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GROWTH TIGHTNESS OF FREE AND AMALGAMATED PRODUCTS

BY ANDREA SAMBUSETTI

ABSTRACT. – We show that every nontrivial free product, different from the infinite dihedral group, is growth tight with respect to any algebraic distance: that is, its exponential growth rate is strictly greater than the corresponding growth rate of any of its proper quotients. A similar property holds for the amalgamated product of residually finite groups over a finite subgroup. As a consequence, we provide examples of finitely generated groups of uniform exponential growth whose minimal growth is not realized by any generating set.

RÉSUMÉ. – Nous montrons que tout produit libre non trivial, différent du groupe diédral infini, est à croissance forte par rapport à n'importe quelle distance algébrique : son taux de croissance exponentielle est toujours strictement supérieur à celui d'un quotient propre quelconque. On prouve une propriété similaire pour les produits amalgamés de groupes résiduellement finis sur un sous-groupe fini. Comme application de ce résultat, nous présentons une classe de groupes à croissance exponentielle uniforme, dont la croissance minimale n'est réalisée par aucune partie génératrice finie.

1. Introduction

The *entropy* of a finitely generated group G, endowed with a finite generating set S, is the limit

$$\mathrm{Ent}(G,S) = \lim_{N\to\infty} N^{-1} \cdot \log \beta_{(G,S)}(N)$$

where $\beta_{(G,S)}(N)$ denotes the number of elements of G which can be represented by words on $S \cup S^{-1}$ of length smaller than N. The growth type of $\beta_{(G,S)}$ does not depend on the particular generating set S, but the number $\mathrm{Ent}(G,S)$ of course does. Accordingly, the *algebraic entropy* of G is defined as the infimum

$$\mathrm{AlgEnt}(G) = \inf_S \mathrm{Ent}(G,S)$$

where S runs over all finite generating sets for G. This is an intrinsic invariant of G; one says that G has uniform exponential growth if $\mathrm{AlgEnt}(G) > 0$.

Let F_n be the free group of rank $n \geqslant 2$, and let $S_n = \{s_1, \ldots, s_n\}$ be a free set of generators. It is easily computed that $\beta_{(F_n,S_n)}(N) = 1 + \sum_{k=1}^{N} 2n(2n-1)^{k-1}$, so that

$$\mathrm{Ent}(F_n,S_n) = \log(2n-1).$$

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On the other hand, if G is any group on n generators $S=\{s_1,\ldots,s_n\}$, one clearly has $\mathrm{Ent}(G,S)\leqslant\log(2n-1)$.

A remarkable rigidity property of free groups is the following

THEOREM 1.1 (Asymptotic characterization of nonabelian free groups). – *Let G be a group on $n\geqslant 2$ generators $S=\{s_1,\ldots,s_n\}$. If $\mathrm{Ent}(G,S)=\log(2n-1)$ then G is free on S.*

More precisely, let $\varepsilon(n,l)=\frac{1}{[l/2]+2}\log\frac{(2n-1)^{[l/2]+2}}{(2n-1)^{[l/2]+2}-1}$, where $[l/2]$ denotes the integral part of $l/2$. One then has:

THEOREM 1.2 (Growth tightness of nonabelian free groups). – *For any nontrivial normal subgroup H of a nonabelian free group F_n, one has $\mathrm{Ent}(F_n,S_n)>\mathrm{Ent}(F_n/H,S_n/H)$, where S_n/H denotes the generating set of F_n/H induced by S_n. Namely,*

$$\mathrm{Ent}(F_n,S_n)-\mathrm{Ent}(F_n/H,S_n/H)\geqslant\varepsilon(n,l),$$

where l is the S_n-length of the smallest nontrivial element of H.

Therefore, if a group G on n generators has entropy ε-close to $\log(2n-1)$, then all relators are large compared to ε. The above property is folklore; we shall give a short proof in Section 2, by way of example (another proof may be found in [4]). The asymptotic characterization 1.1 of free groups clearly follows from Theorem 1.2, since G may be identified to a quotient F_n/H.

One may wonder to what extent this property carries on to more general groups, that is when a group is characterized by its entropy among all of its quotients. In this paper we show that this holds for free products and for a class of amalgamated products (Theorems 1.3 and 1.4). We shall then use this result to answer an open problem about minimal growth of groups.

Let (G,d) be a discrete group endowed with a left-invariant distance. We can consider the exponential growth rate of G with respect to d, that is the invariant

$$\mathrm{Ent}(G,d)=\liminf_{R\to\infty}R^{-1}\cdot\log\#B_{(G,d)}(e,R)$$

where $B_{(G,d)}(e,R)$ is the ball of radius R centred at the identity e (we shall always assume, in order that this definition makes sense, that d has the property that balls of finite radius are finite sets). Clearly, $\mathrm{Ent}(G,S)=\mathrm{Ent}(G,d_S)$ if d_S denotes the word metric of (G,S). When H is a subgroup of G we shall give the left cosets space G/H the quotient metric, that is the G-invariant distance

$$d/H(gH,g'H)=\inf_{h,h'\in H}d(gh,g'h')=d\big(H,g^{-1}g'H\big).$$

We say that (G,d) is *growth tight* if for every infinite normal subgroup $H\lhd G$ one has $\mathrm{Ent}(G,d)>\mathrm{Ent}(G/H,d/H)$. Notice that this is (a priori) a property of the couple (G,d) and not of the group itself. The term growth tightness first appeared[1] in [2], with respect to word metrics of finitely generated groups. However, it seems to be interesting to investigate growth tightness of groups with respect to more general distances (cp. Section 3 and [6]). Here is an algebraic motivation. Let G be any group containing a free group of finite index F_n; since the growth of a group clearly is equivalent to the growth of any subgroup of finite index, and since F_n is growth tight, one would guess that G is growth tight too; actually, for

[1] Our definition is a slight modification of that given in [2], where the authors require that the same inequality holds for every *nontrivial* normal subgroup $H\lhd G$. However, notice that these definitions are equivalent for free products of nontrivial groups, as any finite normal subgroup of $G_1 * G_2$ is necessarily trivial (see [7]).

any finite generating set S and any nontrivial normal subgroup H of G one would say that $\mathrm{Ent}(G,S) = \mathrm{Ent}(F_n, d_S) > \mathrm{Ent}(F_n/(F_n \cap H), d_S) = \mathrm{Ent}(G/H, S/H)$, as $F_n/(F_n \cap H)$ has finite index in G/H. The problem here is that one does not know that F_n is growth tight with respect to the distance induced by d_S (which is not a word metric on F_n).

We shall say that a distance d on a group G is *quasi-algebraic* if G is a finite index subgroup of a finitely generated group $\hat{G}$, and d is the restriction to G of a word metric of $\hat{G}$; thus, d is a G-invariant metric for which entropy is well-defined (the case where $(G,d) = (G,d_S)$ being a particular one).

THEOREM 1.3. – *Every nontrivial free product $G = G_1 * G_2$, different from the infinite dihedral group $\mathbf{Z}_2 * \mathbf{Z}_2$, is growth tight with respect to any quasi-algebraic distance.*

We call $G = G_1 *_F G_2$ a *nontrivial* amalgamated (or free) product when $G_1 \neq F \neq G_2$.

The generality of the distance d in Theorem 1.3 will enable us to conclude

THEOREM 1.4. – *Let $G = G_1 *_F G_2$ be a finitely generated group of exponential growth, which is a nontrivial amalgamated product of residually finite groups G_i over a finite subgroup F. Then, G is growth tight with respect to any word metric.*

Actually, as we shall see in Section 3, the same property holds for every group G which contains a free product as subgroup of finite index (see Theorem 3.1).

Remark 1.5. – For a group G, let us set $G^* = G \setminus \{e\}$. Now, for any nontrivial free product $G = G_1 * G_2$ and any fixed generating set S, the method we use to prove Theorem 1.3 gives an explicitely computable (though not accurate) lower bound for the gap

$$\varepsilon(G,S,H) = \mathrm{Ent}(G,S) - \mathrm{Ent}(G/H, S/H),$$

in terms of the following constants: $c_0 = \inf_{h \in H^*} \|h\|_S$, $c_1 = \inf_{g \in G_1^*} \|g\|_S$, $c_2 = \inf_{g \in G_2^*} \|g\|_S$ and $c = \sup_{s \in S} \ell(s)$, where $\ell(g)$ denotes the length of the normal form of $g \in G_1 * G_2$ (see Section 2).

We discuss now the relationship of the above results with two fairly natural problems about minimal growth of groups:

Question 1. – Do there exist finitely generated groups G such that $\mathrm{Ent}(G,S) > \mathrm{AlgEnt}(G)$ for every S?

In this case we shall say, shortly, that “the minimal growth of G is not achieved”.

Question 2. – Do there exist finitely generated groups G of exponential growth such that $\mathrm{AlgEnt}(G) = 0$?

The first problem, which may be seen as preliminary to the second one, was recently asked by Grigorchuk and de la Harpe [2], and the second dates back (as far as we know) to Gromov [3]. While the last question is still unsettled, growth tightness of free products allows us to answer positively to Problem 1: we can exhibit a large class of groups of uniform exponential growth whose minimal growth in not achieved. Namely:

COROLLARY 1.6. – *Every nontrivial free product whose minimal growth is achieved is hopfian. In particular, if G is the free product of a non-hopfian group with any nontrivial group, the minimal growth of G is not achieved.*

Recall that a group G is called *hopfian* if it is not isomorphic to a proper quotient of itself.

Remark 1.7. – The group $\mathbf{Z}_2 * \mathbf{Z}_2$ is the only nontrivial free product of finitely generated groups which does not have uniform exponential growth (see [1]), and it *is* hopfian.

Proof of Corollary 1.6. – Let S be a generating set for $G(\neq \mathbf{Z}_2 * \mathbf{Z}_2)$ which realizes the minimal growth, that is $\mathrm{EntAlg}(G) = \mathrm{Ent}(G,S)$. Now let $\phi: G \to G$ be a surjective homomorphism. Then, $\ker(\phi) = (e)$ necessarily, otherwise we would have, by Theorem 1.3, $\mathrm{Ent}(G, \phi(S)) < \mathrm{Ent}(G,S)$, which is a contradiction. □

Example 1.8. – Let $G_1 = \langle a, b \mid a^{-1}b^2a = b^3\rangle$ be the Baumslag-Solitar group (which is the simplest finitely generated non-hopfian group). Then, the group

$$G = G_1 * \mathbf{Z}_2 = \left\langle a, b, c \mid a^{-1}b^2a = b^3,\ c^2 = 1\right\rangle$$

does not admit a generating set S which realizes the minimal growth $\mathrm{AlgEnt}(G)$. More precisely, let $\phi: G \to G$ be a homomorphism such that $G/\ker(\phi) \cong G$ (for instance, we could take ϕ defined by $\phi(a) = a, \phi(b) = b^{-1}a^{-1}ba, \phi(c) = c$, see [5]). Then, for any generating set S, one has $\mathrm{Ent}(G, \phi(S)) < \mathrm{Ent}(G,S)$ (and the gap, of course, tends to zero when $\mathrm{Ent}(G,S) \to \mathrm{AlgEnt}(G)$). In particular, defining by recursion $b_1 = b, b_n = b_{n-1}^{-1}a^{-1}b_{n-1}a$, the entropies $\mathrm{Ent}(G, S_n)$ relative to the generating sets $S_n = \{a, b_n, c\}$ form a strictly decreasing sequence.

2. Growth tightness of free products

We start with a quick proof of Property 1.2. For a given metric d on a group G, we shall denote by $\|\ \|_d$ (or simply by $\|\ \|$, when no confusion is possible) the associated norm.

Proof of 1.2. – Let h be a reduced word on $S_n \cup S_n^{-1}$ representing a nontrivial element of H of minimal length l. Write $h = uu'$, with $u = s_{i_1} \cdots s_{i_{[l/2]+1}}$. Then, any word on $S_n \cup S_n^{-1}$ which contains u or u^{-1} as subword is not a geodesic representation of an element of F_n/H. Therefore, if $S(N)$ is the set of reduced words on $S_n \cup S_n^{-1}$ of length N which do not contain neither u nor u^{-1}, it is clear that every $g \in F_n/H$ of length N can be represented by a word in $S(N)$. We may assume that $u = s_{i_1} \cdots s_{i_q}$ with $s_{i_q} \neq s_{i_1}^{-1}$ and $q = [l/2] + 2$, by right multiplying u by $s_i \neq s_{i_1}^{-1}, s_{i_{[l/2]+1}}^{-1}$. Now, for every $N \geqslant 0$, let $N = kq + r$, $0 \leqslant r < q$, and let $\sigma(N) = \#S(N)$. Notice that $\sigma(q) = (2n(2n-1)^{q-1} - 2)$ exactly, and that for $i > 0$ there are at most $(2n-1)^i - 1$ possibilities to extend a word $w \in S(iq)$ to a word $\overline{w}$ of $S((i+1)q)$. Therefore

$$\sigma(N) \leqslant \bigl(2n(2n-1)^{q-1} - 2\bigr)\bigl((2n-1)^q - 1\bigr)^{k-1}(2n-1)^r \leqslant 2\bigl((2n-1)^q - 1\bigr)^{k+1}$$

which implies that

$$\begin{aligned}\mathrm{Ent}(F_n/H, S_n/H) &\leqslant \lim_{K \to +\infty} \frac{1}{Kq} \log\left(\sum_{N=0}^{Kq} \sigma(N)\right) \\ &\leqslant \lim_{K \to +\infty} \frac{1}{Kq} \log\left(2\sum_{k=0}^{K} \bigl((2n-1)^q - 1\bigr)^{k+1}\right) = \frac{1}{q}\log\bigl((2n-1)^q - 1\bigr)\end{aligned}$$

which is exactly equal to $\mathrm{Ent}(F_n, S_n) - \varepsilon(n,l)$. □

The above proof, though very explicit, has the main disadvantage that it cannot be easily adapted to a general group: given some group with a finite generating set (G,S) and a normal subgroup N, counting efficiently the number of reduced words on S representing elements of G/N with respect to words which represents different elements of G is hard for S and N generic. Moreover, the method of counting words does not apply at all when one has distances different from word metrics. That is why we shall turn to a more flexible method (even if less sharp).

We need to introduce some terminology to deal with more general metrics on groups. Recall that a metric space (X,d) is called a *length space* if the distance between any two points x_1,x_2 is equal to the infimum of lengths of (Lipschitz) paths joining x_1 to x_2.

DEFINITION 2.1. – A metric space (X,d) is a δ*-length space* (and d is a δ*-length distance*) if it admits an isometric immersion into some length space $(\hat{X},d)$, such that for every $\hat{x}\in\hat{X}$ there exists $x\in X$ with $d(x,\hat{x})\leqslant\delta$.

Examples 2.2. –

(i) A 0-length space is the same as a length space. A complete metric space (X,d) which is an ε-length space $\forall\varepsilon>0$ is a length space (cp. [3], Théorème 1.8).

(ii) A group G endowed with a finite generating set S is a $\frac{1}{2}$-length space with respect to the word metric d_S (the required isometry being the canonical immersion in its Cayley graph $\mathcal{C}(G,S)$). Similarly, if H is a subgroup of G, the cosets space $(G/H,d_S/H)$ is a $\frac{1}{2}$-length space, via its coset diagram.

(iii) More generally, let (G,d) be a group endowed with a quasi-algebraic distance, i.e. G is a finite index subgroup of a finitely generated group $(\hat{G},\hat{S})$ and $d=d_{\hat{S}}|_G$. Then, (G,d) is a δ-length space, for $\delta=\frac{1}{2}+d_{\mathcal{H}}(G,\hat{G})$, where $d_{\mathcal{H}}$ denotes the Hausdorff distance. Clearly, if H is a subgroup of G, the cosets space $(G/H,d/H)$ again is a δ-length space.

Now, the method of proof of Theorem 1.3 is the following. Let $G=G_1*G_2$ be endowed with a quasi-algebraic distance d, let H be any nontrivial normal subgroup, and let $(\Gamma=G/H$, $\overline{d}=d/H)$ be the quotient space. Let $(\mathbf{Z}_2,l)$ be the finite cyclic group of order 2, endowed with the distance defined by assigning the length $l>0$ to its generator. Consider the group $\Gamma*\mathbf{Z}_2$ (that is, essentially, the space of finite, arbitrarily long sequences of nontrivial elements of Γ) with the "product" metric $\overline{d}*l$: that is, the left-invariant distance associated to the norm

$$\|\gamma_1 1\gamma_2 1\ldots\gamma_{n+1}\|_{\overline{d}*l}=\sum_i\|\gamma_i\|_{\overline{d}}+nl$$

(notice that this is not a word metric). The idea is to show that $\mathrm{Ent}(\Gamma*\mathbf{Z}_2,\overline{d}*l)$ is strictly greater than $\mathrm{Ent}(\Gamma,\overline{d})$, and then to find a contracting immersion (i.e., an injective, Lipschitz map of Lipschitz constant 1) $\Phi:(\Gamma*\mathbf{Z}_2,\overline{d}*l)\hookrightarrow(G,d)$ for $l\gg 0$. Then, R-balls of (G,d) will contain more elements than the corresponding balls of $(\Gamma*\mathbf{Z}_2,\overline{d}*l)$, and this suffices to conclude that

$$\mathrm{Ent}(G,d)\geqslant\mathrm{Ent}(\Gamma*\mathbf{Z}_2,\overline{d}*l)>\mathrm{Ent}(\Gamma,\overline{d}).$$

We state these two main facts:

PROPOSITION 2.3. – *Let* $(\Gamma,\overline{d})$ *be a group endowed with a left-invariant δ-length distance. Assume that* $\mathrm{Ent}(\Gamma,\overline{d})=h>0$. *Then, for every* $l>0$ *one has*

$$\mathrm{Ent}(\Gamma*\mathbf{Z}_2,\overline{d}*l)\geqslant h+\frac{\log[1+\mathrm{e}^{-hL}]}{4L}$$

where $L=\max\{l,\delta\}$.

PROPOSITION 2.4. – *Let* $G = G_1 * G_2 \neq \mathbf{Z}_2 * \mathbf{Z}_2$ *be a nontrivial free product of groups endowed with a quasi-algebraic distance* d. *For any nontrivial normal subgroup* $H \triangleleft G$, *let* $(\Gamma = G/H, \overline{d} = d/H)$ *be the quotient group. Then, there exists a contracting immersion* $\Phi : (\Gamma * \mathbf{Z}_2, \overline{d} * l) \hookrightarrow (G, d)$, *for* $l \gg 0$.

Proof of Theorem 1.3. – The quotient distance $\overline{d} = d/H$ is a left-invariant δ-length distance (Example 2.2(iii)). Since every nontrivial free product different from $\mathbf{Z}_2 * \mathbf{Z}_2$ has exponential growth, we may assume that $\mathrm{Ent}(\Gamma, \overline{d}) = h > 0$ (otherwise the assertion $\mathrm{Ent}(G, d) > \mathrm{Ent}(\Gamma, \overline{d})$ is trivial). Then, Propositions 2.3 and 2.3 clearly imply Theorem 1.3. □

We now start proving Proposition 2.3.

LEMMA 2.5. – *Let* $(\Gamma, \overline{d})$ *be a discrete group endowed with a* δ*-length distance. Let* $B(R)$ *denote the closed ball of* $(\Gamma, \overline{d})$ *of radius* R *centred at the identity, and let*

$$A(R) = B(R + 2\delta) \setminus B(R - 2\delta), \quad \alpha(R) = \# A(R).$$

Then:

(i) *for every* $R_1, R_2 \geqslant 0$ *and for every* $\gamma \in \Gamma$ *with* $\|\gamma\| = R_1 + R_2$, *there exist* $\gamma_1, \gamma_2 \in \Gamma$ *which verify* $\gamma_1 \cdot \gamma_2 = \gamma$ *and such that* $R_1 - \delta \leqslant \|\gamma_1\| \leqslant R_1 + \delta$, $R_2 - \delta \leqslant \|\gamma_2\| \leqslant R_2 + \delta$;

(ii) *if* $\mathrm{Ent}(\Gamma, \overline{d}) = h > 0$, *then* $\alpha(R) \geqslant \mathrm{e}^{hR}$, *for all* $R > 0$.

Proof. – Let $\Gamma \hookrightarrow (\overline{\Gamma}, \overline{d})$ be the isometric immersion which gives the δ-length structure of Γ. Then, simply consider a point $\overline{\gamma}_1 \in \overline{\Gamma}$ on a geodesic joining e to γ, at distance R_1 from e; then take for γ_1 an element of Γ such that $d(\gamma_1, \overline{\gamma}_1) \leqslant \delta$ and set $\gamma_2 = \gamma_1^{-1}\gamma$. As d is Γ-invariant, this proves (i).

Now consider the map $f : A(R_1) \times A(R_2) \to \Gamma$ given by multiplication. If $\gamma \in A(R_1 + R_2)$, one has $\|\gamma\| = (R_1 + \varepsilon) + (R_2 + \varepsilon)$, for $-\delta < \varepsilon \leqslant \delta$; then we can write, by (i), $\gamma = \gamma_1 \cdot \gamma_2$, where $R_i + \varepsilon - \delta \leqslant \gamma_i \leqslant R_i + \varepsilon + \delta$. Thus, $\gamma_i \in A(R_i)$. This shows that $\mathrm{Im}(f) \supset A(R_1 + R_2)$. Therefore $\alpha(R_1 + R_2) \leqslant \alpha(R_1) \cdot \alpha(R_2)$, that is $\log(\alpha)$ is a subadditive function. This implies that $\lim_{R \to \infty} R^{-1} \cdot \log \alpha(R)$ exists and it is equal to h. Best, by submultiplicativity we deduce:

$$\frac{n \log \alpha(R)}{nR} \geqslant \frac{\log \alpha(nR)}{nR}$$

which, by taking limits for $n \to \infty$, yields $\frac{\log \alpha(R)}{R} \geqslant h$, for all $R > 0$. □

Proof of Proposition 2.3. – Let $B(R)$, $A(R)$ and $\alpha(R)$ as in Lemma 2.5. Let moreover $B_l(R)$ denote the closed ball of radius R centred at the identity in $(\Gamma * \mathbf{Z}_2, \overline{d} * l)$, and let $B_{l,n}(R)$ the subset of elements of the form $\gamma_1 1 \gamma_2 1 \ldots \gamma_n 1$ with $\gamma_i \neq e$. Finally, set $R_i = (4k_i - 1)L$, for $k_i \in \mathbf{N}^*$. If $\boldsymbol{\gamma} = \gamma_1 1 \ldots \gamma_n 1$ belongs to the subset $A(R_1)1 \ldots A(R_n)1 \subset \Gamma * \mathbf{Z}_2$ and $\sum k_i = N$, then we have $\|\boldsymbol{\gamma}\|_{\overline{d}*l} \leqslant \sum R_i + nl \leqslant 4LN$, therefore one has a decomposition:

$$B_l(4LN) \supset \bigcup_{n \geqslant 1} B_{l,n}(4LN) \supset \bigcup_{n \geqslant 1} \bigcup_{\substack{k_1, \ldots, k_n \geqslant 1 \\ \Sigma_i k_i = N}} A(R_1)1 \ldots A(R_n)1.$$

Remark that these are disjoint unions, since $R_i > 0$ and $A(R_i) \cap A(R_j) = \emptyset$ if $i \neq j$. Then, by Lemma 2.5(ii), one computes:

$$\#B_l(4LN) \geqslant \sum_{n=1}^{N} \sum_{\substack{k_1,\dots,k_n \geqslant 1 \\ \Sigma_i k_i = N}} \alpha(R_1)\cdots\alpha(R_n) = \sum_{n=1}^{N} \sum_{\substack{k_1,\dots,k_n \geqslant 1 \\ \Sigma_i k_i = N}} \mathrm{e}^{4hLN}\cdot \mathrm{e}^{-hLn}$$
$$= \mathrm{e}^{4hLN} \sum_{n=1}^{N} \binom{N-1}{n-1} \cdot \mathrm{e}^{-hLn} = \mathrm{e}^{-hL}\cdot \mathrm{e}^{4hLN}\cdot (1+\mathrm{e}^{-hL})^{N-1}$$

and, therefore,

$$\mathrm{Ent}(\Gamma * \mathbf{Z}_2, \overline{d} * l) \geqslant \lim_{N\to\infty} \frac{\log \#B_l(4LN)}{4LN} = h + \frac{\log(1+\mathrm{e}^{-hL})}{4L}. \qquad \square$$

Coming to the second step, we need to fix some notations. Let $G = G_1 * G_2$ and, for $i \in \{1,2\}$, define $i^c = 3 - i$. Any $g \in G^*$ may be written uniquely as a word $g[1]\cdots g[n]$ where $g[k] \in G_1^* \cup G_2^*$, and where, if $g[k] \in G_i$, then $g[k+1]$ belongs to G_{i^c}. We refer to the word $g[1]\cdots g[n]$ on the alphabet $G_1^* \cup G_2^*$ as to the *normal form* of g: thus, by $g[i]$ we shall always mean the ith letter of its normal form. Moreover, we can associate, to every $g = g[1]\cdots g[n] \in G^*$, the length of its normal form $\ell(g) = n$ (which must not be confused with its norm $||g||$ with respect to some word metric). Finally, let us denote by $G_{i\cdot}, G_{\cdot i}$ the subsets of elements of G whose normal form is $g[1]\cdots g[n]$ with, respectively, $g[1] \in G_i^*$ and $g[n] \in G_i^*$, and let $G_{ij} = G_{i\cdot} \cap G_{\cdot j}$. For completeness, we set $\ell(e) = 0$ and we add the identity element to the sets G_{12}, $G_{1\cdot}$ and $G_{\cdot 2}$. We say that $g_1, g_2 \in G$ *match well* (in this order) if $g_1 \in G_{\cdot i}$ and $g_2 \in G_{i^c\cdot}$.

LEMMA 2.6. – *Let G be a nontrivial free product, endowed with a quasi-algebraic distance d. Then, for all $g, h \in G$ which match well, one has $||g\cdot h|| \geqslant ||g|| + ||h|| - C$, where C is a constant which does not depend on g, h.*

Proof. – Let G be a finite index subgroup of $(\hat G, \hat S)$ such that $d = d_{\hat S}|_G$, and let $\delta = d_{\mathcal{H}}(G, \hat G)$. Now let γ be a geodesic in $\mathcal{C}(\hat G, \hat S)$ from e to gh, i.e. choose an expression $gh = \hat s_1 \cdots \hat s_r$ of smallest length, $\hat s_i \in \hat S$. Let $\hat t_k \in \hat G$ the points on γ given by $\hat s_1 \cdots \hat s_k$ and choose points $t_k \in G$ such that $d(\hat t_k, t_k) \leqslant \delta$, with $t_1 = e, t_r = gh$. Thus, setting $s_1 = t_1, s_k = t_{k-1}^{-1} t_k \in G$, we also have $gh = s_1 \cdots s_r$ and $t_k = s_1 \cdots s_k$. Notice that

$$\|s_k\| \leqslant d(t_{k-1}, t_k) \leqslant d(t_{k-1}, \hat t_{k-1}) + d(\hat t_{k-1}, \hat t_k) + d(\hat t_k, t_k) \leqslant 1 + 2\delta.$$

Now, since

$$g\cdot h = g[1]\cdots g[l]h[1]\cdots h[m] = s_1[1]\cdots s_1[n_1]\cdots s_r[1]\cdots s_r[n_r]$$

and since g, h match well, we necessarily have $g = s_1[1]\cdots s_{k_0}[j_0]$ for some k_0, j_0 (notice that this expression need not to be the normal form of g). Then, we have

$$d(g, \hat t_{k_0}) \leqslant d(g, t_{k_0}) + d(t_{k_0}, \hat t_{k_0}) \leqslant d\big(s_{k_0}[1]\cdots s_{k_0}[j_0], s_{k_0}\big) + \delta \leqslant c + \delta$$

if $c = \sup\{\sum_k ||g[k]|| \mid g \in G, \|g\| \leqslant 1 + 2\delta\}$. Therefore,

$$||g\cdot h|| = d(e, \hat t_{k_0}) + d(\hat t_{k_0}, g\cdot h) \pm 2d(\hat t_{k_0}, g) \geqslant ||g|| + ||h|| - C$$

for $C = 2(c+\delta)$. $\square$

Proof of Proposition 2.4. – First of all, let $\sim : \Gamma \to G$ be a minimal transversal, that is choose for each $\gamma \in \Gamma$ a representative $\tilde{\gamma}$ such that $\|\tilde{\gamma}\|_d = \|\gamma\|_{\overline{d}}$. The map Φ is defined by

$$\Phi(\gamma_1 1 \gamma_2 1 \dots \gamma_{n+1}) = \tilde{\gamma}_1 \varepsilon_1 \tilde{\gamma}_2 \varepsilon_2 \cdots \tilde{\gamma}_{n+1}$$

(with, possibly, $\gamma_1 = e$ or $\gamma_{n+1} = e$) where ε_i are suitable "separators" of bounded $|| \, ||_d$-norm, which will allow to recover the sequence $\gamma_1, \dots, \gamma_{n+1}$ from $\Phi(\gamma_1 1 \gamma_2 1 \dots \gamma_{n+1})$.

The separators ε_i may be defined explicitely as follows. Assume that $G_1 \neq \mathbf{Z}_2$. Then, notice that, as $H \neq (1)$, one has $H^* \cap G_{12} \neq \emptyset$. In fact, given $h \in H^*$, if $h \in G_{ii^c}$ then $h^{-1} \in H^* \cap G_{i^c i}$; on the other hand, if $h \in G_{ii}$, then for any $g \in G_{i^c}$ we get $h' = ghg^{-1} \in H^* \cap G_{i^c i^c}$, and $hh' \in H^* \cap G_{ii^c}$.

Therefore, let $h = h[1] \cdots h[2r] \in H^* \cap G_{12}$. Up to taking a sufficiently large power of h, we may assume that

$$||h||_d > 2C \tag{1}$$

where $C = C(G, \hat{G}, \hat{S})$ is the constant of Lemma 2.6 (notice that $h^n \neq 1$ for all n, and remember that the balls of finite radius of G are finite). Then, choose an element $g_1 \in G_1^*$ such that $g_1 \neq h[1]$, take any $g_2 \in G_2^*$, and set

$$\eta = (g_1 g_2)^{r+1} = \overbrace{g_1 g_2 \cdot \cdots \cdot g_1 g_2}^{r+1 \text{ times}} \in G_{12}^*. \tag{2}$$

Now the ε_i are defined as:

$$\varepsilon_i = \varepsilon(\tilde{\gamma}_i, \tilde{\gamma}_{i+1}) = \begin{cases} g_2 \eta h^2 & \text{if } \tilde{\gamma}_i \in G_{\cdot 1} \text{ and } \tilde{\gamma}_{i+1} \in G_{1 \cdot}, \\ g_2 \eta h^4 g_1 & \text{if } \tilde{\gamma}_i \in G_{\cdot 1} \text{ and } \tilde{\gamma}_{i+1} \in G_{2 \cdot}, \\ \eta h^6 & \text{if } \tilde{\gamma}_i \in G_{\cdot 2} \text{ and } \tilde{\gamma}_{i+1} \in G_{1 \cdot}, \\ \eta h^8 g_1 & \text{if } \tilde{\gamma}_i \in G_{\cdot 2} \text{ and } \tilde{\gamma}_{i+1} \in G_{2 \cdot}. \end{cases}$$

For any $\boldsymbol{\gamma} = \gamma_1 1 \gamma_2 1 \dots \gamma_{n+1} \in \Gamma * \mathbf{Z}_2$, one clearly has

$$||\Phi(\boldsymbol{\gamma})||_d \leqslant \sum_i ||\gamma_i||_{\overline{d}} + ln = ||\boldsymbol{\gamma}||_{\overline{d} * l}$$

where $l = ||g_1||_d + ||g_2||_d + ||\eta||_d + 8||h||_d$. The lemma below concludes the proof:

LEMMA 2.7. – *The map Φ is injective.*

Before proving the lemma, we shall explain the reasons leading to the expression of ε_i, and the idea of the algorithm which permits to recover the γ_i's from $\Phi(\gamma_1 1 \dots 1 \gamma_{n+1})$. See $w = \Phi(\gamma_1 1 \dots 1 \gamma_{n+1}) = \tilde{\gamma}_1 \varepsilon_1 \tilde{\gamma}_2 \varepsilon_2 \cdots \tilde{\gamma}_{n+1}$ as a word on $G_1^* \cup G_2^*$. The separators ε_i are chosen so as to satisfy the following properties:

(i) *They must permit to obtain from any couple $\tilde{\gamma}_i, \tilde{\gamma}_{i+1}$ a couple $\tilde{\gamma}_i \varepsilon_i, \tilde{\gamma}_{i+1}$ which matches well*, in order not to cancel part of the $\tilde{\gamma}_i$ in the process of enchaining words: this is the reason why they are of four different "types", according to possible types of couples $(\tilde{\gamma}_i, \tilde{\gamma}_{i+1})$;

(ii) *They must contain h as subword*: since h does not appear as a subword of any $\tilde{\gamma}_i$ (if $\|h\|_d \gg 0$, by Lemma 2.6) this will permit to recover approximately the occurence of the first separator ε_1 in the word w, by looking at the first occurence of h;

(iii) *They cannot overlap over h nontrivially*: that is, when two separators $\varepsilon_i, \varepsilon_j$ overlap (as subwords of w) so that the subword h of ε_i overlaps with the subword h of ε_j, then $\varepsilon_i = \varepsilon_j$ and the overlapping must be the trivial one. This condition determines the *exact* position of the first separator ε_1 in the word w. The simplest way to let condition (iii) be satisfied is to define ε_i as $(g_2)\eta h(g_1)$, where η is a sufficiently long word which does not contain $h[1]$ (this is the reason of the choice of $g_1 \neq h[1]$ and of (2));

(iv) Finally, *the type of the couple* $(\tilde{\gamma}_i, \tilde{\gamma}_{i+1})$ *must also be coded in* ε_i: this, in order to recover exactly $\tilde{\gamma}_1$ and not $\tilde{\gamma}_1 g_2$. The type is given by the exponent of h appearing in ε_i, that is by the number of consecutive repetitions of h in w (powers of 2 are needed to compare possible nontrivial overlappings, see cases 2–3 in the proof below).

Let us now give a formal proof of Lemma 2.7.

Proof of Lemma 2.7. – Let $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \Gamma * \mathbf{Z}_2$, and suppose that

$$w = \Phi(\boldsymbol{\gamma}) = \tilde{\gamma}_1 \varepsilon(\tilde{\gamma}_1, \tilde{\gamma}_2) \cdots \tilde{\gamma}_{n+1} = \tilde{\gamma}'_1 \varepsilon(\tilde{\gamma}'_1, \tilde{\gamma}'_2) \cdots \tilde{\gamma}'_{m+1} = \Phi(\boldsymbol{\gamma}') = w'.$$

If $\tilde{\gamma}_i \in G_{a_i b_i}, \tilde{\gamma}'_i \in G_{a'_i, b'_i}$, we have

$$w = \tilde{\gamma}_1 g_2^{2-b_1} \eta h^{4b_1+2a_2-4} g_1^{a_2-1} \tilde{\gamma}_2 \cdots \tilde{\gamma}_{n+1},$$
$$w' = \tilde{\gamma}'_1 g_2^{2-b'_1} \eta h^{4b'_1+2a'_2-4} g_2^{a'_2-1} \tilde{\gamma}'_2 \cdots \tilde{\gamma}_{m+1}$$

and we may assume that $\ell(\tilde{\gamma}'_1) \leqslant \ell(\tilde{\gamma}_1)$.

We have therefore the following possibilities:

(1) $\ell(\tilde{\gamma}'_1 g_2^{2-b'_1} \eta) + 2r \leqslant \ell(\tilde{\gamma}_1)$. Then, $\tilde{\gamma}'_1 g_2^{2-b'_1} \eta h \subset \tilde{\gamma}_1$ as words. Thus, the normal form of $\tilde{\gamma}_1$ would contain h as a subword and, by Lemma 2.6 and by (1), $\tilde{\gamma}_1$ would not be of minimal length in the class γ_1, which is a contradiction.

(2) $\ell(\tilde{\gamma}'_1 g_2^{2-b'_1} \eta) \leqslant \ell(\tilde{\gamma}_1) < \ell(\tilde{\gamma}'_1 g_2^{2-b'_1} \eta) + 2r = \ell$. Then, as $\ell(\eta) = \ell(h) + 2$, one has $\tilde{\gamma}'_1 g_2^{2-b'_1} \eta h h[1] h[2] \subset \tilde{\gamma}_1 g_2^{2-b_1} \eta$ as words. Since $\ell(\tilde{\gamma}_1) < \ell$ we would deduce

$$h[1] = w'[\ell+1] = w[\ell+1] = \begin{cases} \text{either } g_1, \\ \text{or } g_2 \end{cases}$$

which contradicts our choice of $g_1 \neq h[1]$ or the fact that $h[1] \in G_1^*$.

(3) $\ell(\tilde{\gamma}_1) < \ell(\tilde{\gamma}'_1 g_2^{2-b'_1} \eta) = \ell$. Then, as we assumed that $\ell(\tilde{\gamma}'_1) \leqslant \ell(\tilde{\gamma}_1)$, we have

$$\tilde{\gamma}'_1 g_2^{2-b'_1} \eta \subset \tilde{\gamma}_1 g_2^{2-b_1} \eta;$$

but since $w'[\ell+1] = h[1]$ does not appear as a letter of $g_2^{2-b_1} \eta$, we deduce that

$$\tilde{\gamma}'_1 g_2^{2-b'_1} \eta = \tilde{\gamma}_1 g_2^{2-b_1} \eta$$

necessarily. This means that either $\tilde{\gamma}_1 = \tilde{\gamma}'_1 g_2$ or $\tilde{\gamma}_1 = \tilde{\gamma}'_1$. We shall now show that $\tilde{\gamma}_1 = \tilde{\gamma}'_1$ necessarily.

Assume that $\tilde{\gamma}_1 = \tilde{\gamma}'_1 g_2$: then $b_1 = 2$ and $b'_1 = 1$. Thus, $\varepsilon(\tilde{\gamma}_1, \tilde{\gamma}_2)$ contains at least h^6 as subword, while $\varepsilon(\tilde{\gamma}'_1, \tilde{\gamma}'_2)$ contains at most h^4; therefore,

$$w'\left[\ell\left(\tilde{\gamma}'_1 g_2 \eta h^4\right) + 1\right] = w\left[\ell\left(\tilde{\gamma}'_1 g_2 \eta h^4\right) + 1\right] = h[1] \neq g_1,$$

and this implies that $\varepsilon(\tilde\gamma_1', \tilde\gamma_2') = g_2\eta h^2$ precisely. Thus, we have

$$\big(\tilde\gamma_1\eta h^2\big)hh[1] \subset \big(\tilde\gamma_1' g_2\eta h^2\big)\tilde\gamma_2' g_2^{2-b_2'}\eta$$

as words, i.e. $hh[1] \subset \tilde\gamma_2' g_2^{2-b_2'}\eta$. As $\tilde\gamma_2'$ cannot contain h (being minimizing), we should have

$$h[1] = (\tilde\gamma_2' g_2^{2-b_2'}\eta)[2r+1] = \begin{cases} \text{either } g_1, \\ \text{or } g_2 \end{cases}$$

which again gives a contradiction. Therefore, $\tilde\gamma_1 = \tilde\gamma_1'$, which clearly implies that $\gamma_1 = \gamma_1'$. One subsequently shows that $\tilde\gamma_i = \tilde\gamma_i'$ for all i by induction, restarting from the identity $\tilde\gamma_2 \prod_{i=2}^n \varepsilon(\tilde\gamma_i, \tilde\gamma_{i+1})\tilde\gamma_{i+1} = \tilde\gamma_2' \prod_{i=2}^m \varepsilon(\tilde\gamma_i', \tilde\gamma_{i+1}')\tilde\gamma_{i+1}'$. □

3. Growth tightness of amalgamated products

A direct consequence of Theorem 1.3 is:

THEOREM 3.1. – *Let $\hat G$ be a finitely generated group which contains a nontrivial free product $G = G_1 * G_2 \neq \mathbf{Z}_2 * \mathbf{Z}_2$ as a subgroup of finite index. Then, $\hat G$ is growth tight with respect to any word metric.*

Proof. – Let $\hat H$ be an infinite normal subgroup of $\hat G$, and let $H = \hat H \cap G$ be the corresponding normal subgroup of G. Thus, H and G/H are finite index subgroups of $\hat H$ and $\hat G/\hat H$ respectively; notice that $H \neq (e)$, since $\hat H$ is infinite. Now let $\hat S$ be a finite generating set for $\hat G$, let $\hat S/\hat H$ be the induced generating set of $\hat G/\hat H$, and let $d_{\hat S}$, $d_{\hat S/\hat H}$ denote the associated word metric. We shall denote by d_1 the metric on G/H which is the quotient of the metric $d_{\hat S}|_G$ of G; on the other hand, let d_2 be the restriction to G/H of the metric $d_{\hat S/\hat H}$ of $\hat G/\hat H$, i.e.:

$$d_1(g_1H, g_2H) = \inf_{h,h' \in H} d_{\hat S}(g_1h, g_2h') = d_{\hat S}(H, g_1^{-1}g_2),$$

$$d_2(g_1H, g_2H) = \inf_{\hat h,\hat h' \in \hat H} d_{\hat S}(g_1\hat h, g_2\hat h') = d_{\hat S}(\hat H, g_1^{-1}g_2)$$

(as H and $\hat H$ are normal in $G, \hat G$). Clearly $d_1 \geqslant d_2$. Conversely, $d_2 \geqslant d_1 - M$ for

$$M = d_{\mathcal H}(H, \hat H) < \infty$$

(since H has finite index in $\hat H$). Therefore, one has $\mathrm{Ent}(G/H, d_1) = \mathrm{Ent}(G/H, d_2)$. But now Theorem 1.3 yields

$$\mathrm{Ent}(\hat G, \hat S) = \mathrm{Ent}(G, d_{\hat S}) > \mathrm{Ent}(G/H, d_1) = \mathrm{Ent}(G/H, d_2) = \mathrm{Ent}(\hat G/\hat H, \hat S/\hat H)$$

where the first and last inequalities hold since G and G/H have, respectively, finite index in $\hat G$ and $\hat G/\hat H$. □

We can now prove Theorem 1.4 by showing that any amalgamated product, over a finite subgroup, of residually finite groups contains a nontrivial free product of finite index. The construction is similar to that used in [7], Lemma 7.4.

Proof of Theorem 1.4. – Let $G = G_1 *_F G_2$ be a nontrivial amalgamated product, and assume F finite and G_i residually finite, $i = 1, 2$. Let $\rho_i : G_i \to F_i$ be homomorphisms into finite groups, which are injective when restricted to F. Let $X = F_1/\rho_1(F) \times F \times F_2/\rho_2(F)$. We can then define a homomorphism τ of G in the permutation group of the finite set X, which moreover is injective on F, as follows. Let $\overline{f_i}$ denote the class of f_i modulo $\rho_i(F)$, and choose transversals $\sim : F_i/\rho_i(F) \to F_i$ (that is, $\overline{\tilde{\varphi}} = \varphi$, for all $\varphi \in F_i/\rho_i(F)$). These transversals induce bijections $T_i : F_i/\rho_i(F) \times F \xrightarrow{\sim} F_i$. Now, one can define an action of G_i on $F_i/\rho_i(F) \times F \simeq F_i$ letting $g \in G_i$ act on F_i by right multiplication by $\rho_i(g)$: in formulas, if $(\varphi, f) \in F_i/\rho_i(F) \times F$,

$$(\varphi, f) \cdot g = T_i^{-1}\big[T_i(\varphi, f)\rho_i(g)\big] = T_i^{-1}\big(\tilde{\varphi}\rho_i(fg)\big).$$

Then, one obtains an action of $G_1 * G_2$ on elements $(\varphi_1, f, \varphi_2) \in X$ by letting G_1 act on the first two components and G_2 on the last two. In this way, F acts in the same manner on X, both as subgroup of G_1 and as subgroup of G_2 (that is, simply by right muliplication on the second component). Therefore, this action descends to an action of $G_1 *_F G_2$ on X. Moreover, it is clear that τ injects F. By the subgroup theorem for amalgamated products (cp. [7]), the kernel $H = \ker(\tau)$ is the fundamental group of a graph $\mathcal{G}$ of groups which has the double cosets HgG_i as vertices (and $H \cap gG_ig^{-1}$ as corresponding groups), the double cosets HgF as edges (with corresponding groups $H \cap gFg^{-1}$), and with morphisms given by the natural inclusions. Since τ injects F, we have $H \cap gFg^{-1} = (e)$ and therefore H is a free product. Moreover H is a nontrivial free product. In fact, if all vertices groups are trivial, then H is free. Otherwise, let us say that $H \cap G_1 \neq (e)$. Then, H is again a nontrivial free product unless $H \cap G_2 = (e)$ and $HgG_1 = HG_1$ for all g. But in this case (H being the fundamental group of $\mathcal{G}$) we would have $H = H \cap G_1$, hence H would be a normal subgroup of G_1 and $gG_1 = gHG_1 = HgG_1 = HG_1 = G_1$ for all $g \in G$. That is, $G/G_1 = (e)$, which is not possible as G is supposed to be a nontrivial amalgamated product. Notice that, moreover, H is different from $\mathbf{Z}_2 * \mathbf{Z}_2$ since it has finite index in G and G has exponential growth. One then concludes by Theorem 3.1. □

Finally, remark that Theorem 1.4 holds more generally when G_1 and G_2 are only F^*-residually finite, that is when there exist homomorphisms into finite groups $\rho_i : G_i \to F_i$ which are injective when restricted to F.

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Andrea SAMBUSETTI
Dipartimento di Matematica "G. Castelnuovo",
Università di Roma "La Sapienza",
P.le Aldo Moro 2,
00185 Roma, Italy
E-mail: sambuset@mat.uniroma1.it