ON THE ZERO SET OF SEMI-INVARIANTS FOR QUIVERS

BY CHRISTINE RIEDTMANN AND GRZEGORZ ZWARA

ABSTRACT. – Let \( d \) be a prehomogeneous dimension vector for a finite quiver \( Q \). We show that the set of common zeros of all semi-invariants of positive degree for the variety of representations of \( Q \) with dimension vector \( N \cdot d \) under the product of the general linear groups at all vertices is irreducible and a complete intersection for large natural numbers \( N \).

\( \text{© 2003 Elsevier SAS} \)

RéSUMÉ. – Soient \( Q \) un carquois fini et \( d \) un vecteur de dimension pour \( Q \) tels que la variété des représentations de \( Q \) de dimension \( d \) contienne une orbite dense sous l’action du groupe \( \text{Gl}(d) \) des changements de base en chaque sommet. Nous montrons que l’ensemble des zéros des semi-invariants de degré positif sous \( \text{Gl}(N \cdot d) \) sur la variété des représentations de dimension \( N \cdot d \) est irréductible et une intersection complète pourvu que \( N \) soit suffisamment grand.

\( \text{© 2003 Elsevier SAS} \)

1. Introduction

Let \( k \) be an algebraically closed field, and let \( Q = (Q_0, Q_1, t, h) \) be a finite quiver, i.e. a finite set \( Q_0 = \{1, \ldots, n\} \) of vertices and a finite set \( Q_1 \) of arrows \( \alpha : t\alpha \to h\alpha \), where \( t\alpha \) and \( h\alpha \) denote the tail and the head of \( \alpha \), respectively.

A representation of \( Q \) over \( k \) is a collection \( (X(i); i \in Q_0) \) of finite dimensional \( k \)-vector spaces together with a collection \( (X(\alpha) : X(t\alpha) \to X(h\alpha); \alpha \in Q_1) \) of \( k \)-linear maps. A morphism \( f : X \to Y \) between two representations is a collection \( (f(i) : X(i) \to Y(i)) \) of \( k \)-linear maps such that

\[ f(h\alpha) \circ X(\alpha) = Y(\alpha) \circ f(t\alpha) \quad \text{for all } \alpha \in Q_1. \]

The dimension vector of a representation \( X \) of \( Q \) is the vector

\[ \text{dim } X = (\dim X(1), \ldots, \dim X(n)) \in \mathbb{N}^{Q_0}. \]

We denote the category of representations of \( Q \) by \( \text{rep}(Q) \), and for any vector \( d = (d_1, \ldots, d_n) \in \mathbb{N}^{Q_0} \)

\[ \text{rep}(Q, d) = \prod_{\alpha \in Q_1} \text{Mat}(d_{h\alpha} \times d_{t\alpha}, k) \]

is the vector space of representations \( X \) of \( Q \) with \( X(i) = k^{d_i}, \ i \in Q_0 \). The group

\[ \text{Gl}(d) = \prod_{i=1}^{n} \text{Gl}(d_i, k) \]

\( \text{© 2003 Elsevier SAS} \)

0012-9593/06 © 2003 Elsevier SAS. All rights reserved
acts on \( \text{rep}(Q, d) \) by
\[
\left( (g_1, \ldots, g_n) \ast X \right)(\alpha) = g_{\alpha_1} \cdot X(\alpha) \cdot g_{\alpha_1}^{-1}.
\]
Note that the \( \text{Gl}(d) \)-orbit of \( X \) consists of the representations \( Y \) in \( \text{rep}(Q, d) \) which are isomorphic to \( X \).

We call \( d \) a prehomogeneous dimension vector if \( \text{rep}(Q, d) \) contains an open orbit \( \text{Gl}(d) \ast T \).

Such a representation \( T \) is characterized by \( \text{Ext}^1_Q(T, T) = 0 \) [6]. If \( Q \) admits only finitely many indecomposable representations, or equivalently if the underlying graph of \( Q \) is a disjoint union of Dynkin diagrams \( A, D \) or \( E \) [2], every vector \( d \) is prehomogeneous. Indeed, any representation is a direct sum of indecomposables in an essentially unique way by the theorem of Krull–Schmidt, and therefore \( \text{rep}(Q, d) \) contains finitely many orbits, one of which must be open.

Let \( d \) be prehomogeneous, and let \( f_1, \ldots, f_s \in k[\text{rep}(Q, d)] \) be the irreducible monic polynomials whose zeros \( Z(f_1), \ldots, Z(f_s) \) are the irreducible components of codimension 1 of \( \text{rep}(Q, d) \setminus \text{Gl}(d) \ast T \), where \( \text{Gl}(d) \ast T \) is the open orbit. It is easy to see that
\[
g \cdot f_i = \chi_i(g) \cdot f_i
\]
for \( g \in \text{Gl}(d) \), where \( \chi_i : \text{Gl}(d) \to k^* \) is a character. A regular function with this property is called a semi-invariant. By [8], any semi-invariant is a scalar multiple of a monomial in \( f_1, \ldots, f_s \), and \( f_1, \ldots, f_s \) are algebraically independent. We denote by
\[
Z_{Q,d} = \{ X \in \text{rep}(Q, d) ; f_i(X) = 0, \ i = 1, \ldots, s \}
\]
the set of common zeros of all semi-invariants of positive degree. Obviously we have \( \text{codim} Z_{Q,d} \leqslant s \), and equality means that \( Z_{Q,d} \) is a complete intersection.

Our first main result is as follows.

**Theorem 1.1.** – Let \( T_1, \ldots, T_r \) be pairwise non-isomorphic indecomposable representations in \( \text{rep}(Q) \) such that \( \text{Ext}^1_Q(T_i, T_j) = 0 \) for any \( i, j \leqslant r \). Then there is a positive integer \( N \) such that \( Z_{Q,d} \) is a complete intersection and an irreducible variety for any dimension vector \( d = \sum_{i=1}^r \lambda_i \cdot \dim T_i \) with \( \lambda_i \geqslant N, \ i \leqslant r \).

As an immediate consequence we derive the following fact.

**Corollary 1.2.** – Let \( d \) be a prehomogeneous dimension vector in \( \mathbb{N}^{Q_0} \). Then there is a positive integer \( N \) such that \( Z_{c,d} \) is a complete intersection and an irreducible variety for any \( c \geqslant N \).

We will prove in a forthcoming paper that we may choose \( N = 2 \) if \( Q \) is a disjoint union of Dynkin diagrams and \( N = 3 \) if \( Q \) is a disjoint union of Dynkin diagrams and extended Dynkin diagrams.

In order to put our results into the context of invariant theory, we recall a few definitions. We assume that \( k \) is the field of complex numbers. Let \( G \) be a reductive algebraic group acting regularly on a finite dimensional vector space \( V \). By Hilbert’s theorem, the ring \( k[V]^G \) of \( G \)-invariant polynomials on \( V \) is a finitely generated algebra and thus is the algebra of polynomial functions on a variety \( V/\!/G \). The inclusion of \( k[V]^G \) into \( k[V] \) gives rise to a regular surjective map \( \pi : V \to V/\!/G \) which is constant on \( G \)-orbits, the so-called categorical quotient of \( V \) by \( G \) [3]. As \( G \) is completely reducible, the \( G \)-module \( k[V] \) can be decomposed uniquely as a direct sum
\[
k[V] = \bigoplus_{\lambda} M_{\lambda} \otimes_k V_{\lambda}
\]
where \( M_\lambda \) ranges over a set of representatives of the irreducible \( G \)-modules and \( V_\lambda \) is just a vector space, possibly infinite dimensional, which records the multiplicity with which \( M_\lambda \) arises in \( k[V] \). As the action of \( G \) on \( k[V] \) commutes with multiplication by \( G \)-invariants, each \( V_\lambda \) can be viewed as a \( k[V]G \)-module. In fact, (1.1) is a decomposition as \( G-k[V]^G \)-bimodules; the group \( G \) acts only on \( M_\lambda \) and \( k[V]^G \) only on \( V_\lambda \). A covariant of weight \( \lambda \) is a \( G \)-linear map

\[
\varphi : k[V] \to M_\lambda,
\]

or equivalently a linear form on \( V_\lambda \). The pair \( (V, G) \) is called:

- coregular if \( V/G \) has no singularities,
- equidimensional if the fiber \( \pi^{-1}(\pi(0)) \) has the same dimension as the quotient \( V/G \),
- cofree if \( k[V] \) is free as a \( k[V]^G \)-module, or equivalently the module \( V_\lambda^* \) of covariants is free over \( k[V]^G \) for all \( \lambda \).


Let us consider \( V = \text{rep}(Q, \mathfrak{d}) \) for a prehomogeneous \( \mathfrak{d} \) as a representation of the subgroup \( \text{Sl}(\mathfrak{d}) = \prod_{i=1}^n \text{Sl}(d_i) \) of \( \text{GL}(\mathfrak{d}) \). For each arrow \( \alpha \), the set

\[
V_\alpha = \{ X \in V; X(\beta) = 0 \forall \beta \neq \alpha \}
\]

is an irreducible subrepresentation of \( V \), and \( V \) is the direct sum \( V = \bigoplus_{\alpha \in Q_1} V_\alpha \). The ring \( k[V]^{\text{Sl}(\mathfrak{d})} \) of \( \text{Sl}(\mathfrak{d}) \)-invariants is generated by the semi-invariants \( f_1, \ldots, f_s \). The categorical quotient \( V/\text{Sl}(\mathfrak{d}) \) is an \( s \)-dimensional affine space and \( (V, \text{Sl}(\mathfrak{d})) \) is coregular. It is cofree in the cases for which our main results mentioned above hold. Surprisingly, the situation is better for big multiples of a given dimension vector. It can be bad otherwise, as the following example illustrates: For

\[
Q = \begin{array}{cccccccc}
1 & & & & & & & \\
& 2 & & & & & & \\
& & 3 & & & & & \\
& & & & \cdots & & & \\
& & & & & & n & \\
& & & & & & & (n+1)
\end{array}
\]

the dimension vector \( \mathfrak{d} = (1, \ldots, 1, n-1) \) is prehomogeneous. Indeed, the open orbit in \( \text{rep}(Q, \mathfrak{d}) \) consists of those representations \( X \) for which none of the \( (n-1) \times (n-1) \)-minors \( f_1, \ldots, f_n \) of the \( (n-1) \times n \)-matrix \( [X(\alpha_1), \ldots, X(\alpha_n)] \) vanishes. On the other hand, the set \( \mathbb{Z}_{Q,\mathfrak{d}} \) of common zeros of \( f_1, \ldots, f_n \) is the set of \( (n-1) \times n \)-matrices of rank less than \( n-1 \) and thus has codimension 2. In [1], Chang and Weyman examine quivers \( Q \) with underlying graph \( A_n \) and arbitrary dimension vectors. They find that \( (\text{rep}(Q, \mathfrak{d}), \text{Sl}(\mathfrak{d})) \) is always equidimensional, but the set \( \pi^{-1}(\pi(0)) = \mathbb{Z}_{Q,\mathfrak{d}} \) is reducible in general.

2. Notations and preliminaries

We first recall Schofield’s construction of semi-invariants from [9]. The Euler form is the \( \mathbb{Z} \)-bilinear form on \( \mathbb{Z}^{Q_0} \) defined by

\[
\langle \mathfrak{d}, \mathfrak{e} \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha \in Q_1} d_{i\alpha} e_{\alpha}.
\]
For \( d, e \in \mathbb{N}^{Q_0}, X \in \text{rep}(Q, d), Y \in \text{rep}(Q, e) \) we consider the linear map
\[
\mathcal{F}_{X,Y} : \bigoplus_{i \in Q_0} \text{Hom}_k(k^{d_i}, k^{e_i}) \to \bigoplus_{\alpha \in Q_1} \text{Hom}_k(k^{d_{\alpha i}}, k^{e_{\alpha i}}),
\]
which sends \((g_i; i \in Q_0)\) to \((h_\alpha; \alpha \in Q_1)\) with \(h_\alpha = g_{\alpha i} \cdot X(\alpha) - Y(\alpha) \cdot g_{i \alpha}\). Note that
\[
\ker \mathcal{F}_{X,Y} = \text{Hom}_Q(X, Y) \quad \text{and} \quad \text{coker} \mathcal{F}_{X,Y} \cong \text{Ext}_Q^1(X, Y).
\]
This implies that
\[
\langle \dim X, \dim Y \rangle = [X, Y] - 1[X, Y],
\]
where we set
\[
[X, Y] = \dim_k \text{Hom}_Q(X, Y), \quad 1[X, Y] = \dim_k \text{Ext}_Q^1(X, Y).
\]
If we assume that \( (d, e) = 0 \), the linear map \( \mathcal{F}_{X,Y} \) will be represented by a square matrix \( M_{X,Y} \) (with respect to some bases), and the determinant \( \det M_{X,Y} \) is a \( \text{Gl}(d) \times \text{Gl}(e) \)-semi-invariant on \( \text{rep}(Q, d) \times \text{rep}(Q, e) \). It might vanish, however.

For a representation \( U \) of \( Q \), the right perpendicular category \( U^\perp \) and the left perpendicular category \( U^\perp \) are the full subcategories of \( \text{rep}(Q) \) whose objects \( Y \) satisfy
\[
[U, Y] = 1[U, Y] = 0 \quad \text{and} \quad [Y, U] = 1[Y, U] = 0,
\]
respectively. As \( 1[X, Y] = [X, \tau X] \) for any two representations \( X \) and \( Y \) of \( Q \), where \( \tau \) is the Auslander–Reiten translation (see [7] for relevant definitions), we have \( U^\perp = 1(\tau U) \).

Now assume that \( T_1, \ldots, T_r \) are pairwise non-isomorphic with \( 1[T_i, T_j] = 0 \), \( i, j = 1, \ldots, r \) and such that \( T = \bigoplus_{i=1}^r T_i \lambda_i \) is sincere, i.e., \( T(e) \neq 0 \) for all \( e \in Q_0 \). Then the category \( T^\perp \) is equivalent to the category of representations of a quiver \( Q^\perp \) having \((n - r)\) vertices. Choose \( Y \in T^\perp, Y \neq 0 \), and set \( d = \dim T, e = \dim Y \). Observe that
\[
\langle d, e \rangle = [T, Y] - 1[T, Y] = 0,
\]
the dimension of \( \bigoplus_{i \in Q_0} \text{Hom}_k(k^{d_i}, k^{e_i}) \) is positive and \( M_{T,Y} \) is invertible. Thus the \( \text{Gl}(d) \)-semi-invariant \( f_Y = \det M_{X,Y} \) is non-trivial on \( \text{rep}(Q, d) \). It is easy to see that \( f_Y = f_{Y'} \cdot f_{Y''} \) for an exact sequence \( 0 \to Y' \to Y \to Y'' \to 0 \) in \( T^\perp \). If the simple objects of \( T^\perp \) are \( S_1, \ldots, S_{n-r} \), the semi-invariants \( f_{S_1}, \ldots, f_{S_{n-r}} \) are algebraically independent and they generate the algebra of \( \text{Sl}(d) \)-invariants. As a consequence we have
\[
Z_{Q,d} = \{ X \in \text{rep}(Q, d); \ [X, S_j] \neq 0, j = 1, \ldots, n-r \}
\]
\[
= \{ X \in \text{rep}(Q, d); \ [X, Y] \neq 0 \text{ for all } Y \in T^\perp, Y \neq 0 \}.
\]

We will keep the following assumptions and notations throughout the paper: \( T \) is a sincere representation of \( Q \) with \( 1[T, T] = 0 \) and \( \dim T = d \). We can always make \( T \) sincere by considering the full subquiver which supports \( T \) instead of \( Q \). Observe that \( Q \) does not contain oriented cycles. If the decomposition of \( T \) as a direct sum of pairwise non-isomorphic indecomposables is
\[
T = \bigoplus_{i=1}^r T_i^{\lambda_i}, \quad \lambda_i \geq 1,
\]
we set $\lambda = \min\{\lambda_i; \ i = 1, \ldots, r\}$. Note that $Z_{Q, d}$ is defined by $n - r$ polynomial equations. In order to prove it is a complete intersection it suffices to show

$$\operatorname{codim} Z_{Q, d} \geq n - r.$$ 

All varieties we consider will be quasi-projective over $k$, and we will look at codimensions for constructible subsets of affine space only.

3. Proof of Theorem 1.1

Our strategy is to first get rid of the set $\{X \in Z_{Q, d}; [T, X] \neq 0\}$ by showing its codimension is big. In fact, it is for this we need our assumption on the multiplicities $\lambda_i$.

The following lemma will be used several times in our article.

**Lemma 3.1.** Let $d'' \in \mathbb{N}_Q \setminus \{0\}$ be such that $d'' \leq d$, i.e., $d' = d - d'' \in \mathbb{N}_Q$, and let $V = V_1 \oplus \cdots \oplus V_b$ belong to $\operatorname{rep}(Q, d')$, where $V_1, \ldots, V_b$ are indecomposable. Then the set

$$A_V = \{X \in \operatorname{rep}(Q, d); \exists \text{epimorphism } X \to V\}$$

is constructible, irreducible, and $\operatorname{codim} A_V \geq b - (d, d'')$.

**Proof.** Consider the subvariety

$$C = \left\{(X, g = \begin{bmatrix} g' \\ g'' \end{bmatrix}); \ g'' \in \operatorname{Hom}_Q(X, V)\right\}$$

of $\operatorname{rep}(Q, d) \times \operatorname{Gl}(d)$. Note that $g''$ is an epimorphism as $g$ is invertible. This leads to the surjective regular map $\pi: C \to A_V$ given by the first projection. We see that the set $A_V$ is constructible and that

$$\dim \pi^{-1}(X) = [X, V] + \sum_{i \in Q_0} d'_i d_i \geq b + \sum_{i \in Q_0} d'_i d_i$$

for $X \in A_V$, since there exists an epimorphism $X \to V_1 \oplus \cdots \oplus V_b$.

On the other hand, sending $(X, g)$ to $(g \star X, g)$ we obtain an isomorphism from $C$ to the subvariety $D$ of $\operatorname{rep}(Q, d) \times \operatorname{Gl}(d)$ consisting of all pairs $(Y, g)$ for which $Y(\alpha)$ is in the block form

$$Y(\alpha) = \begin{bmatrix} \ast & \ast \\ 0 & V(\alpha) \end{bmatrix}, \ \alpha \in Q_1.$$ 

As $D$ is just the product of an affine space of dimension $\sum_{\alpha \in Q_1} d'_h d_{t\alpha}$ with $\operatorname{Gl}(d)$, we conclude that $A_V$ is irreducible and that

$$\sum_{\alpha \in Q_1} d'_h d_{t\alpha} + \dim \operatorname{Gl}(d) - \dim A_V = \dim C - \dim A_V \geq b + \sum_{i \in Q_0} d'_i d_i.$$ 

Our estimate follows from an easy computation. $\square$

**Corollary 3.2.** Keeping the notations of the preceding lemma, we assume moreover that $V$ is a subrepresentation of $\tau T$. Then we have

$$\operatorname{codim} A_V \geq 1 + \lambda.$$
Proof. – It suffices to show \(-\langle d, d'' \rangle \geq \lambda_i\) for some \(i\). Observe that
\[
[T, V] \leq [T, \tau T] = [T, T] = 0 \quad \text{and} \quad [V, \tau T] = [V, T] > 0.
\]
Consequently, we have \(1[T, V] > 1\) for some \(i\) and thus
\[
-\langle d, d'' \rangle = -[T, V] + [V, T] = [V, T] > 1[T, V] \geq 1[T, V] \geq \lambda_i.
\]
□

For any \(U \in \text{rep}(Q)\), we denote by \(\mathcal{X}_U\) the set
\[
\mathcal{X}_U = \{ X \in \text{rep}(Q, d); [X, U] \neq 0 \}.
\]

**Lemma 3.3.** – Let \(U\) be a non-zero subrepresentation of \(\tau T\). Then we have
\[
\text{codim} \mathcal{X}_U \geq \lambda + 1 - \eta(d \dim U),
\]
where \(\eta(e) = \sum_{i \in Q_0} \lfloor e_i^2/4 \rfloor\), for any \(e \in \mathbb{N}^{Q_0}\), and \([q]\) denotes the largest integer not exceeding \(q\) for \(q \in \mathbb{Q}\).

Proof. – We want to exploit that for any non-zero homomorphism \(\varphi : X \to U\) from \(X \in \text{rep}(Q, d)\) to \(U\), \(X\) belongs to \(\mathcal{A}_U\) for the representation \(V = \varphi(X)\), which is a quotient of \(X\) as well as a subrepresentation of \(\tau T\). Set \(e = \dim U\), and choose \(f \in \mathbb{N}^{Q_0}\) with \(f \leq e, d\). Consider the closed subvariety \(L_f\) of \(\prod_{i \in Q_0} \text{Grass}(k^{e_i}, f_i)\) consisting of sequences \((V_i)_{i \in Q_0}\) such that \(U(\alpha)(V_{i,\alpha}) \subseteq V_{f,\alpha}, \alpha \in Q_1\). In other words, \(L_f\) is the variety of all subrepresentations \(V\) of \(U\) with dimension vector \(f\). Note that \(\dim L_f \leq \eta(e)\) since \(\dim \text{Grass}(k^e, f) = (e - f)f \leq \lfloor e^2/4 \rfloor\). The subset \(\mathcal{F}_f\) of \(L_f \times \text{rep}(Q, d)\) consisting of pairs \((V, X)\) such that there is an epimorphism from \(X\) onto \(V\) in constructible. Indeed, the affine subvariety
\[
\mathcal{H} = \{ (\varphi = (\varphi_i), X); \varphi \in \text{Hom}_{Q}(X, U) \} \subseteq \prod_{i \in Q_0} \text{Hom}_k(k^{d_i}, k^{e_i}) \times \text{rep}(Q, d)
\]
is the disjoint union \(\bigcup_{f \leq e, d} \mathcal{H}_f\) of the locally closed subsets
\[
\mathcal{H}_f = \{ (\varphi, X) \in \mathcal{H}; \text{rk } \varphi_i = f_i, \ i \in Q_0 \},
\]
and \(L_f\) is the image of \(\mathcal{H}_f\) under the regular map sending \((\varphi, X)\) to \(((\im \varphi_i), X)\). Observe that \(\mathcal{X}_U\) is the union \(\bigcup_{0 \neq f \leq d, e} \pi_2(\mathcal{F}_f)\) of the images under the second projection, and thus
\[
\dim \mathcal{X}_U \leq \max_{0 \neq f \leq d, e} \dim \mathcal{F}_f.
\]
Now consider the first projection \(\pi_1 : \mathcal{F}_f \to L_f\). For \(V \in L_f\), we have \(\pi_1^{-1}(V) = \{ V \} \times \mathcal{A}_V\), so we know by Corollary 3.2 that
\[
\dim \pi_1^{-1}(V) \leq \dim \text{rep}(Q, d) - 1 - \lambda,
\]
as \(V \subseteq U\) is a subrepresentation of \(\tau T\) with \(\dim V \leq d\). We conclude that
\[
\dim \mathcal{X}_U \leq \max_{0 \neq f \leq d, e} \dim \mathcal{F}_f \leq \left( \max_{0 \neq f \leq d, e} \dim L_f \right) + \dim \text{rep}(Q, d) - 1 - \lambda
\]
\[
\leq \eta(e) - 1 - \lambda + \dim \text{rep}(Q, d),
\]
which implies our claim.
COROLLARY 3.4. – Let \( c = \max\{\eta(\tau T_i); \ i = 1, \ldots, r\} \). Then the set

\[ \mathcal{E}_d = \{ X \in \text{rep}(Q, d); \ 1[T, X] > 0 \} \]

is either empty or else \( \text{codim} \mathcal{E}_d \geq 1 + \lambda - c \).

**Proof.** – If \( T \) is projective then the set \( \mathcal{E}_d \) is empty. Otherwise, any non-zero map in

\[ \text{Hom}_Q(X, \tau T) \simeq \text{Ext}^1_Q(T, X) \]

induces a non-zero map \( X \to \tau T_i \) for some non-projective \( T_i \), and we see that

\[ \mathcal{E}_d = \bigcup_{T_i \text{ non-projective}} \text{ker} f. \]

The claim follows from Lemma 3.3.

**Remark 3.5.** – For \( \lambda \geq c + n - r \), we have that either \( \mathcal{E}_d \) is empty or that \( \text{codim} \mathcal{E}_d \geq 1 + n - r \).

Now we concentrate on the set \( \mathcal{Z}'_d = \{ X \in \mathcal{Z}_{Q, d}; \ 1[T, X] = 0 \} \).

**Lemma 3.6.** – For \( X \in \mathcal{Z}'_d \), there exists an epimorphism \( X \to S = \bigoplus S_j \), where the sum is taken over the \( n - r \) simple objects of \( T^\perp \).

**Proof.** – We choose a basis \( \{ f_1, \ldots, f_s \} \) of \( \text{Hom}_Q(T, X) \) and we put

\[ f = (f_1, \ldots, f_s) : T^s \to X. \]

Then any homomorphism from \( T \) to \( X \) factors through \( f \). Let \( X' = \text{im} f \) and \( \overline{X} = \text{coker} f \). The exact sequence

\[ 0 \to X' \to X \to \overline{X} \to 0 \]

induces the following long exact sequence

\[ 0 \to \text{Hom}_Q(T, X') \to \text{Hom}_Q(T, X) \to \text{Hom}_Q(T, \overline{X}) \to \text{Ext}^1_Q(T, X') \to \text{Ext}^1_Q(T, X) \to \text{Ext}^1_Q(T, \overline{X}) \to 0. \]

Since there is an epimorphism \( T' \to X' \) and since \( 1[T, T'] = 0 \), we have \( 1[T, X'] = 0 \). Moreover, \( g \) is bijective by the universality of \( f \) and, together with our assumption \( 1[T, X] = 0 \), this implies that \( \overline{X} \in T^\perp \).

Recall that \( [X, Y] \neq 0 \) for all non-zero \( Y \in T^\perp \) as \( X \) lies in \( \mathcal{Z}_{Q, d} \). In particular, \( [X, S_j] \neq 0 \) for \( j = 1, \ldots, n - r \). Mapping the sequence (3.1) to \( S_j \) and using that \( [X', S_j] = [T', S_j] = 0 \), we find that \( [\overline{X}, S_j] \neq 0 \) for all \( j \). But any non-zero morphism \( \overline{X} \to S_j \) is surjective, because \( \overline{X} \in T^\perp \) and \( S_j \in T^\perp \) is simple. We obtain the required epimorphism by composing the projection \( X \to \overline{X} \) with a surjective map \( \overline{X} \to S = \bigoplus_{j=1}^{n-r} S_j \). \( \square \)

Since the set \( \mathcal{Z}_{Q, d} \) is given by \( (n - r) \) equations, each irreducible component of \( \mathcal{Z}_{Q, d} \) has codimension at most \( (n - r) \). Thus Theorem 1.1 follows from Remark 3.5 and the following fact.

**Proposition 3.7.** – If \( \mathcal{Z}'_d \) is not empty, then it is irreducible and \( \text{codim} \mathcal{Z}'_d = n - r \).
Proof. – If $Z'_d$ is non-empty, Lemma 3.6 tells us that $d \geq \dim S$ and that $Z'_d$ lies in

$$A_S = \{ X \in \text{rep}(Q, d); \exists \text{ epimorphism } X \rightarrow S \}.$$ 

By Lemma 3.1, $A_S$ is irreducible, and

$$\text{codim } A_S \geq n - r - \langle d, d'' \rangle = n - r - [T, S] + [1[T, S] = n - r.$$ 

As $[T, X] = 0$ is an open condition, $Z'_d$ is open in $Z_{Q,d}$, and therefore $\text{codim } Z'_d \leq n - r$. Thus $Z'_d$ is open and dense in $A_S$ and consequently irreducible. □

Acknowledgements

The second author gratefully acknowledges support from the Polish Scientific Grant KBN No. 5 PO3A 008 21 and Foundation for Polish Science. He also thanks the Swiss Science Foundation, which gave him the opportunity to spend a year at the University of Bern.

REFERENCES


(Manuscrit reçu le 11 juin 2002 ; accepté le 7 février 2003.)