UNRAMIFIED COHOMOLOGY OF CLASSIFYING VARIETIES FOR CLASSICAL SIMPLY CONNECTED GROUPS✩

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ABSTRACT. – Let $F$ be a field and $G \subset \text{SL}_{n,F}$ an algebraic closed subgroup of $\text{SL}_{n,F}$. Denote by $BG$ the factor variety $\text{SL}_n/G$. The stable $F$-birationnal type of $BG$ is independent on the choice of an embedding $G \subset \text{SL}_{n,F}$. The points of $BG$ classify principal homogeneous spaces of $G$. We compute the degree three unramified Galois cohomology with values in $\mathbb{Q}/\mathbb{Z}(2)$ of the function field of $BG$ for all classical semisimple simply connected groups $G$. As an application, examples of groups $G$ (of types $A_n$ and $D_n$) with stably non-rational over $F$ varieties $BG$ are given.

1. Introduction

Let $G$ be a (smooth) algebraic group defined over a field $F$. Choose an injective homomorphism $\rho: G \to \text{SL}_n$ over $F$ and set $X_\rho = S/\rho(G)$. We call $X_\rho$ a classifying variety of $G$ as $X_\rho$ classifies principal homogeneous spaces of $G$: for every field extension $L/F$ there is a natural bijection [19, Ch. I, §5]

$$H^1(L,G) \simeq X_\rho(L)/S(L).$$

In other words, any principal homogeneous space of $G$ over $L$ is isomorphic to the fiber of the natural morphism $S \to X_\rho$ over some point of $X_\rho$ over $L$. The stable birational type of $X_\rho$ is independent on the choice of $\rho$; we denote it by $BG$.

We consider stable birational invariants of $BG$, namely, the unramified cohomology defined as follows. For every $d \geq 0$ let $H^d_{nr}(F(X_\rho))$ be the intersection of the kernels of residue

✩ Partially supported by the N.S.F.
homomorphisms
\[ \partial_v : H^d(F(X_0), \mathbb{Q}/\mathbb{Z}(d-1)) \longrightarrow H^{d-1}(F(v), \mathbb{Q}/\mathbb{Z}(d-2)) \]
for all discrete valuations \( v \) on \( F(X_0) \) over \( F \). (Here \( \mathbb{Q}/\mathbb{Z}(i) \) is the direct limit of \( \mathbb{Q}/\mathbb{Z}(i) \) taken over all \( n \) prime to the characteristic exponent of \( F \).) The group \( H_{nr}^d(F(X_0)) \) is independent on the choice of \( \rho \) (up to canonical isomorphism) and we denote it by \( H_{nr}^d(BG) \). The natural homomorphism
\[ H^d(F, \mathbb{Q}/\mathbb{Z}(d-1)) \longrightarrow H_{nr}^d(BG) \]
splits by evaluation at the distinguished point of \( BG \), thus,
\[ H_{nr}^d(BG) = H^d(F, \mathbb{Q}/\mathbb{Z}(d-1)) \oplus H_{nr}^d(BG)_{\text{norm}} \]
with the latter group being the group of normalized unramified classes. If the classifying variety \( BG \) is stably rational, then \( H_{nr}^d(BG)_{\text{norm}} = 0 \).

The group \( H_{nr}^d(BG)_{\text{norm}} \) is trivial. Over an algebraically closed field \( F \) the group \( H_{nr}^d(BG) \) has been studied in [14,15,2]. Saltman, for \( G = PGL_n \), and Bogomolov, for \( G \) arbitrary connected reductive, showed that \( H_{nr}^d(BG) \) is trivial (see also [4]). In [16] Saltman has shown that \( H_{nr}^3(BG) = 0 \) for \( G = PGL_n \) and \( n \) odd.

Using [2] (or [4]), one may show that for a (connected) semisimple group \( G \) defined over an arbitrary field \( F \) the group \( H_{nr}^d(BG)_{\text{norm}} \) is trivial. The aim of the paper is to compute the group \( H_{nr}^3(BG)_{\text{norm}} \) for any (connected) semisimple simply connected group \( G \) of classical type defined over an arbitrary field. The idea is to consider the subgroup \( A^0(X_0, H^3) \) of all classes in \( H^3(F(X_0), \mathbb{Q}/\mathbb{Z}(2)) \) unramified only with respect to discrete valuations associated to irreducible divisors of \( X_0 \). This group is also independent of the choice of \( \rho \) and we denote it by \( A^0(BG, H^3) \); thus,
\[ H_{nr}^3(BG) \subset A^0(BG, H^3). \]
Similarly,
\[ A^0(BG, H^3) = H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \oplus A^0(BG, H^3)_{\text{norm}} \]
where \( A^0(BG, H^3)_{\text{norm}} \) is the group of normalized classes.

It was noticed by Rost that the group \( A^0(BG, H^3) \) is canonically isomorphic to the group \( \text{Inv}^3(G, H) \) of dimension 3 cohomological invariants of \( G \), i.e., morphisms of functors
\[ H^1(\ast, G) \longrightarrow H^3(\ast, \mathbb{Q}/\mathbb{Z}(2)) \]
from the category of field extensions of \( F \) to the category of sets. The invariants corresponding to the elements of \( H_{nr}^3(BG) \) (respectively \( A^0(BG, H^3)_{\text{norm}} \)) are called unramified (respectively normalized). The group of normalized invariants \( \text{Inv}^3(G, H)_{\text{norm}} \) has been computed by Rost: if \( G \) is absolutely simple simply connected, that group is cyclic with canonical generator \( r_C \) (called Rost invariant) of certain order \( n_C \) which can be computed in terms of representation theory of \( G \). Thus, in order to compute the group \( H_{nr}^3(BG) \) it suffices to determine all multiples \( mn_C \) of the Rost invariant that are unramified.

It is proved in the paper that if \( G \) is simply connected of type \( B_n \) or \( C_n \), the unramified group \( H_{nr}^3(BG)_{\text{norm}} \) is trivial. On the other hand, for the types \( A_n \) and \( D_n \) the group \( H_{nr}^3(BG)_{\text{norm}} \) is either zero or cyclic of order 2 and can be determined for all groups in terms of the Tits algebras of \( G \). This computation leads to examples of classifying varieties \( BG \) that are not stably rational.
On the other hand, if the Tits algebras of $G$ are trivial (for example, if $G$ is quasi-split or $F$ is separably closed), the group $H^3_{nr}(BG)_{\text{norm}}$ vanishes.

The idea to consider ramification of Rost invariants is due to Rost and Serre (cf. [18]). For the reader’s convenience we include proofs of some basic properties of Rost invariants (Appendix A) due to Rost and a computation of Rost numbers $n_G$ given in [9, §31] without proofs (Appendix B).

2. Classifying varieties

2.1. Definition of classifying varieties

A connected algebraic group $S$ defined over a field $F$ is called special if $H^1(L, S) = 1$ for any field extension $L/F$. Examples of special groups are $\text{SL}_n, \text{Sp}_{2n}, \text{GL}_1(A)$ for a central simple $F$-algebra $A$. Note that the varieties of all these groups are rational.

Let $G$ be an algebraic group over a field $F$. Choose an embedding $\rho: G \hookrightarrow S$ into a special rational group $S$. Consider the variety

$$X_\rho = S/\rho(G),$$

which is called a classifying variety of $G$. Obviously, $X_\rho$ depends on the choice of $\rho$.

Let $\rho': G \to S'$ be another embedding. In order to compare $X_\rho$ and $X_{\rho'}$ consider the diagonal embedding

$$\rho'' = (\rho, \rho'): G \to S \times S',$$

which induces a surjection $\alpha: X_{\rho''} \to X_\rho$. Clearly, $\alpha$ is an $S'$-torsor over $X_\rho$. Since $S'$ is special, this torsor is trivial at the generic point of $X_\rho$, hence $F(X_{\rho''}) \cong F(X_\rho)(S')$. The group $S'$ is rational, so that $X_\rho$ is stably birationally equivalent to $X_{\rho''}$. Similarly, $X_{\rho'}$ is stably birationally equivalent to $X_{\rho''}$, hence $X_\rho$ and $X_{\rho'}$ are stably birationally equivalent. We denote by $BG$ the variety $X_\rho$ for some $\rho$. The stable birational type of $BG$ is well defined.

2.2. Homotopy invariant functors

Let

$$J: \text{Fields}/F \to \text{Ab}$$

be a functor from the category of field extensions of $F$ to the category of abelian groups. We say that $J$ is homotopy invariant if for any field extension $L/F$, the map $J(L) \to J(L(t))$ is an isomorphism.

**Proposition 2.1.** – Let $J$ be a homotopy invariant functor, $G$ an algebraic group over $F$. Then the group $J(F(X_\rho))$ depends only on $G$ and does not depend (up to canonical isomorphism) on the choice of an embedding $\rho$.

**Proof.** – In the notation of (2.1), the field extension $F(X_{\rho''})/F(X_\rho)$ is purely transcendental, hence the map $J(F(X_\rho)) \to J(F(X_{\rho''}))$ is an isomorphism. Similarly, we have an isomorphism $J(F(X_{\rho'})) \to J(F(X_{\rho''}))$. □

We denote the group $J(F(X_\rho))$ by $J(BG)$. The group $J(BG)$ can detect stable non-rationality of a classifying variety $X_\rho$: if the natural homomorphism $J(F) \to J(BG)$ is not an isomorphism, then the variety $X_\rho$ is not stably rational.
Let \( \alpha : G \to G' \) be a group homomorphism. Consider two embeddings \( \rho : G \hookrightarrow S \) and \( \rho' : G' \hookrightarrow S' \) with \( S \) and \( S' \) special rational groups and the embedding \( \rho'' = (\rho, \rho') : G \hookrightarrow S \times S' \). The projection \( S \times S' \to S' \) induces a dominant morphism \( X_{\rho''} \to X_{\rho'} \) and hence a group homomorphism

\[
J(BG') = J(F(X_{\rho'})) \longrightarrow J(F(X_{\rho''})) = J(BG)
\]

for a homotopy invariant functor \( J \). Thus, the assignment \( G \to J(BG) \) is a contravariant functor from the category of algebraic groups over \( F \) to the category of abelian groups.

### 2.3. Cycle modules

A cycle module \( M \) over a field \( F \) is an object function \( E \mapsto M^*(E) \) from the category \( \text{Fields}/F \) to the category of \( \mathbb{Z} \)-graded abelian groups together with some data and rules [13, §2]. The data include a graded module structure on \( M \) under the Milnor ring of \( F \), a degree 0 homomorphism \( i_v : M(E) \to M(L) \) for any field homomorphism \( i_v : E \to L \) over \( F \), a degree 0 homomorphism (norm map) \( j_v : M(L) \to M(E) \) for any finite field homomorphism \( j_v : E \to L \) over \( F \) and also a degree \(-1\) residue homomorphism \( \partial_v : M(E) \to M(F(v)) \) for a discrete, rank one, valuation \( v \) on \( E \) over \( F \) with residue field \( F(v) \).

**Example 2.2.** – We will be considering the cycle module \( H \) given by Galois cohomology [13, Remark 2.5]

\[ H^d(E) = H^d(E, \mathbb{Q}/\mathbb{Z}(d-1)) \overset{\text{def}}{=} \lim_{\pi} H^d(E, \mu_n^{\otimes (d-1)}), \]

where the limit is taken over all \( n \) prime to the characteristic exponent of \( F \).

Let \( M \) be a cycle module over \( F \), \( L/F \) a finite field extension, \( v \) a discrete valuation of \( L \) over \( F \). An element \( a \in M^d(L) \) is called unramified with respect to \( v \) if \( a \) belongs to the kernel of the residue homomorphism

\[ \partial_v : M^d(L) \longrightarrow M^{d-1}(F(v)). \]

An element \( a \in M^d(L) \) is unramified over \( F \) if it is unramified with respect to all discrete valuations of \( L \) over \( F \). We denote the subgroup in \( M^d(L) \) of all unramified over \( F \) elements by \( M^d_{\text{nr}}(L) \) (cf. [5]).

Let \( i_v : E \to L \) be a field homomorphism over \( F \), \( v \) a discrete valuation of \( L \) over \( F \), \( v' \) the restriction of \( v \) on \( E \). Assume that an element \( a \in M^d(E) \) is unramified with respect to \( v' \) (if \( v' \) is not trivial). By rules R3a and R3c in [13], the element \( i_v(a) \in M^d(L) \) is unramified with respect to \( v \). Hence, \( i_v \) takes \( M^d_{\text{nr}}(E) \) into \( M^d_{\text{nr}}(L) \), making \( M^d_{\text{nr}} \) a functor from \( \text{Fields}/F \) to \( \text{Ab} \).

**Proposition 2.3.** – The functor \( M^d_{\text{nr}} \) is homotopy invariant.

**Proof.** – Let \( L/F \) be a field extension and \( i_v : L \to L(t) \) the inclusion. By homotopy property [13, 2.2(H)], the homomorphism \( i_v : M^d(L) \to M^d(L(t)) \) is injective and the image of \( i_v \) consists of all elements in \( M^d(L(t)) \) that are unramified with respect to all discrete valuation on \( L(t) \) over \( L \). Therefore, for every \( m \in M^d_{\text{nr}}(L(t)) \) there is (unique) \( m' \in M^d(L) \) such that \( i_v(m') = m \), and we need to show that \( m' \in M^d_{\text{nr}}(L) \).

Let \( v \) be any discrete valuation of \( L \) over \( F \) with residue field \( F(v) \) and let \( v' \) be an extension of \( v \) to \( L(t) \) with ramification index 1 and residue field \( F(v)(t) \). Denote by \( j \) the inclusion
\( F(v) \to F(v)(t) \). By rule R3a in [13], the diagram

\[
\begin{array}{ccc}
M^d(L) & \xrightarrow{i_*} & M^d(L(t)) \\
\partial_\psi & & \partial_\psi \\
M^{d-1}(F(v)) & \xrightarrow{j_*} & M^{d-1}(F(v)(t))
\end{array}
\]

commutes. Since \( j_* \) is injective and \( \partial_\psi(m') = 0 \), it follows that \( \partial_\psi(m') = 0 \), i.e., \( m' \) is unramified. □

**Corollary 2.4.** For any algebraic group \( G \), the group \( M^d_{nr}(BG) \) is well defined.

Let \( X_\rho \) be a classifying variety of a group \( G \) with respect to an embedding \( \rho: G \hookrightarrow S \) with \( S \) a special rational group. Consider the group \( A^0(X_\rho, M^d) \) consisting of all elements in \( M^d(F(X_\rho)) \) unramified with respect to discrete valuations associated to all irreducible divisors of \( X_\rho \) [13, §2]. Thus, we have

\[
M^d_{nr}(BG) = M^d_{nr}(F(X_\rho)) \subset A^0(X_\rho, M^d).
\]

By Corollary A.2, the group \( A^0(X_\rho, M^d) \) does not depend on the choice of \( \rho \) if \( S \) is a split semisimple simply connected group (for example, \( S = \text{SL}_n \) or \( \text{Sp}_{2n} \)). We denote by \( A^0(BG, M^d) \) the group \( A^0(X_\rho, M^d) \) with such a choice of \( S \). We have

\[
M^d_{nr}(BG) \subset A^0(BG, M^d).
\]

The unramified group \( M^d_{nr}(BG) \) has nice functorial properties with respect to field extensions. Namely, for any field extension \( L/F \) there is a well defined restriction homomorphism

\[
\text{res}: M^d_{nr}(BG) \longrightarrow M^d_{nr}(BG_L),
\]

where \( BG_L = BG \times_{\text{Spec } F} \text{Spec } L \). If \( L/F \) is finite, the rule R3b in [13] implies the existence of the corestriction homomorphism

\[
\text{cor}: M^d_{nr}(BG_L) \longrightarrow M^d_{nr}(BG).
\]

Denote by \( A^0(BG, M^d)_{\text{norm}} \) the kernel of the evaluation (pull-back) homomorphism [13, §12]

\[
i^*: A^0(BG, M^d) \longrightarrow A^0(\text{Spec } F, M^d) = M^d(F)
\]

induced by the distinguished point \( i: \text{Spec } F \to BG \). Thus,

\[
A^0(BG, M^d) = M^d(F) \oplus A^0(BG, M^d)_{\text{norm}}.
\]

Also set

\[
M^d_{\text{nr}}(BG)_{\text{norm}} = M^d_{\text{nr}}(BG) \cap A^0(BG, M^d)_{\text{norm}}.
\]

Thus,

\[
M^d_{\text{nr}}(BG) = M^d(F) \oplus M^d_{\text{nr}}(BG)_{\text{norm}}.
\]

Note that if \( BG \) is stably rational, then \( M^d_{\text{nr}}(BG)_{\text{norm}} = 0 \) by Proposition 2.3.
3. Unramified invariants of algebraic groups

Let $G$ be an algebraic group defined over a field $F$ and let $M$ be a cycle module over $F$. An invariant of $G$ in $M$ of dimension $d$ is a morphism

$$H^1(\ast, G) \longrightarrow M^d(\ast)$$

of functors from the category Fields/F to the category of sets [20, §6]. All the invariants of $G$ in $M$ of dimension $d$ form an abelian group $\text{Inv}^d(G, M)$.

An element in $M^d(F)$ defines a constant invariant of $G$ in $M$. Thus, there is an inclusion

$$M^d(F) \subset \text{Inv}^d(G, M).$$

An invariant is called normalized if it takes the distinguished element in $H^1(F, G)$ to zero (i.e., it can be considered as a morphism of functors with values in the category of pointed sets). We denote the subgroup of normalized invariants by $\text{Inv}^d(G, M)_{\text{norm}}$. Clearly,

$$\text{Inv}^d(G, M) = M^d(F) \oplus \text{Inv}^d(G, M)_{\text{norm}}.$$

Let $X = X_\rho$ be a classifying variety of $G$ with respect to an embedding of $G$ into a special rational group $S$. An invariant $u \in \text{Inv}^d(G, M)$ defines for any field extension $L/F$ the composition

$$\tilde{u}_L: X(L) \longrightarrow H^1(L, G) \overset{u_L}{\longrightarrow} M^d(L),$$

which is constant on orbits of the $S(L)$-action on $X(L)$.

Let $\xi \in X(F(X))$ be the generic point. The image $\tilde{u}_{F(X)}(\xi)$ is an element of the group $M^d(F(X))$. A proof of the following Proposition 3.1 and Theorem 3.2 can be found in Appendix A.

**Proposition 3.1 (Rost, Serre [18]).** – The element $\tilde{u}_{F(X)}(\xi)$ is unramified with respect to the discrete valuation associated to every irreducible divisor of $X$, i.e., $\tilde{u}_{F(X)}(\xi) \in A^0(X, M^d)$.

Thus, by Proposition 3.1, we get a homomorphism

$$\theta: \text{Inv}^d(G, M) \longrightarrow A^0(X, M^d), \quad u \mapsto \tilde{u}_{F(X)}(\xi).$$

**Theorem 3.2 (Rost).** – The map $\theta$ is injective. If the special group $S$ is split semisimple simply connected, $\theta$ is an isomorphism.

Thus, for any algebraic group $G$, we have a canonical isomorphism

$$\theta: \text{Inv}^d(G, M) \overset{\sim}{\longrightarrow} A^0(BG, M^d).$$

We say that an invariant $u \in \text{Inv}^d(G, M)$ is unramified if $\theta(u) \in M^d_{\text{ur}}(BG)$ and normalized if $u(1) = 0$. We denote the groups of unramified (respectively normalized) invariants by $\text{Inv}^d_{\text{ur}}(G, M)$ (respectively $\text{Inv}^d_{\text{norm}}(G, M)$).

**Lemma 3.3.** – Let $E/F$ be a field extension with $\text{tr.deg}(E/F) \geq \dim X + \dim S$. Then for every point $x \in X(E)$ there is $s \in S(E)$ such that the point $sx \in X(E)$, considered as a morphism $\text{Spec} E \to X$, is dominant.
Proof. – Let $Y$ be the closure of the image of $x: \text{Spec } E \rightarrow X$. The function field $F(Y)$ can be considered as a subfield in $E$. Since $\text{tr.deg}(E/F) \geq \dim X + \dim S$, there is a field between $F(Y)$ and $E$, purely transcendental over $F(Y)$ of degree $\dim S$. Since $S$ is rational, we can embed the function field $F(S \times Y)$ into $E$ over $F(Y)$. The composition

$$f: \text{Spec } E \longrightarrow \text{Spec } F(S \times Y) \longrightarrow S \times Y$$

is dominant and defines a point $s \in S(E)$. The point $sx$ is given by the composition

$$sx: \text{Spec } E \xrightarrow{f} S \times Y \xrightarrow{m} X,$$

where $m$ is the restriction of the action morphism. Since $S$ acts transitively on $X$, $m$ is dominant and therefore so is $sx$. □

The following proposition provides a useful tool to determine whether a given invariant is unramified.

**Proposition 3.4.** – An invariant $u \in \text{Inv}^d(G,M)$ is unramified if and only if for any field extension $L/F$ and for every point $y \in H^1(L((t)), G)$ the element $u(y) \in M^d(L((t)))$ is unramified with respect to the canonical discrete valuation on $L((t))$ over $L$.

**Proof.** – Assume that $u(y) \in M^d(L((t)))$ is unramified for any field extension $L/F$ and every $y \in H^1(L((t)), G)$. Let $X$ be a classifying variety of $G$, $v$ a discrete valuation on $F(X)$ over $F$. The completion $E$ of $F(X)$ with respect to $v$ is isomorphic to $L((t))$, where $L$ is the residue field of $v$. Let $y \in H^1(E, G)$ be the image of the generic point $\xi$ under the composition

$$X(F(X)) \longrightarrow H^1(F(X), G) \rightarrow H^1(E, G)$$

induced by the embedding $i: F(X) \hookrightarrow E$. By assumption, the element $u(y)$ is unramified with respect to the extension $v'$ on $E$ of the valuation $v$. The composition

$$M^d(F(X)) \xrightarrow{i_*} M^d(E) \xrightarrow{\partial_{v'}} M^{d-1}(L)$$

coincides with $\partial_v$. Hence,

$$\partial_v(u(\xi)) = \partial_{v'}(i_*u(\xi)) = \partial_{v'}(u(y)) = 0,$$

i.e., $u$ is unramified.

Conversely, assume that $u$ is unramified. Let $L/F$ be a field extension and $y \in H^1(L((t)), G)$. Choose a point $x \in X(L((t)))$ representing $y$. By Lemma 3.3, we may assume that the point $x$, considered as a morphism $\text{Spec } L((t)) \rightarrow X$, is dominant. Thus, the function field $F(X)$ is isomorphic to a subfield in $L((t))$. The natural homomorphism induced by the field extension $L((t))/F(X)$,

$$X(F(X)) \longrightarrow X(L((t)))$$

takes the generic point $\xi$ to $x$, hence the map

$$M^d(F(X)) \longrightarrow M^d(L((t)))$$

takes $u(\xi)$ to $u(y)$. Since $u(\xi)$ is unramified, so is $u(y)$. □
4. Rost invariants

We will be considering the following cohomological cycle module $H$ over $F$ (Example 2.2):

$$H^d(L) = H^d(L, Q/\mathbb{Z}(d - 1))$$

for a field extension $L/F$. We shall compute the unramified groups

$$H^3_{nr}(BG) \simeq \text{Inv}^3_{nr}(G, H)$$

for every (connected) semisimple simply connected group $G$. The following propositions reduce the problem to the case of an absolutely simple simply connected group $G$. By Corollary B.3, in this case the group $\text{Inv}^3_{nr}(G, H)$ is finite cyclic with a canonical generator $r_G$ (Rost invariant). In the following sections we consider all absolutely simple groups of classical types $A_n, B_n, C_n$ and $D_n$.

An arbitrary simply connected group $G$ is a product of simple simply connected groups $G_1 \times G_2 \times \cdots \times G_k$ [21, 3.1.2]. The functorial properties of $H^3_{nr}$ considered in (2.2) and Corollary B.4 imply

**Proposition 4.1.** $H^3_{nr}(BG)_{\text{norm}} \simeq \prod_{i=1}^k H^3_{nr}(BG_i)_{\text{norm}}$.

Any simply connected group $G$ is of the form $R_{L/F}(G')$, where $L/F$ is a finite separable field extension and $G'$ is an absolutely simple simply connected group over $L$ [21, 3.1.2]. By Corollary B.5, the two compositions $j^* \circ \text{res}_{L/F}$ and $\text{cor}_{L/F} \circ i^*$ in the diagram

$$H^3_{nr}(BG)_{\text{norm}} \xrightarrow{\text{res}_{L/F}} H^3_{nr}(BG_{L})_{\text{norm}} \xrightarrow{j^*} H^3_{nr}(BG')_{\text{norm}}$$

are isomorphisms inverse to each other.

**Proposition 4.2.** $H^3_{nr}(BG)_{\text{norm}} \simeq H^3_{nr}(BG')_{\text{norm}}$.

We will need the following lemmas. The definition and properties of the numbers $n_\alpha$, $n_G$, $n'_G$ and the Rost invariant $r_G$ are collected in Appendix B.

**Lemma 4.3.** Let $\alpha : H \to G$ be a homomorphism of absolutely simple simply connected groups with $n_\alpha = 1$. If $H^3(\norm{BH}) = 0$ and $n'_G = n_G$, then $H^3_{nr}(BG)_{\text{norm}} = 0$.

**Proof.** The image of $r_G$ in $A^0(\norm{BH})$ is equal to $r_H$ since $n_\alpha = 1$. Assume that $nr_G$ is unramified for some $m \in \mathbb{Z}$. It follows from $H^3_{nr}(BH)_{\text{norm}} = 0$ that $nr_H = 0$ and hence $m$ is divisible by $n'_G = n_G$. Therefore, $nr_G = 0$. □

**Lemma 4.4.** Let $G$ be an absolutely simple simply connected group over $F$. Assume that for a field extension $L/F$, $n'_{GL} = n'_G$ and $H^3_{nr}(BG_{L})_{\text{norm}} = 0$. Then $H^3_{nr}(BG)_{\text{norm}} = 0$.

**Proof.** Assume that $mr_G$ is unramified for some $m \in \mathbb{Z}$. Since over $L$ this element becomes trivial, $m$ is divisible by $n'_{GL} = n'_G$. Hence, $nr_G = 0$. □

In the following sections we compute the groups $H^3_{nr}(BG)_{\text{norm}}$ for all absolutely simple simply connected algebraic groups $G$ of classical types. We follow the classification of simple groups given in [9, Ch. 6].
5. Type $A_{n-1}$

5.1. Inner type

Let $G$ be a simply connected group of inner type $A_{n-1}$, i.e., $G = SL_1(A)$ for a central simple $F$-algebra $A$ of degree $n$. We embed $G$ into the special rational group $GL_1(A)$ with the classifying variety $X = G_m$. Since $X$ is rational, $M_{nr}(BG)_{\text{norm}} = 0$ for any cycle module $M$ over $F$.

THEOREM 5.1. – Let $G$ be a simply connected group of inner type $A_n$. Then a classifying variety $BG$ is stably rational and $M_{nr}(BG)_{\text{norm}} = 0$ for any cycle module $M$ over $F$.

5.2. Outer type

Let $G$ be a simply connected group of outer type $A_{n-1}$, i.e., $G = SU(B, \tau)$, where $B$ is a central simple algebra of degree $n \geq 3$ with unitary involution $\tau$ over a quadratic separable field extension $K/F$.

Over $K$, the group $G_K$ is isomorphic to $SL_1(B)$ and by B.3.1, a value of the Rost invariant $r_{G_K}$ over a field extension $L/K$ is of the form $(x) \cup [B_L]$ for some $x \in L^\times$. Hence, taking the norm of the invariant $r_{G_K}$ in the extension $K/F$, we conclude that a value of the invariant $2r_G$ over a field extension $E/F$ is of the form
\[
N_{K \otimes E/F}((x) \cup [B_{K \otimes E}]) \in H^3(E, Q/Z(2))
\]
for some $x \in (K \otimes E)^\times$.

LEMMA 5.2. – If $\exp(B)$ is even, the invariant $\exp(B)r_G$ is unramified.

Proof. – Let $L/F$ be a field extension. By Proposition 3.4, it suffices to show that for every $y \in H^1(L((t)), G)$ the element $\exp(B)r_G(y)$ in $H^3(L((t)), Q/Z(2))$ is unramified with respect to the canonical valuation $v$ of $L((t))$. Consider two cases.

Case 1: $K \otimes L$ is not a field, i.e., the group $G_{L((t))}$ is of inner type. By (B.3.1), the order of the Rost invariant over $L((t))$ is equal to $\exp(B_{L((t))})$, hence $\exp(B)r_G$ is zero over $L((t))$ and obviously $\exp(B)r_G(y) = 0$ is unramified.

Case 2: $KL = K \otimes L$ is a field. Since $\exp(B)$ is even, it suffices to show that $\partial_v(2r_G(y)) = 0$.

We have by (1),
\[
2r_G(y) = N_{KL((L))/L((t))}(x \cup [B_{KL((L))}])
\]
for some $x \in KL((t))^\times$. Then for the valuation $v'$ of $KL((t))$,
\[
\partial_{v'}(2r_G(y)) = \partial_{v'}(N_{KL((L))/L((t))}(x \cup [B_{KL((L))}])) = N_{KL/L}[B_{KL}]^{v'(x)} = 0
\]
since $B_{KL}$ has unitary involution $\tau_{KL}$ and therefore
\[
N_{KL/L}[B_{KL}] = 0 \in H^2(L, Q/Z(1))
\]
by [9, Th. 3.1]. \qed

Denote by $D = D(B, \tau)$ the discriminant algebra of $(B, \tau)$ [9, §10].

THEOREM 5.3. – Assume that $\text{char}(F) \neq 2$. Let $G = SU(B, \tau)$, where $B$ is a central simple algebra of degree $n \geq 3$ with unitary involution $\tau$ over a quadratic field extension $K/F$. Then
the group $H^3_{nr}(BG)_{\text{norm}}$ is cyclic of order 2 generated by $\exp(B)r_G$, except for the following cases (when this group is trivial):

(1) $\exp(B)$ is odd;
(2) $n$ is a 2-power and $\exp(B) = n$;
(3) $n$ is a 2-power, $\exp(B) = n/2$ and the discriminant algebra $D$ is split.

Proof. – Assume that the invariant $mr_G$ is unramified for some $m \in \mathbb{Z}$. Since over $K$, the group $H^3_{nr}(BG_K)_{\text{norm}}$ is trivial by Theorem 5.1, the class $mr_G$ vanishes over $K$. It is shown in B.3.1 that over $K$, $r_G$ has order $\exp(B) \neq 1$, therefore $\exp(B) \neq m$. Thus, since $\text{char}(F) \neq 2$ and $\exp(B) | n_G = 2 \exp(B)$ by Theorem B.20, the group $H^3_{nr}(BG)_{\text{norm}}$ consists of at most two elements and it is cyclic of order 2 if and only if $n_G = 2 \exp(B)$ and the invariant $\exp(B)r_G$ is unramified.

Assume first that $B$ splits, i.e., $B = \text{End}_K(V)$, where $V$ is a vector space over $K$ of dimension $n \geq 3$. The involution $\tau$ is adjoint with respect to a hermitian form $h$ on $V$ over $K/F$ [9, Th. 4.2]. Theorem B.20 gives $n_G = 2$ over any field extension of $F$. By Lemma 4.4, it is sufficient to prove that $H^3_{nr}(BG_L)_{\text{norm}} = 0$ for some field extension $L/F$.

We claim that over a field extension $E/F$, there is a non-degenerate subform $(V_0, h_0)$ in $(V \otimes_F E, h \otimes_F E)$ of dimension 2 and non-trivial discriminant $\text{disc}(h_0)$. To prove the claim we consider two cases. If $h$ is anisotropic, we can take $E = F$ and $h_0$ an arbitrary subform of $h$ of dimension 2. Assume that $h$ is isotropic, $h = h' \perp H$, where $H$ is a hyperbolic plane. Let $a \in F^\times$ be a value of $h'$. The form $H$ is universal, hence the form $h_0 = (a, t)$ is a subform of $h$ over $E = F(t)$. The discriminant $\text{disc}(h_0)$ is not trivial as $at$ is not a norm in the quadratic extension $K(t)/F(t)$.

Now we can replace $F$ by $E$ and consider the subgroup $H = \text{SU}(V_0, h_0) \subset G$. The group $H$ is a simply connected group of (inner) type $A_1$, therefore, $H = \text{SL}_1(Q)$ for a quaternion algebra $Q$ [9, Th. 26.9]. Since the discriminant of $h_0$ is not trivial, $H$ is not split and hence $Q$ does not split. Thus, $n_H = 2$ by Theorem B.17. Let $\rho: H \hookrightarrow G$ be the embedding. By Example B.6, $n_{\rho} = 1$. Hence, the inner case 5.1 and Lemma 4.3, applied to the embedding $\rho$, imply that $H^3_{nr}(BG)_{\text{norm}} = 0$.

Now assume that $\exp(B)$ is odd. We have $n_G = 2 \exp(B)$ by Theorem B.20. The first part of the proof shows that the non-trivial invariant $\exp(B)r_G$ ramifies over any field extension of $F$ which splits $B$ but not $K$ (for example, the function field of the variety $R_{K/F}(\text{SB}(B))$, where $\text{SB}(B)$ is the Severi–Brauer variety of $B$ [9, §1.C]). Hence, $\exp(B)r_G$ already ramifies over $F$ and therefore the group $H^3_{nr}(BG)_{\text{norm}}$ is trivial.

Finally, assume that $\exp(B)$ is even. By Lemma 5.2, $\exp(B)r_G$ is unramified, i.e., the group $H^3_{nr}(BG)_{\text{norm}}$ is cyclic of order 2 if and only if $n_G = 2 \exp(B)$. The result follows from Theorem B.20. □

Corollary 5.4. – Assume that $\text{char}(F) \neq 2$. Let $G = \text{SU}(B, \tau)$ with $\exp(B)$ even and $\text{deg}(B) \geq 4$. Assume in addition that if $\text{deg}(B) = 4$, the discriminant algebra $D(B, \tau)$ does not split. Then a classifying variety $BG$ is not stably rational.

Proof. – Consider the field extension $L = F(R_{K/F}(\text{SB}(B^{\otimes 2})))$. By the index reduction formula [17, §3], $D(B, \tau)$ is not split over $L$ and $\exp(B_L) = 2$, so that, extending the base field to $L$, we may assume that $\exp(B) = 2$. Then, by Theorem 5.3, the unramified group $H^3_{nr}(BG)_{\text{norm}}$ is not trivial. □

Remark 5.5. – Examples of stably non-rational classifying varieties $BG$ with simply connected $G$ of type $A_n$ exist for every odd $n \geq 3$. Every number field can be taken for the base field $F$. 

4$^{e}$ SÉRIE – TOME 35 – 2002 – N° 3
6. Type $B_n$

Let $G$ be a simply connected group of type $B_n$, $n \geq 2$, i.e., $G = \text{Spin}(V,q)$, where $(V,q)$ is a non-degenerate quadratic form of dimension $2n + 1$.

If $n = 2$, we have $B_2 = C_2$ and $H^3_{nr}(BG)_{\text{norm}} = 0$ by Theorem 7.1.

Assume that $n \geq 3$. We claim that over some field extension of $F$, $(V,q)$ contains a non-degenerate subform $(V',q')$ of dimension 5 and of Witt index at most 1. To prove the claim, we may assume first that $q$ is hyperbolic. Let $f$ be anisotropic 3-dimensional form over some field extension $L/F$. Since $\dim q \geq 7$, $f$ is isomorphic to a subform of $q_L$. The Witt index of any 5-dimensional form $q'$ such that $f \subset q' \subset q_L$ is at most 1. The claim is proved.

The group $H = \text{Spin}(V',q')$ is a subgroup of $G_L$ of type $B_2 = C_2$. We have $n_H = n_G = nG_L = 2$ by Theorem B.22 and $H^3_{nr}(BH)_{\text{norm}} = 0$ by the case $n = 2$. Example B.10 shows that $n_\rho = 1$ for the embedding $\rho: H \hookrightarrow G$. Lemma 4.3 implies that $H^3_{nr}(BG_L)_{\text{norm}} = 0$ and by Lemma 4.4, $H^3_{nr}(BG)_{\text{norm}} = 0$.

**Theorem 6.1.** – Let $G$ be a simply connected group of type $B_n$. Then $H^3_{nr}(BG)_{\text{norm}} = 0$.

7. Type $C_n$

Let $G$ be a simply connected group of type $C_n$, $n \geq 2$, i.e., $G = \text{Sp}(A,\sigma)$, where $A$ is a central simple $F$-algebra of degree $2n$ with a symplectic involution $\sigma$. We consider the canonical embedding

$$\rho: \text{Sp}(A,\sigma) \hookrightarrow \text{SL}_1(A).$$

The map $x \mapsto \sigma(x)x$ establishes an isomorphism between the classifying space $X_\rho$ and the open subvariety in the linear space

$$\text{Symd}(A,\sigma) = \{a + \sigma(a), \ a \in A\},$$

consisting of all invertible elements [9, 29.24]. This variety is rational (being an open subset of an affine space), hence $M^d_{\text{nr}}(BG)_{\text{norm}} = 0$ for any cycle module $M$ over $F$.

**Theorem 7.1.** – Let $G$ be a simply connected group of type $C_n$. Then a classifying variety $BG$ is stably rational and $M^d_{\text{nr}}(BG)_{\text{norm}} = 0$ for any cycle module $M$ over $F$.

8. Type $D_n$

We assume $\text{char } F \neq 2$. Let $G$ be a simply connected group of type $D_n$ (we exclude groups of trialitarian type in $D_4$), i.e., $G = \text{Spin}(A,\sigma)$ for a central simple algebra $A$ of degree $2n$ over $F$ with an orthogonal involution $\sigma$. The standard isogeny

$$\alpha: G \longrightarrow \text{O}^+(A,\sigma)$$

induces a map

$$\alpha_*: H^1(F,G) \longrightarrow H^1(F,\text{O}^+(A,\sigma)).$$

Let $X$ be the variety of pairs

$$(a,x) \in \text{Sym}(A,\sigma) \times F^\times$$

**Annales Scientifiques de l’École Normale Supérieure**
such that $\text{Nrd}(a) = x^2$. The morphism

$$\text{GL}_1(A) \longrightarrow X, \quad g \mapsto (g\sigma(g), \text{Nrd}(g))$$

induces an isomorphism of varieties $\text{GL}_1(A)/\mathbb{O}^+(A,\sigma) \cong X$ making $X$ a classifying variety of $\mathbb{O}^+(A,\sigma)$ and identifying the set $H^1(F, \mathbb{O}^+(A,\sigma))$ with the factor set of $X(F)$ modulo the action of the group $\text{GL}_1(A)$ given by $g(a,x) = (g\sigma(g), \text{Nrd}(g)x)$ [9, 29.27].

The embedding

$$\beta : \mathbb{O}^+(A,\sigma) \hookrightarrow \text{SL}_1(A)$$

induces the morphism $X \to \mathbb{G}_m = \text{GL}_1(A)/\text{SL}_1(A)$ taking a pair $(a,x)$ to $x$. Thus, the map

$$\beta_* : H^1(F, \mathbb{O}^+(A,\sigma)) \longrightarrow H^1(F, \text{SL}_1(A)) = F^\times / \text{Nrd}(A^\times)$$

takes the class represented by a pair $(a,x)$ to $x \text{Nrd}(A^\times)$.

By Example B.11, $n_{\beta\alpha} = 2$, hence the Rost invariant for $\text{SL}_1(A)$ corresponds to $2r_G$ under $\beta \circ \alpha$ and therefore, by B.3.1, for any field extension $L/F$ and every $y \in H^1(L,G)$,

$$(2) \quad 2r_G(y) = (x) \cup [A_L] \in H^3(L, \mathbb{Q}/\mathbb{Z}(2)),$$

provided the class $\alpha_* (y)$ is represented by a pair $(a,x) \in X(F)$.

Let $Q$ be a quaternion division algebra and let $(V,h)$ be a $(-1)$-hermitian forms over $Q$ with respect to the canonical (symplectic) involution on $Q$. Assume that discriminant of $h$ (i.e., discriminant of the adjoint involution $\sigma_h$ on $\text{End}_Q(V)$) is trivial. Then the Clifford algebra $C(\text{End}_Q(V), \sigma_h)$ is a product of two central simple $F$-algebras $C^+(h)$ and $C^-(h)$ [9, §8]. If in addition $\dim_Q V$ is even, exponent of the algebras $C^\pm$ is at most 2 [9, Th. 9.13].

**Lemma 8.1.** Let $k$ and $l$ be two $(-1)$-hermitian forms over $Q$ with respect to the canonical involution on $Q$. Assume that $\text{rank}(k) + \text{rank}(l)$ is even and the discriminant of the form $k_F(\{t\}) \perp tl_F(\{t\})$ over $F(\{t\})$ is trivial. Then

$$\partial_v [C^\pm (k_F(\{t\}) \perp tl_F(\{t\})] = \text{disc}(l) \in F^\times / F^\times 2,$$

where $v$ is the discrete valuation on $F(\{t\})$ and $\partial_v : 2\text{Br} F(\{t\}) \to F^\times / F^\times 2$ is the residue homomorphism.

**Proof.** We can split $Q$ generically (by the function field of the conic curve corresponding to $Q$) and assume that we are given two quadratic forms $f$ and $g$ of even dimension such that the form $f_F(\{t\}) \perp t^2 F(\{t\})$ has trivial discriminant. Denote by $IF$ the fundamental ideal in the Witt ring of $F$ [10]. The commutativity of the diagram

$$\begin{array}{ccc}
I^2F(\{t\}) & \to & 2\text{Br} F(\{t\}) \\
\partial_v & \downarrow & \partial_v \\
IF & \text{disc} & F^\times / F^\times 2
\end{array}$$

and description of the residue homomorphisms in [10] yield the result. $\square$

**Proposition 8.2.** If $n = 4$ and $\text{disc}(\sigma)$ is trivial, then the invariant $2r_G$ is unramified.
Proof. – By Proposition 3.4, it suffices to prove that for any field extension $L/F$ and every $y \in H^1(L(\ell), G)$, the residue $\partial_\ell(2r_G(y))$ is trivial. We may assume that $L = F$.

We have by (2),

$$2r_G(y) = (x) \cup [A_{F(\ell)}] \in H^3\left(F(\ell), \mathbb{Q}/\mathbb{Z}(2)\right)$$

with $x \in F(\ell)^\times$ such that $x^2 = \text{Nrd}(a)$ for some $a \in \text{Sym}(A_{F(\ell)}, \sigma g_{F(\ell)})$. Hence

$$\partial_\ell(2r_G(y)) = [A]^v(x) \in H^2(F, \mathbb{Q}/\mathbb{Z}(1)) = \text{Br} F.$$ 

Thus, we may assume that $A$ is not split. Since then $\exp(A) = 2$, it suffices to prove that $v(x)$ is even. Assume that $v(x)$ is odd. The integer $v(\text{Nrd}(a))$ is divisible by $\text{ind}(A)$, $v(x) = v(\text{Nrd}(a))/2$ is divisible by $\text{ind}(A)/2$. Therefore, $\text{ind}(A) = 2$, i.e., $A$ is similar to a quaternion division algebra $Q$ over $F$, $A \simeq M_4(Q)$.

By [9, Th. 4.2], the involution $\sigma$ is adjoint to a $(-1)$-hermitian form $h$ of rank 4 over $Q$ with respect to the canonical involution on $Q$. The symmetric element $a$ gives rise to another $(-1)$-hermitian form $h'$ of rank 4 over $Q_{F(\ell)}$ with trivial discriminant. We diagonalize this form by choosing an element $g \in \text{GL}_4(A_{F(\ell)})$ such that $g a \sigma(g)$ is the diagonal matrix $\text{diag}(t^4 q_1, t^2 q_2, t^3 q_3, t^4 q_4)$, where $q_i \in Q^\times$ are pure quaternions and $e_i = 0$ or 1. We have

$$t^2 \sum e_i \prod \text{Nrd}(q_i) = \prod \text{Nrd}(t^{e_i} q_i) = \text{Nrd}(g)^2 \text{Nrd}(a) = \text{Nrd}(g)^2 x^2.$$ 

Since $v(\text{Nrd}(g))$ is divisible by $\text{ind}(A)$ and hence even and $v(x)$ is odd, the sum of the $e_i$ is odd. There are two cases:

Case 1: $\sum e_i = 1$. We may assume that $e_1 = 1$ and $e_2 = e_3 = e_4 = 0$. The pair $(a, x)$ belongs to the image of

$$H^1(F(\ell), G) \longrightarrow H^1(F(\ell), O^+(A, \sigma)).$$

By [8], one of the components $C^+$ and $C^-$ of the Clifford algebra of the form $h_{F(\ell)} \perp h'$ splits. By Lemma 8.1, $\text{disc}(q_2)$ is trivial, i.e., $-\text{Nrd}(q_2) = y^2$ for some $y \in F^\times$ [9, 7.2]. Hence $\text{Nrd}(y + q_1) = 0$, a contradiction, since $Q$ is a division algebra.

Case 2: $\sum e_i = 3$. We may assume that $e_1 = e_2 = e_3 = 1$ and $e_4 = 0$. As in case 1, by Lemma 8.1, $\text{disc}(q_1, q_2, q_3)$ is trivial, i.e., $-\text{Nrd}(q_1 q_2 q_3)$ is a square in $F^\times$. Since the form $h'$ has trivial discriminant $\text{disc}(h') = \text{Nrd}(q_1 q_2 q_3 q_4)$, it follows that $-\text{Nrd}(q_4)$ is also a square in $F^\times$, a contradiction as in case 1. \hfill \Box

Lemma 8.3. – Assume $A$ is not split, $n \geq 4$ and in the case $n = 4$ the discriminant of $\sigma$ is not trivial. Then there is a field extension $L/F$ and an element $y \in H^1(L(\ell), G)$ such that $2r_G(y)$ ramifies.

Proof. – Denote by $S$ the generalized Severi–Brauer variety $SB(2, A)$ [9, 1.16]. Replacing $F$ by $F(S)$, we can get $A$ similar to a quaternion division algebra $Q = (a, b)$, $A \simeq M_n(Q)$ by [1]. Let $W$ be the quadric hypersurface given by the quadratic form $(1, 1, -a, -b, ab)$. The field $F(W)$ does not split $Q$ by [10, Ch. IX]. Thus, we may replace $F$ by $F(W)$ and therefore assume that there is an element $q \in Q$ with $\text{Nrd}(q) = -1$. Every element of $Q$ is a product of two pure quaternions. Hence there are pure quaternions $q_1, q_2$ and $q_3$ such that $q_1 q_2 q_3 = q$.

The involution $\sigma$ is adjoint to a $(-1)$-hermitian form $h$ of rank $n$ over $Q$. We claim that there is a $(-1)$-hermitian form $h''$ of rank $n - 3$ over $Q$ (maybe over some field extension of $F$ which does not split $Q$) such that discriminants of the $(-1)$-hermitian forms

$$h' = h''_{F(\ell)} \perp t\langle q_1, q_2, q_3 \rangle$$

ANNALES SCIENTIFIQUES DE L’ÉCOLE NORMALE SUPÉRIEURE
and $h_{F((t))}$ over $F((t))$ coincide, i.e., $\text{disc}(h'') = \text{disc}(h)$. Consider two cases.

Case 1: $n \geq 5$. The determinant of a $(-1)$-hermitian form is the product of reduced norms of pure quaternions of a diagonalization. Every element of $Q$ is a product of two pure quaternions, hence every value of the reduced norm of $Q$ can be the determinant of a $(-1)$-hermitian form of rank at least 2. This is the case with $h''$, since $\text{rank}(h'') = n - 3 \geq 2$.

Case 2: $n = 4$. Let $i, j$ be the generators of $Q$, $i^2 = a, j^2 = b, ij = -ji$. Consider projective quadric hypersurface given by the equation

$$aX^2 + bY^2 - abZ^2 = cT^2,$$

where $c \in F^\times$ represents $\text{disc}(h) = \text{disc}(\sigma)$. Let $L$ be the function field of the quadric. Since $c$ is not a square in $F^\times$, the field $L$ does not split $Q$ by [10, Ch. IX]. Now we can set $h'' = (q'')$, where $q'' = Xi + Yj + Zij \in Q_L$. Then

$$\text{disc}(h'') = -\text{Nrd}(q'') = cL^2 = \text{disc}(h_L).$$

We replace $F$ by $L$. The claim is proved.

Thus, the hermitian form $h'$ represents an element in $H^1(F((t)), \text{O}(A, \sigma))$, i.e., a pair $(a, x) \in \text{Sym}(A_{F((t))}, \sigma_{F((t))})$ such that $\text{Nrd}(a) = x^2$. Let $H$ (respectively $H'$) be the matrix of $h$ (respectively $h'$). By [8], $\text{Nrd}(a) = \text{Nrd}(H) \text{Nrd}(H')^{-1}$, hence

$$v(\text{Nrd} a) = v(\text{Nrd} H) - v(\text{Nrd} H') = 0 - 6 = -6.$$

Therefore $v(\pm x) = -3$ and the class $([A_{F((t))}] \cup \pm x)$ ramified since

$$\partial_v([\pm x] \cup [A_{F((t))}]) = [A]^v(\pm x) = [A] \neq 1.$$

It suffices to lift $(a, x)$ or $(a, -x)$ to an element $y$ in the set $H^1(F((t)), G)$ (maybe over an extension of $F$ which does not split $A$). By [8], we need to split one of the components $C^+$ and $C^-$ of the Clifford algebra of the form $h_{F((t))}$ ramified over $A$. Hence, the function field of the Severi–Brauer variety of $D^+$ splits $D^+$ and also $C^+$ but does not split $A$. □

Now Proposition 3.4 yields

**Corollary 8.4.** – Assume $A$ is not split, $n \geq 4$ and in the case $n = 4$ that the discriminant of $\sigma$ is not trivial. Then the invariant $2r_G(x)$ ramifies.

**Theorem 8.5.** – Let $(A, \sigma)$ be a central simple algebra over a field $F$ (char $F \neq 2$) of degree $2n \geq 6$ with an orthogonal involution $\sigma$, $C = C(A, \sigma)$ the Clifford algebra, $G = \text{Spin}(A, \sigma)$. Then $H^3_{\text{nr}}(BG)$ is trivial, except for the following cases (when this group is cyclic of order 2 generated by $2r_G$):

1. $n = 3$, $\text{disc}(\sigma)$ is not trivial, $A$ is not split and $\exp(C) = 2$;
2. $n = 4$, $\text{disc}(\sigma)$ is trivial, $A$ is not split and neither component $C^+$ nor $C^-$ of $C$ splits.
Proof. – The case \( n = 3 \) follows from Theorem 5.3 since \( D_3 = A_3 \). Under this equality, the Clifford algebra \( C \) coincides with the algebra \( B \) considered in Section 5.1 and the algebra \( A \) coincides with the discriminant algebra \( D \) [9, §26].

Consider the case \( n \geq 4 \). Assume \( A \) splits, i.e., \( G = \text{Spin}(V, q) \) for a non-degenerate quadratic form \((V, q)\) of dimension \( 2n \).

Suppose first that \( n = 4 \). Since \( n_G = 2 \) by Theorem B.26, it suffices to show that the invariant \( r_G \) ramifies. Extending \( F \), we assume that \( q \) is hyperbolic. The image of the map

\[
H^1(L, G) \to H^1(L, O^+(V, q))
\]

for a field extension \( L/F \) is identified with the set of isomorphism classes of quadratic forms over \( L \) of dimension 8 with trivial discriminant and trivial Clifford invariant [9, 31.41], i.e., with the set of isomorphism classes of forms similar to 3-fold Pfister forms \((a, b, c)\) over \( L \). The Rost invariant \( r_G \) associates to this form its Arason invariant [9, 31.42]

\[
(a) \cup (b) \cup (c) \in H^3(L, Q/Z(2)).
\]

Let \( L/F \) be a field extension having a non-split quaternion algebra \( Q = (a, b) \). Consider the form \( q = \langle t, a, b \rangle \) over \( L((t)) \). It follows from

\[
\partial_v (r_G(q)) = \partial_v ((t) \cup (a) \cup (b)) = [Q] \neq 1,
\]

that \( r_G(q) \) ramifies. By Proposition 3.4, the Rost invariant \( r_G \) ramifies.

For arbitrary \( n \geq 4 \) we can find a non-degenerate subform \((V_0, q_0)\) in \((V, q)\) of dimension 8. Then \( H = \text{Spin}(V_0, q_0) \) is a subgroup in \( G = \text{Spin}(V, q) \). By Theorem B.27, \( n_H = n_G = 2 \). It follows from the case \( n = 4 \) of the proof, Example B.10 and Lemma 4.3 that \( H_{3\text{norm}}^1(BG) = 0 \).

Now assume that \( A \) is not split. By Theorems B.26 and B.27, \( n_G \) divides 4. Let \( L \) be any splitting field for \( A \). As shown above, the Rost invariant \( r_G \) ramifies over \( L \) and hence ramifies over \( F \). Hence the group \( H_{3\text{norm}}^1(BG) \) is nontrivial if and only if the invariant \( 2r_G \) is nontrivial and unramified. Now the statement follows from Proposition 8.2, Corollary 8.4 and Theorems B.26 and B.27.

\( \square \)

Corollary 8.6. – A classifying variety \( BG \) for the group \( G = \text{Spin}(A, \sigma) \) is not stably rational in the following cases:

1. \( n = 3 \), \( \text{disc}(\sigma) \) is not trivial and the algebras \( A \) and \( C \) are not split;
2. \( n = 4 \), \( \text{disc}(\sigma) \) is not trivial and the algebras \( A_Z \) and \( C \) are not split (\( Z/F \) being the discriminant quadratic field extension of \( (A, \sigma) \));
3. \( n = 4 \), \( \text{disc}(\sigma) \) is trivial and the algebras \( A, C^+, C^- \) are not split.

Proof. – The case \( n = 3 \) follows from Corollary 5.4 since \( D_3 = A_3 \). If \( n = 4 \), the variety \( BG \) is not stably rational even over the discriminant quadratic field extension \( Z/F \) by Theorem 8.5 since \( H_{3\text{norm}}^1(BG_Z) \neq 0 \).

Remark 8.7. – Examples of stably non-rational classifying varieties \( BG \) with simply connected \( G \) of type \( D_n \) exist for \( n = 3 \) and \( n = 4 \) over every number field \( F \).

Appendix A. Invariants of algebraic groups

A.1. Proof of Proposition 3.1

(The proof is different from the one in [18].) Let \( m : S \times X \to X \) be the action morphism. For any field extension \( L/F \) and every \( s \in S(L) \), \( x \in X(L) \), we have \( \tilde{u}_L(sx) = \tilde{u}_L(x) \). Now let
\( L = F(S \times X) \). Denote by \( \eta \in S(L) \) the image of the generic point of \( S \) under the embedding \( F(S) \hookrightarrow L \) induced by the projection \( p_1 : S \times X \to S \) and by \( \xi \in X(L) \) the image of the generic point \( \xi \) of \( X \) under the embedding \( F(X) \hookrightarrow L \) induced by the projection \( p_2 : S \times X \to X \). Then \( \eta \xi \in X(L) \) is the image of \( \xi \) under the embedding \( i : F(X) \hookrightarrow L \) induced by \( m \).

Choose a point \( x \in X \) of codimension 1. We need to show that

\[
(A.1) \quad \partial_x(\tilde{u}_{F(X)}(\xi)) = 0 \in M^{d-1}(F(x)).
\]

Consider the point \( y \in S \times X \) of codimension 1 with the closure \( S \times \{x\} \). Since \( S \) acts transitively on \( X \), \( m(y) \) is the generic point of \( X \). Hence the restriction of the discrete valuation on \( L \) associated to the point \( y \) is trivial on \( i(F(X)) \). Therefore, by rule R3 in [13],

\[
(A.2) \quad \partial_y(\tilde{u}_L(\xi')) = \partial_y(\tilde{u}_L(\eta\xi')) = \partial_y(i_\ast \tilde{u}_{F(X)}(\xi)) = 0 \in M^{d-1}(F(y)).
\]

Let \( k : F(x) \to F(y) \) be the field homomorphism induced by the projection \( p_2 : S \times X \to X \). By the rule R3a in [13] and (A.2),

\[
k_* \partial_x(\tilde{u}_{F(X)}(\xi)) = \partial_y(p_2_\ast(\tilde{u}_{F(X)}(\xi))) = \partial_y(\tilde{u}_L(\xi')) = 0 \in M^{d-1}(F(y)).
\]

The field \( F(y) \) is isomorphic to \( F(x)(S) \). Since the smooth variety \( S \) has a rational point, the map

\[
k_* : M^{d-1}(F(x)) \to M^{d-1}(F(y))
\]

is injective (cf. [11, Lemma 1.3]) and hence (A.1) holds.

**A.2. Proof of Theorem 3.2**

**Lemma A.1 (Specialization principle).** – Let \( x_1 \) and \( x_2 \) be two points of \( X \) such that \( x_2 \) is regular and of codimension 1 in \( \{x_1\} \). We also consider the \( x_i \) as a point of \( X(F(x_i)) \).

Suppose that for an invariant \( u \in \text{Inv}^d(G, M) \) we have \( \tilde{u}_{F(x_1)}(x_1) = 0 \in M^d(F(x_1)) \). Then \( \tilde{u}_{F(x_2)}(x_2) = 0 \in M^d(F(x_2)) \).

**Proof.** – Denote by \( A \) the local ring of the point \( x_2 \) in the variety \( \{x_1\} \). By assumption, \( A \) is a discrete valuation ring with quotient field \( F(x_1) \) and residue field \( F(x_2) \). Let \( \tilde{A} \) be the completion of \( A \), so that

\[
\tilde{A} \simeq F(x_2)[[t]]
\]

[23, Ch. VIII, Th. 27]. Denote by \( E \) the quotient field of \( \tilde{A} \), the completion of the field \( F(x_1) \), thus, \( E = F(x_2)((t)) \). We have the following diagram of maps induced by natural morphisms

\[
\begin{array}{ccc}
H^1_{\text{ét}}(X, G) & \longrightarrow & H^1_{\text{ét}}(\tilde{A}, G) \\
& \downarrow{i} & \downarrow{j} \\
H^1(F(x_1), G) & \longrightarrow & H^1(E, G)
\end{array}
\]

with the bijections \( i \) and \( j \) inverse to each other [6, Exp. XXIV, Prop. 8.1]. Considering images in all the sets of the diagram of the class in \( H^1_{\text{ét}}(X, G) \) representing the universal \( G \)-torsor \( S \to X \), we get

\[
\tilde{u}_{F(x_2)}(x_2)_E = \tilde{u}_E(x_2) = \tilde{u}_E(x_1) = \tilde{u}_{F(x_1)}(x_1)_E = 0,
\]
i.e., the class $\tilde{u}_F(x_2)$ splits over $E$. It remains to notice that the map $M^d(F(x_2)) \to M^d(E)$ is injective (being split by a specialization homomorphism [13, p. 329]). 

Assume that for $u \in \text{Inv}^d(G, M)$ we have $\tilde{u}_F(\xi) = 0$. For a field extension $L/F$ consider any point $p \in X(L), i.e., a morphism $p: \text{Spec}(L) \to X$. We need to show that $\tilde{u}_L(p) = 0$. Denote by $x \in X$ the only point in the image of $p$. There is a sequence of points $\xi = x_1, x_2, \ldots, x_m = x$ such that $x_i+1$ is regular of codimension 1 in the closure $\overline{\{x_i\}}$ for all $i = 1, 2, \ldots, m - 1$. By Lemma A.1, $\tilde{u}_F(x)(\xi) = 0$. The element $p$ is the image of $x$ under $X(F(x)) \to X(L)$, induced by the natural homomorphism $F(x) \to L$, hence $\tilde{u}_L(p) = 0$, being the image of $\tilde{u}_F(x)$ under $M^d(F(x)) \to M^d(L)$. Thus, $u = 0$, i.e., $\theta$ is injective.

Assume now that $S$ is split semisimple simply connected. Let $v \in A^0(X, M^d)$ and $x \in X(L)$ be a point over a field extension $L/F$. We define the class $v(x) \in M^d(L)$ as the image of $v$ under the pull-back homomorphism

$$x^*: A^0(X, M^d) \to A^0(\text{Spec} L, M^d) = M^d(L)$$

with respect to $x: \text{Spec} L \to X$. Thus, we get a map

$$\tilde{u}_L: X(L) \to M^d(L), \quad x \mapsto v(x).$$

In order to show that $\tilde{u}_L$ defines an invariant $u \in \text{Inv}^d(G, M)$ with $\theta(u) = v$ it suffices to prove that the map $v$ is constant on orbits of the $(S(L))$-action on $X(L)$.

Let $s \in S(L), x \in X(L)$. Then $v(sx) \in M^d(L)$ is the image of $v$ under the pull-back homomorphism with respect to the composition

$$\text{Spec} L \xrightarrow{(s,x)} S \times X \xrightarrow{m} X,$$

where $m$ is the action morphism. The element $v(x) \in M^d(L)$ is the image of $u$ under the pull-back homomorphism with respect to the composition

$$\text{Spec} L \xrightarrow{(s,x)} S \times X \xrightarrow{p_2} X,$$

where $p_2$ is the projection. Thus, it suffices to show that $m$ and $p_2$ induce the same homomorphism

$$m^* = p_2^*: A^0(X, M^d) \to A^0(S \times X, M^d).$$

Consider the map $i: X \to S \times X$, $i(x) = (1, x)$. Since $p \circ i = \text{id}_X = m \circ i$, we have $i^* \circ p_2^* = \text{id} = i^* \circ m^*$. Hence, it is sufficient to prove that $p_2^*$ is an isomorphism.

The spectral sequence associated to $p_2$ [13, §8]:

$$E_1^{p,q} = \prod_{x \in X^{(i)}} A^q(S_{F(x)}, M^{d-p}) \Rightarrow A^{p+q}(S \times X, M^d)$$

induces an exact sequence

$$0 \to A^0(S \times X, M^d) \xrightarrow{r} A^0(S_{F(X)}, M^d) \to \prod_{x \in X^{(i)}} A^0(S_{F(x)}, M^{d-1}).$$
The group $S$ is split simply connected, hence, by [7, Th. 4.7(i)], the natural homomorphism

$$M^{k}(F(x)) \rightarrow A^{0}(S_{F(x)}, M^{k})$$

is an isomorphism for every $x \in X$ and $k \in \mathbb{Z}$. By [13, Prop. 8.1], the kernel of $\partial$ is isomorphic to $A^{0}(X, M^{d})$ and the map $k$ induces an isomorphism

$$j: A^{0}(S \times X, M^{d}) \rightarrow A^{0}(X, M^{d})$$

such that the composition $j \circ p_{2}$ is the identity. Hence $p_{2}^{*}$ is an isomorphism. □

**Corollary A.2.** The group $A^{0}(X_{p}, M^{d})$ does not depend on the choice of an embedding $\rho: G \hookrightarrow S$ into a split semisimple simply connected group $S$.

**Appendix B. Rost numbers**

Let $G$ be a split simply connected group defined over a field $F$, $T \subset G$ a split maximal torus over $F$, $W$ the Weyl group. The $W$-invariant elements $S^{2}(T^{*})^{W}$ in the symmetric square of the character group $T^{*} = \text{Hom}(T, G_{m})$ are $W$-invariant integral quadratic forms on the vector space $V = T_{*} \otimes \mathbb{R}$ of the co-root system, where $T_{*} = \text{Hom}(G_{m}, T)$ is the co-character lattice. By [3, Ch. VI, §1, Prop. 7], $S^{2}(T^{*})^{W}$ is a free abelian group with a canonical basis given by positive definite forms $q_{1}, q_{2}, \ldots, q_{k}$ corresponding to the $k$ connected components of the Dynkin diagram of $G$. In particular, if $G$ is simple, the group $S^{2}(T^{*})^{W}$ is cyclic with the canonical generator $q_{G}$ being a (unique) integral-valued positive definite $W$-invariant quadratic form on $T_{*}$. Since $G$ is simply connected, the lattice of co-characters $T_{*}$ is generated by the co-roots of the root system dual to the root system of $G$. A quadratic form on the space $V$ taking value 1 on short co-roots is integral, hence it coincides with $q_{G}$. Thus, $q_{G}(\beta) = 1$ for every short co-root $\beta$.

**Example B.1.** Let $G = \text{SL}_{n}$, $n \geq 2$. A split maximal torus $T$ of $G$ is isomorphic to the kernel of the product homomorphism

$$(G_{m})^{n} \rightarrow G_{m}.$$ 

Hence the group of co-characters $T_{x}$ can be identified with the subgroup in $\mathbb{Z}^{n}$ consisting of all $n$-tuples $x = (x_{1}, x_{2}, \ldots, x_{n})$ with trivial sum of the $x_{i}$ [9, §24]. The Weyl group $W = S_{n}$ acts by permutations of the $x_{i}$. Clearly, the $W$-invariant integral quadratic form

$$q_{G}(x) = \frac{1}{2} \sum_{i=1}^{n} x_{i}^{2} = - \sum_{i<j} x_{i}x_{j}$$

is the canonical generator of $Q(G)$. It takes value 1 on the (short) co-roots $\pm(e_{i} - e_{j})$ for $i \neq j$.

Now let $G$ be a (not necessarily split) simply connected group defined over a field $F$. Choose a maximal torus $T \subset G$ over $F$. The absolute Galois group $\text{Gal}(F) = \text{Gal}(F_{\text{sep}}/F)$ acts on $S^{2}(T_{\text{sep}}^{*})^{W}$ by permuting the basis forms $q_{i}$, thus $S^{2}(T_{\text{sep}}^{*})^{W}$ is a permutation $\text{Gal}(F)$-module. In particular, if $G$ is absolutely simple, the group $S^{2}(T_{\text{sep}}^{*})^{W}$ is cyclic with the canonical generator $q_{G}$ and trivial $\text{Gal}(F)$-action. Clearly, the form $q_{G}$ does not change under field extensions.

We denote the group $(S^{2}(T_{\text{sep}}^{*})^{W})^{\text{Gal}(F)}$ by $Q(G)$. If $G$ is absolutely simple, $Q(G) = \mathbb{Z}q_{G}$.

A homomorphism $\rho: G \rightarrow G'$ of simply connected groups induces a homomorphism $Q(\rho): Q(G') \rightarrow Q(G)$ [9, p. 433].

Let $p$ be the characteristic exponent of $F$. 

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A. MERKURJEV

4e série – tome 35 – 2002 – n° 3
Theorem B.2 ((Rost) [7, Appendix B, Cor. C.2(b)]). There is a natural surjective homomorphism
\[ \gamma_G : Q(G)[1/p] \longrightarrow A^0(BG, H^3 \text{norm}). \]
The kernel of \( \gamma_G \) is generated over \( \mathbb{Z}[1/p] \) by the elements \( Q(\alpha)(q_{SL_n}) \) for all irreducible representations \( \alpha : G \rightarrow SL_n \) defined over \( F \).

Let \( \rho : G \rightarrow G' \) be a homomorphism of absolutely simple simply connected groups. Then \( Q(\rho)(q_{G'}) = n_{\rho} \cdot q_G \) for a uniquely determined integer \( n_{\rho} \geq 0 \). We set
\[ n_G = \gcd n_\alpha \]
with the \( \gcd \) taken over all irreducible representations \( \alpha : G \rightarrow SL_n \) of the group \( G \). Let \( n_G' \) be the greatest divisor of \( n_G \) prime to \( p \). Thus, \( n_G = n_G' \) if \( n_G \) is relatively prime to \( p \).

Denote by \( r_G \) the element \( \gamma(q_G) \in A^0(BG, H^3 \text{norm}) \). The corresponding invariant \( \theta^{-1}(r_G) \in \text{Inv}^3(G,H) \text{norm} \) (Theorem 3.2) we also denote by \( r_G \) and call it the Rost invariant of \( G \).

Corollary B.3. Let \( G \) be an absolutely simple simply connected group. Then \( A^0(BG, H^3 \text{norm}) \) is a cyclic group generated by \( r_G \) of order \( n_G' \).

Note that \( r_G \) (but not \( n_G \)) does not change under field extensions: for a field extension \( L/F \), \( r_{GL} \) is the image of \( r_G \) under the canonical homomorphism
\[ A^0(BG, H^3 \text{norm}) \longrightarrow A^0(BG_L, H^3 \text{norm}). \]

An arbitrary simply connected group \( G \) is a product of simply connected groups \( G_1 \times G_2 \times \cdots \times G_k \). The group \( Q(G) \) splits obviously into a direct sum of the \( Q(G_i) \). Hence, Theorem B.2 implies

Corollary B.4 ([9, Cor. 31.38]).

\[ A^0(BG, H^3 \text{norm}) \cong \prod_{i=1}^{k} A^0(BG_i, H^3 \text{norm}). \]

Any simple simply connected group \( G \) is of the form \( R_{L/F}(G') \), where \( L/F \) is a finite separable field extension and \( G' \) is an absolutely simple simply connected group over \( L \). The group \( G' \) is a canonical direct factor of \( G_L \), therefore, there are canonical homomorphisms
\[ G_L \xrightarrow{i} G' \xleftarrow{j} G. \]

By naturality of the homomorphism \( \gamma_G \) in Theorem B.2, the following diagrams commute (with right and left arrows respectively)
\[ Q(G) \xrightarrow{\gamma_G} Q(G)_L \xleftarrow{i} Q(G') \]
\[ A^0(BG, H^3 \text{norm}) \xrightarrow{\gamma_G} A^0(BG_L, H^3 \text{norm}) \xleftarrow{i} A^0(BG', H^3 \text{norm}). \]
Clearly, the two compositions $j^* \circ \text{res}_{L/F}$ and $\text{cor}_{L/F} \circ i^*$ in the top row of the diagram are isomorphisms inverse to each other. We have proved

**Corollary B.5** ([9, Cor. 31.39]). — The two compositions $j^* \circ \text{res}_{L/F}$ and $\text{cor}_{L/F} \circ i^*$ in the bottom row of the diagram

$$A^0(BG, H^3)_{\text{norm}} \xrightarrow{\sim} A^0(BG', H^3)_{\text{norm}}$$

are isomorphisms inverse to each other.

### B.1. The numbers $n_\rho$

Let $\rho: G \to G'$ be a homomorphism of absolutely simple simply connected groups. Clearly,

$$n_G \mid n_\rho \cdot n_{G'}.$$  

(\text{B.1})

Let $\beta: G_m \to G$ be a short co-root of $G$. Then

$$n_\rho = n_\rho \cdot q_G(\beta) = q_{G'}(\rho \circ \beta).$$  

(\text{B.2})

In particular, if $\rho \circ \beta$ is a short co-root of $G'$, then $n_\rho = 1$.

The number $n_\rho$ does not change under field extensions. If $\rho': G' \to G''$ is another homomorphism of absolutely simple simply connected groups, then

$$n_{\rho' \circ \rho} = n_{\rho'} \cdot n_\rho.$$  

**Example B.6.** – For the standard inclusion $\rho: \text{SL}_n \hookrightarrow \text{SL}_m$ ($m > n$) we have $n_\rho = 1$ since the co-roots of $\text{SL}_n$ are also co-roots of $\text{SL}_m$ and have the same length.

**Example B.7.** – Let $\rho: \text{Sp}_{2n} \hookrightarrow \text{SL}_{2n}$ be the standard embedding. The embedding of maximal tori is defined by

$$(t_1, t_2, \ldots, t_n) \mapsto (t_1, t_2, \ldots, t_n, t_1^{-1}, t_2^{-1}, \ldots, t_n^{-1}).$$

Hence, the map of co-character groups takes $(x_1, x_2, \ldots, x_n)$ to

$$(x_1, x_2, \ldots, x_n, -x_1, -x_2, \ldots, -x_n).$$

The image of the short co-root $\pm e_i$ is the short co-root $\pm (e_i - e_{n+i})$, hence $n_\rho = 1$.

**Example B.8.** – Let $\rho: \text{Spin}_{2n} \hookrightarrow \text{Spin}_{2n+1}$, $n \geq 3$, be the standard embedding. A maximal torus of the first group is also maximal in the second. The short co-roots $\pm e_i \pm e_j$ of $G'$ correspond to the same short co-roots of $G$, hence $n_\rho = 1$.

**Example B.9.** – Let $\rho: \text{Spin}_{2n+1} \hookrightarrow \text{Spin}_{2n+2}$, $n \geq 2$, be the standard embedding. The homomorphism of co-character groups of maximal tori is induced by the canonical inclusion $\mathbb{Z}^n \hookrightarrow \mathbb{Z}^{n+1}$. The co-characters $\pm e_i \pm e_j$ are the short co-roots in both the groups, hence $n_\rho = 1$.

**Example B.10.** – Let $\rho: \text{Spin}_n \hookrightarrow \text{Spin}_m$ be the standard embedding, $m > n \geq 5$. By Examples B.8 and B.9, $n_\rho = 1$. 

\[4^e \text{ SÉRIE – TOME 35 – 2002 – N° 3}\]
Example B.11. – Let \( \rho \) be the composition

\[
\text{Spin}_{2n} \xrightarrow{\alpha} \text{O}^+_{2n} \xrightarrow{\eta} \text{SL}_{2n},
\]

where \( \alpha \) is the standard isogeny. The co-character group of the maximal torus of \( \text{Spin}_{2n} \) is contained in \( \mathbb{Z}^n \) with the co-roots \( \pm e_i \pm e_j \) of the same length. The corresponding co-character of the maximal torus of \( \text{SL}_{2n} \) is \( \pm (f_i - g_i) \pm (f_j - g_j) \) if we identify the group of all co-characters with a subgroup in \( \prod \mathbb{Z}f_i \oplus \prod \mathbb{Z}g_i \). By Example B.1 and formula (B.2), \( n_\rho = 2 \).

### B.2. The numbers \( n_G \)

Let \( G \) be a simply connected semisimple group defined over a field \( F \), \( \alpha : G \to \text{SL}(V) \) a representation over \( F \), \( T \subset G \) a maximal torus defined over \( F \). The space \( V_{\text{sep}} = V \otimes_F F_{\text{sep}} \) splits into a direct sum of one-dimensional eigenspaces with some eigenvalues \( \chi_1, \chi_2, \ldots, \chi_m \in T_{\text{sep}}^* \). Then, by Example B.1 and (B.2),

\[
n_\alpha = \frac{1}{2} \sum_i \langle \chi_i, \eta_G \rangle^2 \in \mathbb{Z},
\]

where \( \eta_G \in T_{\text{sep}}^* \) is a short co-root of \( G \).

For an algebraic group \( H \) denote by \( R(H) \) the representation ring of \( H \). Consider the following additive group homomorphism

\[
\Phi_G : R(T_{\text{sep}}) = \mathbb{Z}[T_{\text{sep}}^*] \longrightarrow \frac{1}{2} \mathbb{Z}, \quad \sum \chi_i \mapsto \frac{1}{2} \sum \langle \chi_i, \eta_G \rangle^2.
\]

Thus, for a representation \( \alpha : G \to \text{SL}(V) \) we have

(B.3) \[
n_\alpha = \Phi_G(\alpha|_{T_{\text{sep}}}^*).
\]

The Weyl group \( W \) of \( G_{\text{sep}} \) acts naturally on \( R(T_{\text{sep}}) \). The absolute Galois group \( \text{Gal}(F) \) acts on \( R(T_{\text{sep}}) \) through the \( * \)-action on \( T_{\text{sep}}^* \) defined in [21, 2.3]. The semidirect product \( \Delta \) of \( W \) and \( \text{Gal}(F) \) acts naturally on \( R(T_{\text{sep}}) \).

Denote by \( \Lambda^+ \subset T_{\text{sep}}^* \) the cone of dominant characters (with respect to some system of simple roots). The group \( \text{Gal}(F) \) leaves \( \Lambda^+ \) invariant. The field of definition of a dominant character \( \chi \in \Lambda^+ \), denoted by \( F(\chi) \), is the field corresponding to the stabilizer of \( \chi \) in \( \text{Gal}(F) \) by Galois theory.

Let \( C \) be the center of \( G \). For a character \( \chi \in T_{\text{sep}}^* \), denote by \( \bar{\chi} \in C_{\text{sep}}^* \) its restriction on \( C \). For a dominant character \( \chi \in \Lambda^+ \), the field of definition \( F(\bar{\chi}) \) of \( \bar{\chi} \) is contained in \( F(\chi) \). Denote by \( A_\chi \) a Tits algebra associated to \( \bar{\chi} \) [9, §27], [22, §4], so that \( A_\chi \) is a central simple algebra over \( F(\bar{\chi}) \) uniquely determined up to Brauer equivalence over \( F(\chi) \). For every character \( \chi \in \Lambda^+ \) set

\[
A_\chi = A_\chi \otimes_{F(\chi)} F(\chi).
\]

The algebra \( A_\chi \) is a central simple over \( F(\chi) \). The index of \( A_\chi \) depends only on the \( \text{Gal}(F) \)-orbit of \( \chi \).

Let \( \chi \in \Lambda^+ \) be a dominant character. Denote by \( \Delta(\chi) \in R(T_{\text{sep}}) \Delta \) the sum in \( R(T_{\text{sep}}) = \mathbb{Z}[T_{\text{sep}}^*] \) of all (finitely many) characters in the \( \Delta \)-orbit of \( \chi \).

**Theorem B.12** ([22, Th. 3.3]). – The restriction homomorphism \( R(G) \to R(T_{\text{sep}}) \) is an injection. The elements \( \Delta(\chi) \cdot \text{ind}(A_\chi) \), for all \( \chi \in \Lambda^+ \), form a \( \mathbb{Z} \)-basis of \( R(G) \).
The formula (B.3) then implies

**Corollary B.13.** — For a simply connected group $G$,

$$n_G = \gcd_{\chi \in \Lambda^+} [\Phi_G(\varDelta(\chi)) \cdot \text{ind}(A_{\chi})].$$

**B.3. Groups of type $A_n-1$**

We compute the number $n_G$ for a simply connected group $G$ of type $A_{n-1}$ over a field $F$, $n \geq 2$. Let $T \subset G$ be a maximal torus defined over $F$. The group of characters $T_{sep}^*$ can be identified with $\mathbb{Z}^n/\mathbb{Z}$ (with $\mathbb{Z}$ embedded diagonally) and $T_{sep}^*$ with the subgroup of $\mathbb{Z}^n$ of elements with the zero sum of the components. All the co-roots $\pm(e_i - e_j)$ have the same length and we can take $\eta_G = e_1 - e_2$. The Weyl group is the symmetric group $S_n$ which permutes the $e_i$ (see Example B.1). The restriction homomorphism to the center $C$ of $G$

$$\mathbb{Z}^n/\mathbb{Z} \rightarrow C_{sep} = \mathbb{Z}/n\mathbb{Z},$$

takes $(x_1, x_2, \ldots, x_n) + \mathbb{Z}$ to $\sum x_i + n\mathbb{Z}$.

We choose the set of simple roots $\{e_i \mid 1 \leq i \leq n\}$ which repeat $r_1, r_2, \ldots, r_k$ times respectively, so that $n = \sum r_i$. Note that the $a_i$ can be modified by adding an integer to all the $a_i$. We denote the character $\chi$ by $(r_1, \ldots, r_k; a_1, \ldots, a_k)$ or simply by $(r, a)$.

The stabilizer of $\chi = (r, a)$ in the Weyl group $S_n$ is isomorphic to $S_{r_1} \times S_{r_2} \times \cdots \times S_{r_k}$. Hence the number of characters in the $S_n$-orbit of $\chi$ is equal to

$$\frac{n!}{r_1!r_2!\cdots r_k!}.$$

For a pair of distinct indices $(i, j)$ the number of characters in the $S_n$-orbit with first two components $(a_i, a_j)$ is equal to

$$\frac{(n-2)!}{r_1!r_2!\cdots r_k!} r_i r_j.$$
B.3.1. Inner type

Let $G$ be a simply connected group of inner type $A_{n-1}$, i.e., $G = \text{SL}_1(A)$, where $A$ is a central simple algebra of degree $n$ over $F$. We have $\Delta = W = S_n$. The Tits algebra of a character $(r, a)$ is similar to $A^{\otimes ra}$ by [9, §27.B]. Hence, by Corollary B.13,

\begin{equation}
B.4 \quad n_G = \gcd([r, a] \cdot \text{ind}(A^{\otimes ra})),
\end{equation}

where the gcd is taken over all $(r, a)$ such that $\sum r_i = n$.

Denote by $v_p$ the $p$-adic valuation on $\mathbb{Z}$. For any integer $c \geq 0$, let $s_p(c)$ be the sum of the digits in the base $p$ expansion of $c$.

**Lemma B.14 ([12, Lemma 5.4(a)]).** – If $c = c_1 + c_2 + \cdots + c_k$, $c_i \geq 0$, then

\[ v_p\left(\frac{c!}{c_1!c_2! \cdots c_k!}\right) = \sum s_p(c_i) - s_p(c) - k. \]

**Lemma B.15.** – Let $p$ be a prime integer, $n = r_1 + r_2 + \cdots + r_k$, $r_i \geq 0$, $l = \min v_p(r_i)$ and $v_p(r_j) = l$ for some $j$. Then

\[ v_p\left(\frac{n!}{r_1!r_2! \cdots r_k!}\right) \geq v_p(n) - l, \]

and the equality holds if and only if

\[ s_p(n-1) = s_p(r_1) + \cdots + s_p(r_j - 1) + \cdots + s_p(r_k). \]

**Proof.** – We have

\[ \frac{n!}{r_1!r_2! \cdots r_k!} = \frac{n}{r_j} \cdot \frac{(n-1)!}{r_1! \cdots (r_j-1)! \cdots r_k!}, \]

and the second factor of the r.h.s. is integral, whence the inequality. The second statement follows from Lemma B.14 applied to the second factor. \qed

**Lemma B.16.** – For every dominant character $(r, a)$, $\gcd(n, ra)$ divides $[r, a]$.

**Proof.** – Let $p$ be a prime divisor of $n$, $l = \min v_p(r_i)$. Obviously,

\[ v_p\left(n \cdot \sum r_i a_i^2\right) \geq v_p(n) + l, \quad v_p(ra^2) \geq v_p(ra) + l. \]

By Lemma B.15,

\[ v_p\left(\frac{(n-2)!}{r_1!r_2! \cdots r_k!}\right) \geq -l. \]

Hence,

\[ v_p([r, a]) \geq -l + \min\{v_p(n) + l, v_p(ra) + l\} = \min\{v_p(n), v_p(ra)\}. \]

**Theorem B.17.** – $n_G = \exp(A)$.
Proof. – We prove first that $n_G$ divides $\exp(A)$. In view of (B.4), it suffices to show that for every prime integer $p$ there is a dominant character $(r, a)$ such that
\[ v_p([r, a] \cdot \ind(A^{\otimes ra})) = v_p(\exp(A)) \]

Let $a = v_p(\exp(A))$. We have $v_p(n) \geq a$ since $\exp(A) | n$.

Case 1: $v_p(n) > a$. Consider the character $(r, a) = (p^a, n - p^a; 1, 0)$. Then
\[ [r, a] = \left( \frac{n - 2}{p^a - 1} \right), \quad ra = p^a, \quad v_p(\ind(A^{\otimes ra})) = 0. \]

Clearly, $s_p(n - 2) = s_p(n - p^a - 1)$. Hence, by Lemma B.14,
\[ v_p([r, a] \cdot \ind(A^{\otimes ra})) = \frac{s_p(p^a - 1)}{p - 1} = a = v_p(\exp(A)). \]

Case 2: $v_p(n) = a$. Since $\exp(A) | \ind(A) | n$, it follows that $v_p(\ind(A)) = a$. Consider the character $(r, a) = (1, n - 1; 1, 0)$. We have $[r, a] = 1 = ra$ and
\[ v_p([r, a] \cdot \ind(A^{\otimes ra})) = v_p(\ind(A)) = a = v_p(\exp(A)). \]

It remains to prove that $\exp(A)$ divides $[r, a] \cdot \ind(A^{\otimes ra})$ for every dominant character $(r, a)$. By Lemma B.16,
\[ \exp(A^{\otimes [r, a]}) | \exp(A^{\otimes \gcd(n, ra)}) = \exp(A^{\otimes ra}), \]
and hence
\[ \exp(A) | [r, a] \cdot \exp(A^{\otimes [r, a]}) | [r, a] \cdot \ind(A^{\otimes ra}). \]

By [9, Cor. 29.4], $H^2(F, G) = F^X / \Nrd(A^X)$. Consider the normalized invariant $r'_G$ of $G$ defined by
\[ r'_G(a \Nrd(A^X)) = (a) \cup [A], \]
where $[A]$ is the class of the algebra $A$ in the group
\[ H^2(F, \mathbb{Q}/\mathbb{Z}(1)) = \Br(F)[1/p], \]
($p$ is the characteristic exponent of $F$). The residue of the value
\[ r'_G(t \Nrd(A^X_{F(\ell)})) = (t) \cup [A_{F(\ell)}] \]
equals $[A]$. Hence the order of $r'_G$ is divisible by $\exp(A)'$, the greatest divisor of $\exp(A)$ prime to $p$. It follows from Theorem B.17 that the invariants $r_G$ and $r'_G$ are two generators of $\Inv^X(G, H)_{\text{norm}}$. In particular, any value of the Rost invariant $r_G$ over a field $L$ is the cup-product $(x) \cup [A_L]$ for some $x \in L^X$. It seems plausible that $r_G$ coincides with $r'_G$ (up to sign).

B.3.2. Outer type
Let $G$ be a simply connected group of outer type $A_{n-1}$, i.e., $G = \text{SU}(B, \tau)$, where $B$ is a central simple algebra of degree $n \geq 3$ with a unitary involution $\tau$ over a separable quadratic
field extension \( K/F \). Over the quadratic extension \( K/F \) the group \( G \) is isomorphic to \( \text{SL}_1(B) \). Under the field extension map the Rost invariant \( r_G \) maps to the Rost invariant \( r_{G,K} \) of order \( \exp(B) \) by Theorem B.17. The corestriction map for the field extension \( K/F \) takes \( r_{G,K} \) to \( 2r_G \). Hence

\[
\exp(B) \mid n_G \mid 2\exp(B).
\]  

**B.3.3.** Consider first the case when \( B \) splits, i.e., \( G = \text{SU}(V, h) \), where \( (V, h) \) is a non-degenerate hermitian form over \( K/F \) of dimension \( n \). Let \( (V, h) \) be the associated quadratic form over \( F \) of dimension \( 2n \). The natural homomorphism \( G \to \text{Spin}(V, h) \) together with the Arason invariant give a non-trivial invariant of \( G \) (see [9, Example 31.44]). Hence \( n_G = 2 \) by (B.5).

**B.3.4.** Assume now that the exponent of \( B \) is odd. By (B.3.3), over a field extension of \( F \), which splits \( B \) (but not \( K \) ), the number \( n_G \) is equal to \( 2 \). Hence \( n_G = 2\exp(B) \) by (B.5). (The Rost invariant in this case is considered in [9, Example 31.45].)

**B.3.5.** Consider now the general case. We may assume that \( n \) is even and set \( m = n/2 \). The problem is to decide whether \( n_G = \exp(B) \) or \( n_G = 2\exp(B) \). Thus, it is sufficient to trace only the 2-part of these integers.

The Galois group \( \text{Gal}(F) \) acts on \( T^*_{\text{sep}} \) through \( \text{Gal}(K/F) \) by the involution

\[
\kappa : (x_1, x_2, \ldots, x_n) \mapsto (-x_n, \ldots, -x_2, -x_1) + \mathbb{Z}.
\]

A dominant character \( \chi = (r, a) \in \Lambda^+ \) is called symmetric if it does not change under \( \kappa \), that is, the sequence \( (r_1, r_2, \ldots, r_k) \) is symmetric and the sum \( a_i + a_{k+1-i} \) does not depend on \( i \).

Let \( \chi \in \Lambda^+ \) be a dominant character. If \( \chi \) is symmetric, then \( \Delta(\chi) = W(\chi) \). Otherwise, the \( \Delta \)-orbit of \( \chi \) is twice longer than the \( W \)-orbit of \( \chi \). It is also clear that if \( (r', a') = \kappa(r, a) \), then \( [r', a'] = [r, a] \).

If \( \chi = (r, a) \) is symmetric, then \( ra \) is divisible by \( m \). The corresponding Tits algebra \( A_\chi \) is equivalent to \( D^{\otimes ra} \), where \( D \) is the discriminant algebra of \( (B, \tau) \). If \( \chi = (r, a) \) is not symmetric, then the Tits algebra \( A_\chi \) is equivalent to \( B^{\otimes ra} \) by [9, §27.B].

It follows from Corollary B.13 that \( n_G \) is the \( \gcd \) of two integers \( n'_G \) and \( n''_G \):

\[
n'_G = 2\gcd([r, a] \cdot \text{ind}(B^{\otimes ra})),
\]

where the \( \gcd \) is taken over all non-symmetric dominant characters \( (r, a) \) and

\[
n''_G = \gcd([r, a] \cdot \text{ind}(D^{\otimes ra})),
\]

where the \( \gcd \) is taken over all symmetric characters \( (r, a) \).

Note that the algebra \( D^{\otimes ra} \otimes_K K \) is similar to \( B^{\otimes ra} \) [9, Prop. 10.30], hence

\[
\text{ind}(D^{\otimes ra}) \mid 2\text{ind}(B^{\otimes ra}).
\]

Therefore, we can modify the integer \( n'_G \) by including in the \( \gcd \) also symmetric characters, without changing the \( \gcd \) of \( n'_G \) and \( n''_G \). It follows from (B.4) and Theorem B.17 (applied to the algebra \( B \) instead of \( A \)) that \( n''_G = 2\exp(B) \). We get

\[
n_G = \gcd\left[2\exp(B), \gcd([r, a] \cdot \text{ind}(D^{\otimes ra}))\right],
\]

\[\text{ANNALES SCIENTIFIQUES DE L’ÉCOLE NORMALE SUPÉRIEURE} \]
where the gcd inside the brackets is taken over all symmetric characters \((r, a)\). Finally,

\[
\nu_G = \begin{cases} 
2 \exp(B) & \text{if } 2 \exp(B) \mid \text{ind}(D \otimes \mathbb{F}_q) \\
\exp(B) & \text{otherwise.}
\end{cases}
\]

Thus, we need to consider divisibility properties of the integers \([r, a] \cdot \text{ind}(D \otimes \mathbb{F}_q)\) for all symmetric characters \([r, a]\). We need only to look at the 2-part of these integers.

Let \(\chi = (r, a)\) be a symmetric character. We consider two cases.

**Case 1:** The integer \(\frac{ra}{m}\) is even, i.e., \(ra\) is divisible by \(n\).

We will show (Proposition B.18) that \([r, a]\) is 2-divisible by \(2n\) and hence by \(2 \exp(B)\), i.e., the term \([r, a] \cdot \text{ind}(D \otimes \mathbb{F}_q)\) does not contribute to the gcd.

**Proposition B.18.** – If \(n\) is even, \(ra \) is divisible by \(n\), then \(v_2([r, a]) \geq v_2(n) + 1\).

**Proof.** – Let \(ra = nq\) for some \(q\). We have

\[
[r, a] = \frac{1}{n - 1} \cdot \frac{n!}{r_1! r_2! \ldots r_k!} \left( \sum r_i a_i^2 - nq^2 \right).
\]

Since \(n - 1\) is odd, by Lemma B.15, it suffices to prove that

\[
v_2 \left( \sum r_i a_i^2 - nq^2 \right) \geq l + 1,
\]

where \(l = \min v_2(r_i)\). We have

\[
\sum r_i a_i^2 - nq^2 = \sum r_i a_i (a_i + 1) - nq(q + 1).
\]

Since \(v_2(r_i) \geq l\) and \(v_2(n) \geq l\), the r.h.s. is divisible by \(2^{l+1}\).

**Case 2:** The integer \(\frac{ra}{m}\) is odd.

**Proposition B.19.** – If \(n\) is even, \(ra = nq\) with odd \(q\), then \(v_2([r, a]) \geq v_2(m)\). The equality holds if and only if \(n\) is a 2-power, \(k = 2\) and \(r = (m, m)\).

**Proof.** – By Lemma B.15,

\[
\text{(B.6)} \quad v_2 \left( \frac{(n - 2)!}{r_1! r_2! \ldots r_k!} \right) \geq -l,
\]

where \(l = \min v_2(r_i)\). Since \(q\) is odd, it follows that \(v_2(m) = v_2(ra) \geq l\). Thus, to prove the inequality it is sufficient to show that

\[
v_2 \left( n \sum r_i a_i^2 - m^2 q^2 \right) \geq v_2(m) + l.
\]

It is obvious since \(v_2(r_i) \geq l\) and \(v_2(m) \geq l\).

If \(n\) is a 2-power, \(k = 2\) and \(r = (m, m)\), \(a = (a_1, a_2)\) then

\[
ra = ma_1 + ma_2 = m(a_1 + a_2) = mq,
\]

\(4^e\) SÉRIE – TOME 35 – 2002 – N° 3
hence \( a_1 + a_2 \) is odd. We have

\[
[r, a] = \frac{(2m - 2)!}{(m - 1)!^2} (a_1 - a_2)^2
\]

and by Lemma B.14, since \( a_1 - a_2 \) is odd,

\[
v_2([r, a]) = v_2\left(\frac{(2m - 2)!}{(m - 1)!^2}\right) = 2s_2(m - 1) - s_2(2m - 2) = s_2(m - 1) = v_2(m).
\]

Thus, the equality holds.

Conversely, assume the equality. Then the first part of the proof shows that the equality in (B.6) holds, or equivalently,

\[
v_2\left(\frac{n!}{r_1!r_2!\cdots r_k!}\right) = v_2(n) - l,
\]

and hence by Lemma B.15,

\[
s_2(n - 1) = s_2(r_1) + \cdots + s_2(r_j - 1) + \cdots + s_2(r_k),
\]

where \( j \) satisfies \( v(r_j) = l \). This means that when we consequently add (in any order, in base 2) the integers \( r_1, \ldots, r_j - 1, \ldots, r_k \) we never carry over units. In particular, all these integers are pairwise distinct. Thus, the sequence \( r \) can have at most one pair of equal terms. But the character \( (r, a) \) is symmetric, hence the sequence \( r \) is symmetric. It follows that \( k \leq 3 \). If \( k = 2 \), then \( r = (m, m) \) and \( s_2(2m - 1) = s_2(m) + s_2(m - 1) \), i.e., when we add \( m \) and \( m - 1 \) (in base 2) we don’t carry over units. It is possible only if \( m \) (and hence \( n \)) is a 2-power.

Finally, assume \( k = 3 \), i.e., \( r = (r_1, r_2, r_3) \) with \( r_1 = r_3 \). Then by symmetry, \( a_1 + a_3 = 2a_2 \), hence

\[
mq = r_1a_1 + r_2a_2 + r_3a_3 = r_1(a_1 + a_3) + r_2a_2 = (2r_1 + r_2)a_2 = na_2,
\]

therefore, \( q = 2a_2 \), a contradiction since \( q \) is odd. This case does not occur.

\[
\text{Theorem B.20.} \quad \text{Let } G = SU(B, \tau), \text{ where } B \text{ is a central simple algebra of degree } n \text{ with a unitary involution } \tau \text{ over a separable quadratic field extension } K/F. \text{ Let } D = D(B, \tau) \text{ be the discriminant algebra of } (B, \tau). \text{ Then}
\]

\[
\nu_G = \begin{cases} 
\exp(B) & \text{if } n \text{ is a 2-power and } \exp(B) = n; \\
2\exp(B) & \text{if } n \text{ is a 2-power, } \exp(B) = n/2, \text{ and } D \text{ is split; otherwise.}
\end{cases}
\]

\[
\text{Proof:} \quad \text{By (B.5) we may assume that } n \text{ is even. We know from the cases 1 and 2 considered above that } \nu_G = \exp(B) \text{ if and only if there exists a symmetric character } (r, a) \text{ such that } ra = mq \text{ with } q \text{ odd and}
\]

\[
(B.7) \quad v_2([r, a] \cdot \text{ind}(D)) = v_2(\exp(B)).
\]

By Proposition B.19, for such a character \( (r, a) \),

\[
(B.8) \quad v_2([r, a]) \geq v_2(m),
\]
hence equality (B.7) implies
\[ v_2(m) + 1 = v_2(n) \geq v_2(\exp(B)) \geq v_2(m) + v_2(\text{ind}(D)) \geq v_2(m). \]

There are two cases:

Case 1: \( v_2(\exp(B)) = v_2(n) = v_2(m) + 1 \).

The algebra \( D_K \) is similar to \( B^{\otimes m} \) and hence is not split. Index \( \text{ind}(D) \) divides 4 [9, Prop. 10.30], therefore, \( \text{ind}(D) \) is even. It follows then from (B.7) that
\[ v_2([r, a]) = v_2(\exp(B)) - v_2(\text{ind}(D)) \leq v_2(\exp(B)) - 1 = v_2(m), \]
i.e., we have equality in (B.8). By Proposition B.19, \( n \) is a 2-power and hence \( \exp(B) = n \).

Conversely, if \( n \) is a 2-power, \( \exp(B) = n \), then \( D \) has index 2 by [9, Prop. 10.30]. It follows from Proposition B.19 that for the character \((r, a) = (m, m; 1, 0)\) we have
\[ v_2([r, a] \cdot \text{ind}(D)) = v_2([r, a]) + 1 = v_2(m) + 1 = v_2(n) = v_2(\exp(B)), \]
i.e., (B.7) holds.

Case 2: \( v_2(\exp(B)) = v_2(m) \). Comparing (B.7) and (B.8), we deduce that \( D \) splits and the equality in (B.8) holds. Hence again by Proposition B.19, \( n \) is a 2-power and \( \exp(B) = m = n/2 \). Conversely, if \( n \) is a 2-power, \( \exp(B) = n/2 \) and \( D \) is split, then for the same character \((r, a)\) as in case 1, by Proposition B.19,
\[ v_2([r, a] \cdot \text{ind}(D)) = v_2([r, a]) = v_2(m) = v_2(\exp(B)), \]
i.e., (B.7) holds. \( \square \)

Remark B.21. – Inspection of the proof shows that the only source of reduction of the value of \( n_G \) from 2 \( \exp(B) \) to \( \exp(B) \) is the divisibility property (B.1) for the canonical representation \( \rho: G \to G' = \text{SL}_1(D) \).

B.4. Groups of type \( B_n \)

Let \( G \) be a simply connected group of type \( B_n \), \( n \geq 1 \), i.e., \( G = \text{Spin}(V, q) \) for a non-degenerate quadratic form \((V, q)\) of dimension \( 2n + 1 \). Consider the composition
\[ \alpha: \text{Spin}(V, q) \longrightarrow \text{O}^+(V, q) \hookrightarrow \text{SL}(V). \]
Since \( n_G = 2 \) (Example B.11), we have \( n_G \) \( \mid \) \( 2 \).

**Theorem B.22.** – Let \( G \) be a simply connected group of type \( B_n \), \( n \geq 1 \), i.e., \( G = \text{Spin}(V, q) \) for a non-degenerate quadratic form \((V, q)\) of dimension \( 2n + 1 \). Then
\[ n_G = \begin{cases} 1 & \text{if } n = 1 \text{ or } 2 \text{ and } q \text{ has maximal Witt index } n, \\ 2 & \text{otherwise}. \end{cases} \]

**Proof.** – The case \( n = 1 \) follows from Theorem B.17 since \( G \simeq \text{SL}_1(C_0) \), where \( C_0 \) is the even Clifford algebra of \((V, q)\) by [9, Th. 15.2] and \( q \) is isotropic if and only if \( C_0 \) is split. If \( n = 2 \) and \( q \) is of Witt index 2, then \( G \) splits and hence \( G \simeq \text{Sp}_4 \) (since \( B_2 \simeq C_2 \)) and the latter group is special, therefore \( n_G = 1 \).
Assume that $n \geq 2$ and the Witt index of $q$ is less than 2 if $n = 2$. The image of the map $H^1(F, G) \to H^1(F, O^+(V, q))$ classifies quadratic forms $q'$ on $V$ such that $q \perp q' \in I^1$ (cf. [9, 31.41]). The invariant taking $q'$ to the Arason invariant of $q \perp -q'$ is non-trivial, hence it coincides with $r_G$ and therefore $n_G = 2$. \hfill $\square$

### B.5. Groups of type $C_n$

Let $G$ be a simply connected group of type $C_n$, $n \geq 1$, i.e., $G = \text{Sp}(A, \sigma)$ for a central simple algebra $A$ of degree $2n$ with a symplectic involution $\sigma$. Let

$$\alpha : \text{Sp}(A, \sigma) \hookrightarrow \text{SL}_1(A)$$

be the natural embedding. Since $n_\alpha = 1$ (Example B.7), it follows from Theorem B.17 and (B.1) that

$$n_G | n_\alpha \cdot n_{\text{SL}_1(A)} = \exp(A).$$

In the case $A$ splits we have then $n_G = 1$, and in general, $n_G | 2$ since $\exp(A) | 2$.

**Theorem B.23.** Let $G$ be a simply connected group of type $C_n$, $n \geq 1$, i.e., $G = \text{Sp}(A, \sigma)$ for a central simple algebra $A$ of degree $2n$ with a symplectic involution $\sigma$. Then

$$n_G = \begin{cases} 1 & \text{if } A \text{ splits,} \\ 2 & \text{otherwise.} \end{cases}$$

**Proof.** We may assume that $A$ is not split. Suppose first that $A$ is similar to a quaternion division algebra $Q$. By [9, Th. 4.2], the involution $\sigma$ is adjoint to a hermitian form $(V, h)$ of rank $n$ over $Q$ with respect to the canonical involution on $Q$. Let $(V_0, h_0)$ be a non-degenerate subform of $h$ of rank 1. We have an embedding

$$\beta : \text{SL}_1(Q) = \text{Sp}(V_0, h_0) \hookrightarrow \text{Sp}(V, h) = G$$

with $n_\beta = 1$ (Example B.7) and $2 = n_{\text{SL}_1(Q)} | n_\beta \cdot n_G = n_G$ by Theorem B.17 and (B.1), hence $n_G = 2$.

In general, choose a field extension $L/F$ such that $A_L$ is similar to a quaternion algebra. Since $2 = n_{G_L} | n_G$, it follows that $n_G = 2$. \hfill $\square$

### B.6. Groups of type $D_n$

Let $G$ be a simply connected group of (classical) type $D_n$, $n \geq 4$, i.e., $G = \text{Spin}(A, \sigma, f)$ for a central simple algebra $A$ of degree $2n$ with a quadratic pair $(\sigma, f)$ (simply $G = \text{Spin}(A, \sigma)$ if $\text{char } F \neq 2$). Consider the composition

$$\rho : \text{Spin}(A, \sigma, f) \xrightarrow{\alpha} \text{O}^+(A, \sigma, f) \hookrightarrow \text{SL}_1(A),$$

where $\alpha$ is the standard isogeny. Since by Example B.11, $n_\rho = 2$, it follows from Theorem B.17 and (B.1) that

$$n_G | n_\alpha \cdot n_{\text{SL}_1(A)} = 2 \exp(A).$$

In the case $A$ splits it implies then $n_G | 2$, and in general, $n_G | 4$ since $\exp(A) | 2$.

If $A$ splits, i.e., $G = \text{Spin}(V, q)$ for a quadratic form $(V, q)$ of dimension $2n \geq 8$, there is a non-trivial Arason invariant, hence $n_G = 2$. 

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**UNRAMIFIED COHOMOLOGY OF CLASSIFYING VARIETIES**

473

**ANNALES SCIENTIFIQUES DE L’ÉCOLE NORMALE SUPÉRIEURE**
Let $Z/F$ be the discriminant quadratic extension (the center of the Clifford algebra $C(A, \sigma, f)$). It is an étale quadratic extension of $F$.

The character group $T_{sep}^*$ can be identified with $\mathbb{Z}^{n} + \mathbb{Z} e\varepsilon$, where

$$\varepsilon = \frac{e_1 + e_2 + \cdots + e_n}{2}.$$  

The group of co-characters $T_{sep,*}$ is identified with the subgroup in $\mathbb{Z}^n$ of the elements with even sum of the components. All the co-roots $\pm e_i \pm e_j$ have the same length and we can take $\eta_G = e_1 - e_2$. The Weyl group $W$ is a semidirect product of $H = (\mathbb{Z}/2\mathbb{Z})^{n-1}$ and the symmetric group $S_n$; the elements of $H$ change signs in even number of places and $S_n$ permutes the $e_i$. The Galois group $\text{Gal}(F)$ acts on $T_{sep}$ through $\text{Gal}(Z/F)$ by the involution

$$\kappa: (x_1, \ldots, x_{n-1}, x_n) + \mathbb{Z} \mapsto (x_1, \ldots, x_{n-1}, -x_n) + \mathbb{Z}.$$ 

We choose the set of simple roots $e_1 - e_2$, $e_2 - e_3$, $\ldots$, $e_{n-1} - e_n$, $e_n - e_1$. The corresponding cone of dominant characters $\Lambda^+$ consists of all characters $(x_1, x_2, \ldots, x_n)$ such that $x_1 \geq x_2 \geq \cdots \geq x_{n-1} \geq |x_n|$.

Let $C$ be the center of $G$. The group $C^*$ consists of 4 elements: $0$, $\lambda$, $\lambda^+$ and $\lambda^-$, where $\lambda$ is trivial on the kernel of the isogeny $\alpha$. The corresponding Tits algebra $A_\lambda$ is similar to $A$ [9, 27.B]. The restriction of $\chi$ of a character $\chi = (x_1, x_2, \ldots, x_n)$ to $C$ satisfies

$$\bar{\chi} = \begin{cases} 
0 & \text{if all the } x_i \text{ are integers and } \sum x_i \text{ is even}, \\
\lambda & \text{if all the } x_i \text{ are integers and } \sum x_i \text{ is odd}, \\
\lambda^+ \text{ or } \lambda^- & \text{if all the } x_i \text{ are semi-integers}.
\end{cases}$$

B.6.1. Inner case

Assume that $Z$ splits. Then $C(A, \sigma, f) = C^+ \times C^-$, where $C^+$ and $C^-$ are central simple algebras over $F$ being Tits algebras of $\lambda^+$ and $\lambda^-$. Denote by $n_0$, $n_1$, $n^+$ and $n^-$ the $\gcd(\Phi_G(W(\chi)))$ for all dominant characters $\chi$ restricting to $0$, $\lambda$, $\lambda^+$ and $\lambda^-$ respectively. We have

$$n_G = \gcd(n_0, n_1 \cdot \text{ind}(A), n^+ \cdot \text{ind}(C^+), n^- \cdot \text{ind}(C^-)). \tag{B.9}$$

Consider a dominant character $\chi = (x_1, \ldots, x_n)$ with integer components. Assume first that only one of the components is nonzero, i.e., $\chi = (a, 0, \ldots, 0) = ac\varepsilon_1$ with $a > 0$. The $W$-orbit of $\chi$ consists of the characters $\pm ac\varepsilon_1$, hence $\Phi_G(W(\chi)) = 2a^2$. In particular, $n_1 \geq 2$.

Assume now that $\chi$ has at least two nonzero components, i.e., $\chi = (a, b, \ldots)$ with $a > b > 0$. We claim that $\Phi_G(W(\chi))$ is divisible by 4. Consider the subgroup $W' \subset W$ being the semidirect product of $H$ and $S_2$ interchanging the first two components. It suffices to show that $\Phi_G(W'(\chi'))$ is divisible by 4 for every $\chi' \in W\chi$. Each orbit $W\chi'$ is the union of the following sets of characters:

- $\pm c, \pm d, \ldots$ and $\pm c, \pm d, \ldots$ for nonzero $c \neq d$;
- $\pm c, \ldots$;
- $\pm c, 0, \ldots, \pm d, \ldots$ and $0, \pm c, \ldots, \pm d, \ldots$ for nonzero $c, d$.

One easily checks that the value $\Phi_G$ of the sum of characters in each set is divisible by 4. We have proved

**Lemma** B.24. – The integer $n_0$ is divisible by 4 and $n_1 = 2$.

Now consider the integers $n^+$ and $n^-$. All the coordinates of a character $\chi$ restricting to $\lambda^+$ or $\lambda^-$ are semi-integers, and in particular are nonzero.
Lemma B.25. \( n^+ = n^- = 2^{n-3} \).

Proof. Clearly, \( \Phi_G(W(e)) = 2^{n-3} \). We claim that \( \Phi_G(W(\chi)) \) is divisible by \( 2^{n-3} \) for every character \( \chi \) with semi-integer components. It suffices to show that \( \Phi_G(W'(\chi)) \) is divisible by \( 2^{n-3} \) for every character \( \chi \) with semi-integer components. We split the orbit \( W' \chi \) into a union of the pairs \( \chi_1 = ae_1 + be_2 + \cdots, \chi_2 = -ae_1 + be_2 + \cdots \) with semi-integers \( a \) and \( b \). Then

\[
\Phi_G(\chi_1 + \chi_2) = 2(a^2 + b^2) \in \frac{1}{2} \mathbb{Z}
\]

and the number of pairs in the orbit is \( 2^{n-2} \), whence the claim. \( \square \)

Lemmas B.24 and B.25 and (B.9) give then the following theorem.

Theorem B.26. Let \( G \) be a simply connected group of classical type \( D_n, \ n \geq 4 \), i.e., \( G = \text{Spin}(A, \sigma, f) \) for a central simple algebra \( A \) of degree \( 2n \) with a quadratic pair \( (\sigma, f) \). If \( \text{disc}(\sigma, f) \) is trivial,

\[
n_G = \begin{cases} 
2 & \text{if } A \text{ splits;} \\
2 & \text{if } n = 4 \text{ and one of the algebras } C^+ \text{ and } C^- \text{ splits;} \\
4 & \text{otherwise.}
\end{cases}
\]

B.6.2. Outer case

The group \( \Delta \) is a semidirect product of \( (\mathbb{Z}/2\mathbb{Z})^n \) and \( S_n \).

Theorem B.27. Let \( G \) be a simply connected group of classical type \( D_n, \ n \geq 4 \), i.e., \( G = \text{Spin}(A, \sigma, f) \) for a central simple algebra \( A \) of degree \( 2n \) with a quadratic pair \( (\sigma, f) \). If \( \text{disc}(\sigma, f) \) is nontrivial,

\[
n_G = \begin{cases} 
2 & \text{if } A \text{ splits;} \\
4 & \text{otherwise.}
\end{cases}
\]

Proof. As in the inner case we prove that \( n_G | 2 \exp(A) \). If \( A \) splits, nontriviality of the Arason invariant implies that \( n_G = 2 \). Assume that \( A \) is not split. It suffices to show that for every character \( \chi \in \Lambda^+ \), the integer

\[
\Phi_G(\Delta(\chi)) \cdot \text{ind}(A_\chi)
\]

is divisible by \( 4 \).

Assume first that only one of the components of \( \chi \) is nonzero, i.e., \( \chi = ae_1 \) with positive integer \( a \). The \( \Delta \)-orbit of \( \chi \) consists of the characters \( \pm ae_i \), hence \( \Phi_G(\Delta(\chi)) = 2a^2 \). Note that \( \chi \) is stable under the involution \( \kappa \), hence \( F(\chi) = F \) and \( A_\chi = A^{\otimes a} \) [9, 27.A]. If \( a \) is odd, then the algebra \( A^{\otimes a} \) does not split, \( \text{ind}(A^{\otimes a}) \) is even and hence the integer (B.10) is divisible by \( 4 \).

If \( \chi \) has at least two nonzero components, then as in the inner case we see that even \( \Phi_G(\Delta(\chi)) \) is divisible by \( 4 \).

Finally assume that all the components of \( \chi \) are semi-integers. The orbit \( \Delta \chi \) is twice longer than in the inner case, hence as in the proof of Lemma B.25 we see that \( \Phi_G(\Delta(\chi)) \) is divisible by \( 2^{n-2} \) and therefore by \( 4 \) since \( n \geq 4 \). \( \square \)

REFERENCES


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