because $H_{1,k} = \emptyset$ for every k, there exists a sequence $h_{1,k} \in G_1$ (resp. $h_{2,k} \in G_2$) converging uniformly to the identity for which $h_{1,k}(p_1) > p_1$ (resp. $h_{2,k}(p_2) > p_2$). Hence Proposition (5.1) guarantees the analycity of u in a certain non empty interval. Thus Proposition (4.3) gives the analycity of u everywhere. The theorem is proved. $\square$

COROLLARY 5.4. – *Let $(M_j^n, \mathcal{F}_j)$ $(j = 1, 2)$ be real analytic oriented foliations, where M_j^n is a S^1-bundle and $\mathcal{F}_j$ is transverse to its fibers. Assume also that the global holonomy* *of $\mathcal{F}_j$ $(j = 1,2)$ is non solvable and generated by elements very close to the identity in* $\mathrm{Diff}^{\omega}(S^1)$. *If $\overline{u} : (M_1^n, \mathcal{F}_1) \to (M_2^n, \mathcal{F}_2)$ is a foliation-preserving homeomorphism, then $\overline{u}$ is transversaly real analytic. In particular $\overline{u}^*(GV(\mathcal{F}_2)) = GV(\mathcal{F}_1)$, where $GV(\mathcal{F}_j)$ stands for the Godbillon-Vey class of $\mathcal{F}_j$.*

Proof. – Since $\overline{u}$ conjugates in a natural way the holonomy groups of $\mathcal{F}_1$ and $\mathcal{F}_2$, Theorem C shows that $\overline{u}$ is transversaly real analytic. To the second part of the statement, let us recall that the Godbillon-Vey class $GV(\mathcal{F})$ of $\mathcal{F}$ may be defined by the pull back $\rho(\mathcal{F})^*c$ of a cocycle $c \in H^3(B\Gamma_{\mathbf{R}}^{\infty}, \mathbf{R})$ of the classifying space $B\Gamma_{\mathbf{R}}^{\infty}$ of the pseudogroup $\Gamma_{\mathbf{R}}^{\infty}$ given by the orientation preserving $C^{\infty}-$diffeomorphism of open subsets of $\mathbf{R}$ by the classifying map $\rho(\mathcal{F}) : M \to B\Gamma_{\mathbf{R}}^{\infty}$ (see [B-H]). For $\overline{u}(\mathcal{F}_1) = \mathcal{F}_2$ and $\overline{u}$ is transversaly real analytic, it follows $\rho(\mathcal{F}_2) \circ \overline{u} = \rho(\mathcal{F}_1)$ which in its turn gives $GV(\mathcal{F}_1) = \overline{u}^*(GV(\mathcal{F}_2))$. The corollary is proved. □

6. Ergodicity

In this final section, we provide the proof of Theorem A.

We denote by μ the normalized Lebesgue measure of S^1. Unless we state the contrary, any consideration done in this section is relative to (normalized) Lebesgue measures. As usual, if ν is a measure, we say a property holds $\nu-$almost everywhere (or $\nu-$a.e. or simply a.e. when no misunderstanding is possible) if the set where this property fails has $\nu-$measure *zero*.

Let G be a subgroup of $\mathrm{Diff}^{\omega}(S^1)$. The group G is called $\mu-$*ergodic* if and only if for every borelian $\mathcal{B} \subseteq S^1$ $\mu-$a.e. invariant by G, one has $\mu(\mathcal{B}) = 0$ or 1.

Theorem A is a straighforward consequence of Proposition (6.1) below. Once more one considers a pseudogroup Γ of mappings from open subsets of B (the unit disc) into $\mathbf{C}$ as in section 3 and 5 (i.e. verifying assumptions 1, 2 and 3 of section 3). Let $R \subset B$ be a rectangle equipped with a nowhere *zero* vector field $\mathcal{X}$ which is in the closure of Γ (relative to R). The existence of such an open (non empty) rectangle verifying $R \cap \mathbf{R} \neq \emptyset$ follows from Proposition (3.1). We denote by Γ_R the pseudogroups consisting of mappings from open subsets of R into $\mathbf{C}$ defined by restriction of elements in Γ.

Consider $I = R \cap \mathbf{R}$ endowed with its normalized Lebesgue measure denoted by ν. Let Γ_I be a pseudogroup of mappings from open subsets of I into $\mathbf{R}$. We shall say Γ_I is $\nu-$ergodic on I, for any borelian $\mathcal{B} \subseteq I$ with positive $\nu-$measure, the subset of I defined by

$$\mathcal{B}_{\Gamma_I} = \left(\bigcup_{\gamma \in \Gamma_I} \gamma(\mathcal{B}) \right)$$

has total $\nu-$measure in I. In the same setting $\mathcal{B}$ is called $\nu-$a.e. invariant if $\nu(\mathcal{B}_{\Gamma_I} \setminus \mathcal{B}) = 0$.

PROPOSITION 6.1. – *Let Γ and R be as above and put $I = R \cap \mathbf{R}$. Let ν be the normalized Lebesgue measure of I and let Γ_I be the pseudogroup of mappings from I into $\mathbf{R}$ defined by restriction of elements in Γ. Then Γ_I is $\nu-$ergodic on I.*

Proof. – Let us suppose that the statement is false. Thus there exists a borelian $\mathcal{B}$ $\nu-$a.e. invariant under Γ_I and satisfying $0 < \nu(\mathcal{B}) < 1$ $(1 = \nu(I))$.

We consider a nowhere *zero* vector field $\mathcal{X}$ defined on R and contained in the closure (relative to R) of Γ_R. We set Φ^t the real flow arisen from $\mathcal{X}$ on R. Since $\mathcal{X}$ is non singular on I, to any two points $p, q \in I$, it corresponds a number $t_{pq} \in \mathbf{R}$ such that $\Phi^{t_{pq}}(p) = q$.

Because $0 < \nu(\mathcal{B}) < 1$ $(1 = \nu(I))$, one can find points $p \in \mathcal{B}$ and $q \in I \setminus \mathcal{B}$ which are respectively points of Lebesgue density for $\mathcal{B}$ and $I \setminus \mathcal{B}$. After the observation above, there is $t_{pq} \in \mathbf{R}$ such that $\Phi^{t_{pq}}(p) = q$.

Now recalling that $\mathcal{X}$ is in the closure of Γ_R, one knows that there are a neighborhood $U_p \in \mathbf{C}$ of p and a sequence $\{g_i\}_{i\in\mathbf{N}} \subseteq \Gamma_R$ verifying:

a) $g_i(U_p) \subseteq R$ for all $i \in \mathbf{N}$ and $\Phi^{t_{pq}}(U_p) \subseteq R$.

b) $\Phi^{t_{pq}}$ is the uniform limit (on U_p) of the sequence $\{g_i\}$.

One concludes from Cauchy's Integral Formula that the restriction of $\{g_i\}_{i\in\mathbf{N}}$ to $I = R \cap \mathbf{R}$ converges C^r (for every $r = 1, 2 \ldots$) on I to the restriction of $\Phi^{t_{pq}}$ to I. Let $\{I_p^j\}$ ($I_p^j \subset \mathbf{R}$ for every j) be a sequence of intervals centered at p and with diameter going to *zero* when j goes to infinity. Since p is a Lebesgue density point for $\mathcal{B}$, it results that

$$\lim_{j\to\infty} \frac{\nu(\mathcal{B} \cap I_p^j)}{\nu(I_p^j)} = 1\,.$$

On other hand, if $\{I_q^s\}$ is a sequence of intervals containing q and such that the diameter of I_q^s goes to *zero* when s goes to infinity, since q is a Lebesgue density point for $I \setminus \mathcal{B}$, one has

$$\lim_{s\to\infty} \frac{\nu(\mathcal{B} \cap I_q^s)}{\nu(I_q^s)} = 0\,.$$

Now to each fixed j, we assign $i(j)$ so that $g_{i(j)}(I_p^j)$ is a neighborhood of q. Furthermore, using that the sequence $\{g_i\}$ is C^1 uniformly bounded, one obtains

$$\frac{\nu(g_i(\mathcal{B} \cap I_p^j))}{\nu(g_i(I_p^j))} > Const \frac{\nu(\mathcal{B} \cap I_p^j)}{\nu(I_p^j)}\,,$$

for some positive (and uniform) constant $Const$. Now thanks to the ν−a.e. invariance of $\mathcal{B}$ under Γ_I one gets

$$\lim_{j\to\infty} \frac{\nu(\mathcal{B} \cap g_{i(j)}(I_p^j))}{\nu(g_{i(j)}(I_p^j))} \geq \lim_{j\to\infty} \frac{\nu(g_{i(j)}(\mathcal{B} \cap I_p^j))}{\nu(g_{i(j)}(I_p^j))} \geq Const \frac{\nu(\mathcal{B} \cap I_p^j)}{\nu(I_p^j)} > 0,$$

which contradicts the fact q is a Lebesgue density point for $I \setminus \mathcal{B}$ and proves the statement. □

Proof of Theorem A. – Assume first G has no finite orbits. Then one can find a finite open covering $\{U_i\}_{i=1}^l$ of $S^1 \subset \mathbf{C}$ such that each U_i is endowed with a nowhere *zero* vector field $\mathcal{X}_1$ contained in the closure of the pseudogroup obtained by restriction to U_i of elements in G. Hence G is ergodic (Proposition (6.1)) on each $I_i = U_i \cap S^1$. Indeed, since any two points in S^1 can be "linked" by a "finite path" of flows associated to some vector fields $\mathcal{X}_i$, it is easy to verify that G is ergodic on S^1 in this case.

Now assume G has finite orbits and let P be the finite set obtained by the union of all these orbits. To show G is ergodic on the connected components of $S^1 \setminus P$, it is sufficient to apply Proposition (4.4) combined with the same argument above. Theorem A is proved. □

REMARK 6.2. – Let Γ be a non solvable pseudogroup of local holomorphic diffeomorphisms fixing $0 \in \mathbf{C}$. Let $\Sigma(\Gamma)$ be the separatrix of Γ passing through $0 \in \mathbf{C}$ whose existence is stated by one of Nakai's theorems. If U is a small neighborhood of $0 \in \mathbf{C}$, then each connected component of $U \setminus \Sigma(\Gamma)$ is equipped with two Nakai vector fields $\mathcal{X}$ and $\mathcal{Z}$ which are linearly independent at any point of this component (these components are called Nakai sectors). We want to stand out the same argument used in Proposition (6.1) can be easily extended to show that Γ is ergodic (with respect to the normalized Lebesgue measure) on each Nakai sector.

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