SP(2N)-COVERS FOR SELF-CONTRAGredient
Super cuspidal Representations of GL(N)

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ABSTRACT. – Let \( F \) be a non-archimedean local field of odd residual characteristic. Let \((J, \tau)\) be a maximal simple type in \( GL_N(F) \) for the inertial class \([GL_N(F), \pi]_{GL_N(F)}\) of a self-contragredient supercuspidal irreducible representation \( \pi \) of \( GL_N(F) \). Identify \( GL_N(F) \) to the standard Siegel Levi subgroup in \( Sp_{2N}(F) \). We construct, in \( Sp_{2N}(F) \), a type for the inertial class \([GL_N(F), \pi]_{Sp_{2N}(F)}\), as a \( Sp_{2N}(F) \)-cover of \((J, \tau)\), strongly related to the \( GL_{2N}(F) \)-cover of \((J \times J, \tau \circ \tau)\) in \( GL_{2N}(F) \) constructed by Bushnell and Kutzko and which induces to a simple type in \( GL_{2N}(F) \). In the process, we show that if \( \tau \) has positive level, then the maximal simple type \((J, \tau)\) may be attached to a simple stratum \([\mathfrak{A}, n, 0, \beta]\) where the field \( F[\beta] \) is a quadratic extension of \( F[\beta^2] \), and to a simple character \( \theta \) in \( C(\mathfrak{A}, 0, \beta) \) Galois conjugate of its inverse.

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RéSUMÉ. – Soit \( F \) un corps local non archimédien de caractère résiduelle impaire. Soit \((J, \tau)\) un type maximal dans \( GL_N(F) \) pour la classe d’inertie \([GL_N(F), \pi]_{GL_N(F)}\) d’une représentation irréductible supercuspidale autoduale \( \pi \) de \( GL_N(F) \). Identifions \( GL_N(F) \) au sous-groupe de Levi de Siegel standard de \( Sp_{2N}(F) \). Nous construisons, dans \( Sp_{2N}(F) \), un type pour la classe d’inertie \([GL_N(F), \pi]_{Sp_{2N}(F)}\), sous forme d’une paire couvrante de \((J, \tau)\) fortement reliée à la paire couvrante de \((J \times J, \tau \circ \tau)\) dans \( GL_{2N}(F) \) construite par Bushnell et Kutzko et qui s’induit en un type simple de \( GL_{2N}(F) \). En cours de route nous montrons que si \( \tau \) est de niveau strictement positif, le type simple maximal \((J, \tau)\) peut être attaché à une strate simple \([\mathfrak{A}, n, 0, \beta]\) telle que le corps \( F[\beta] \) soit extension quadratique de \( F[\beta^2] \) et à un caractère simple \( \theta \) de \( C(\mathfrak{A}, 0, \beta) \) égal au conjugué de son inverse par l’élément de Galois correspondant.

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Let \( G \) be the group of \( F \)-points of a connected reductive algebraic group defined over \( F \), a local non-archimedean field. The goal of the theory of types is the description of direct summands of the category \( \mathcal{R}(G) \) of smooth complex representations of \( G \) as categories of modules over Hecke algebras.

More precisely, the Bernstein decomposition of this category states that it is the direct sum, over the set of inertial classes in \( G \), of full subcategories \( \mathcal{R}^{[M, \pi]}(G) \) attached to each inertial class. Recall that an inertial class in \( G \) is the equivalence class \([M, \pi]_G\) of a pair \((M, \pi)\) made up of a \( F \)-Levi subgroup \( M \) of \( G \) and an irreducible supercuspidal representation \( \pi \) of \( M \); the equivalence relation includes \( G \)-conjugacy and twisting of \( \pi \) by an unramified character of \( M \). The subcategory \( \mathcal{R}^{[M, \pi]}(G) \) consists of representations each of whose irreducible subquotients is a subquotient of a representation parabolically induced to \( G \) from an unramified twist of \( \pi \).

Finding a type \((J, \lambda)\) for this subcategory means finding a compact open subgroup \( J \) of \( G \) and a smooth irreducible representation \( \lambda \) of \( J \) such that the subcategory \( \mathcal{R}^{[M, \pi]}(G) \) consists of representations generated by their isotypic component of type \( \lambda \) under \( J \). If \((J, \lambda)\) is a type for
particular, obtaining \( [M, \pi]_G \), i.e. for \( \mathcal{R}^{[M, \pi]}(G) \), this subcategory is then equivalent to the category of non-degenerate modules over the Hecke algebra \( \mathcal{H}(G, J, \lambda) \) (for all this see [12]).

The problem of finding types in \( G \) naturally breaks into two pieces which are very different in nature. One is finding types for the inertial classes of supercuspidal representations of \( G \). The other is finding types for inertial classes \([M, \pi]_G\) where \( M \) is a proper Levi subgroup of \( G \). C.J. Bushnell and P.C. Kutzko in [12] have developed a method to address this second problem, based on the definition of covers.

We say that \((J, \lambda)\) is a \( G \)-cover of \((J_M, \lambda_M)\), an analogous pair in \( M \), if there is an \( F \)-parabolic subgroup \( P \) of \( G \) with unipotent radical \( U \) and Levi decomposition \( P = MU \) such that:

(i) \((J, \lambda)\) is a decomposed pair with respect to \((M, P)\), i.e.
- \( J = (J \cap U^-)(J \cap M)(J \cap U) \), where \( U^- \) is the unipotent radical of the parabolic subgroup \( P^- \) opposite of \( P \) relative to \( M \), and
- \( \lambda \) is trivial on \( J \cap U^- \) and \( J \cap U \);

(ii) \( J \cap M = J_M \) and \( \lambda|_{J_M} \simeq \lambda_M \);

(iii) for any smooth irreducible representation \((\sigma, V)\) of \( G \), the restriction to \( V^\lambda \) of the Jacquet functor \( \tau_U \) is injective.

Note that the definition in [12] requires those properties to hold for any such parabolic subgroup \( P \); nonetheless it follows from [8] that one may restrict the definition to just one parabolic subgroup.

C.J. Bushnell and P.C. Kutzko have shown that:

if \((J_M, \lambda_M)\) is a type for \([M, \pi]_M\) in \( M \) and if \((J, \lambda)\) is a \( G \)-cover of \((J_M, \lambda_M)\), then \((J, \lambda)\) is a type for \([M, \pi]_G\) in \( G \).

Let now \( \overline{G} = \text{Sp}_{2N}(F) \) where \( F \) has odd residual characteristic, let \( \overline{\mathcal{P}} \) be the Siegel parabolic subgroup, and let \( \overline{M} \) be the Siegel Levi subgroup, which we identify with \( \text{GL}_N(F) \) (see Section 1). Let \( \pi \) be an irreducible supercuspidal representation of \( \text{GL}_N(F) \) and \((J_{\text{ST}, \lambda_{\text{ST}}})\) be a Bushnell–Kutzko type for \([\text{GL}_N(F), \pi]_{\text{GL}_N(F)}\) in \( \overline{M} \). Observe that the non-trivial element \( s \) in \( N_{\overline{M}}(\overline{M})/\overline{M} \) stabilizes the inertial class \([\overline{M}, \pi]_{\overline{M}}\) if and only if \( \pi \) and its contragredient representation are equivalent up to twisting by an unramified character of \( \text{GL}_N(F) \) – yet, since any unramified character of \( \text{GL}_N(F) \) is a square, \( \pi \) and its contragredient representation are in the same inertial class if and only if this class actually contains a self-contragredient representation.

If this is not the case, it should follow from [12], Theorem 12.1, that the Hecke algebra \( \mathcal{H}(\overline{G}, J, \lambda) \) of a \( \overline{G} \)-cover \((J, \lambda)\) of \((J_{\text{ST}, \lambda_{\text{ST}}})\) is commutative, isomorphic to \( \mathcal{H}(\overline{M}, J_{\text{ST}}, \lambda_{\text{ST}}) \), and the corresponding subcategories are equivalent. In any case, a recent result of A. Roche [19, Theorem 3.1] states, in our present setting, that parabolic induction from \( \mathcal{R}^{[\overline{M}, \pi]}(\overline{M}) \) to \( \mathcal{R}^{[\overline{M}, \pi]}(\overline{G}) \) is an equivalence of categories if and only if \( s \) does not stabilize \([\overline{M}, \pi]_{\overline{G}}\).

Hence, although the question of existence of \( \overline{G} \)-covers is interesting in itself, the most interesting case is the case when \( \pi \) is self-contragredient. Indeed, given a \( \overline{G} \)-cover \((J, \lambda)\) of \((J_{\text{ST}, \lambda_{\text{ST}}})\), one expects the description of \( \mathcal{H}(\overline{G}, J, \lambda) \) to give insight into reducibility problems for parabolically induced representations in \( \mathcal{R}^{[\overline{M}, \pi]}(\overline{G}) \) (see e.g. [3] for details in the case of \( \text{Sp}_4(F) \); although, for this group, the results concerning reducibility were already known). In particular, obtaining \( \overline{G} \)-covers and their Hecke algebras for all such representations \( \pi \) should lead to an exact knowledge of the real numbers \( \alpha \) such that the parabolically induced representation to \( \overline{G} \) of the twisted representation \( \pi \otimes \det(\alpha) \) is reducible (those numbers are known to belong to \( \{0, \pm 1/2\} \) if \( N > 1 \), by the work of Shahidi [20,21]).
We construct in this paper $\overline{G}$-covers for Bushnell–Kutzko types attached to inertial classes $[\text{GL}_N(F), \pi]_{\text{GL}_N(F)}$ where $\pi$ is self-contragredient, which is the first step in the above program. We do not compute the corresponding Hecke algebras. The principle of the construction is to start with a well chosen $\text{GL}_{2N}(F)$-cover attached to the inertial class $[\text{GL}_N(F) \times \text{GL}_N(F), \pi \otimes \pi]_{\text{GL}_{2N}(F)}$ and then restrict it to $\text{Sp}_{2N}(F)$. In the process we need some strong properties of simple types attached to self-contragredient supercuspidal representations. We prove the following in Section 2 (Corollary 2.3):

**Theorem 1.** Let $(\Gamma, \gamma)$ be a maximal simple type (in the sense of [11]) in $\text{GL}_N(F)$ such that the corresponding inertial class contains a self-contragredient representation. Then either $(\Gamma, \gamma)$ has level zero, or the simple character $\theta_0$ attached to $\gamma$ can be attached to a simple stratum $[\mathfrak{A}_0, \mathfrak{r}_0, 0, \beta]$ in $\text{M}_N(F)$ with the following properties:

1. The field $F[\beta]$ is a quadratic extension of $F[\beta^2]$ (in particular $N$ is even).
2. Let $\nu$ be an element in $\mathfrak{A}_0^\times$ realizing the Galois conjugation of $F[\beta]$ over $F[\beta^2]$. The character $\theta_0$ satisfies $\theta_0(\nu x \nu^{-1}) = x \theta_0(x^{-1}) (x \in H^1(\beta, \mathfrak{A}_0)).$

This property of self-contragredient supercuspidal representations was known in the tame case: such a representation is then attached to an admissible character of a maximal field extension contained in $\text{GL}_N(F)$ and Adler [1] proved that this character is trivial on the group of norms relative to a quadratic subextension (or $N = 1$ and the character is quadratic). In loc. cit. Adler also gives a full description of level zero self-contragredient supercuspidal representations (which exist only if $N$ is even or $N = 1$).

Let $g \mapsto \overline{\tau} g$ be the transposition relative to the anti-diagonal. Theorem 1 essentially amounts to saying that, for a suitable order $\mathfrak{A}$ in $\text{M}_N(F)$ related to the order $\mathfrak{A}_0$ above, the stratum $\mathcal{A} = [\mathfrak{A}, 2n_0, 0, \{0 \times -1, \gamma \}]$ in $\text{M}_2(N)(F)$ is simple. Let $G = \text{GL}_{2N}(F)$ and $P$ be the maximal parabolic subgroup in $G$ of upper block-diagonal matrices with Levi subgroup $\mathcal{M}$ isomorphic to $\text{GL}_N(F) \times \text{GL}_N(F)$. The process in [11], §7.2, provides us (Corollary 2.2) with a $G$-cover $(J_P, \lambda_P)$ of $(\Gamma \times \overline{\tau} \Gamma, \gamma \otimes \overline{\gamma})$, with $\gamma(x) = \overline{\gamma}(\tau x^{-1})$, attached to the stratum $\mathcal{A}$. It will lead us (Theorem 3.1) to the cover we are looking for:

**Theorem 2.** Let $(\Gamma, \gamma)$ be as in Theorem 1 and $(J_P, \lambda_P)$ be as above. The unique representation $\omega$ of $\Omega = J_P \cap \mathcal{M}$ such that $(\Omega, \omega)$ is a decomposed pair with respect to $(\mathcal{M}, F)$ with $\Omega \cap \overline{\mathcal{M}} = \Gamma$ and $\omega | \Omega \cap \overline{\mathcal{M}} = \gamma$ is a $\overline{G}$-cover of $(\Gamma, \gamma)$.

In the case when $(\Gamma, \gamma)$ has level zero, the cover given by Theorem 2 has previously been obtained by L. Morris in [18]. Also recall that J.-L. Kim [14] has constructed a set of types in classical groups, under the assumption that the characteristic of $F$ is 0 and the residual characteristic is “big enough”. The types in her work that correspond to our present setting need not be the same as those above, in particular they may not be $\overline{G}$-covers (see [2]).

The main goal of this paper is Theorem 2, while Theorem 1 appears as a necessary tool. In Section 1, we establish notation and explain the basic mechanism allowing one to build decomposed pairs in $\text{Sp}_{2N}(F)$ from the restriction of decomposed pairs in $\text{GL}_{2N}(F)$. In Section 2, we detail the structure of the maximal simple type $(\Gamma, \gamma)$ and of suitable $\text{GL}_{2N}(F)$-covers of $(\Gamma \times \overline{\tau} \Gamma, \gamma \otimes \overline{\gamma})$. This leads us to a proof of Theorem 1, as a corollary of Theorem 2.3. In Section 3, we first build a periodic infinite sequence $(\Omega_i, \omega_i)$ of decomposed pairs in $\mathcal{G}$, with $\Omega_i \cap \overline{\mathcal{M}} = \Gamma$ and $\omega_i | \Omega_i \cap \overline{\mathcal{M}} = \gamma$, then we show that certain sufficient criteria for this sequence to be a sequence of $G$-covers are satisfied. Section 4 is devoted to the proof of an intertwining property (Proposition 4.1) that has been assumed in Section 3.

From Section 2.2 on, we assume that the order $\mathfrak{A}_0$ is standard. Any maximal simple type $(\Gamma, \gamma)$ in $\text{GL}_N(F)$ is conjugate to a maximal simple type satisfying this property, hence Theorems 1 and 2 hold without this restriction (see the remarks after Corollary 2.3 and Theorem 3.1).
1. Framework and basic tool

1.1. Notations

Let $F$ be a non-archimedean local field of residual characteristic $p$ different from 2, let $\mathfrak{o}_F$ or $\mathfrak{p}$ be its ring of integers, $p_F$ or $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}_F$, $\mathfrak{o}_F$ or $\mathfrak{p}$ a uniformizing element and $k_F = \mathfrak{o}_F/p_F$ the residue class field, of cardinality $q_F$. We will be working with the group $G = \text{GL}_{2N}(F)$ and its subgroup $\Gamma = \text{Sp}_{2N}(F)$ viewed as the symplectic group of the $F$-vector space $V = F^{2N}$ equipped with the symplectic form $\langle \cdot , \cdot \rangle$ with matrix $\begin{pmatrix} 0 & -w_N \\ w_N & 0 \end{pmatrix}$ in the canonical basis $\{ e_1, \ldots, e_{2N} \}$, where

$$w_N = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$ 

Most matrices written below will be $2 \times 2$ block matrices with $N \times N$ blocks. Hence:

$$\text{Sp}_{2N}(F) = \left\{ g \in \text{GL}_{2N}(F); \begin{pmatrix} 0 & -w_N \\ w_N & 0 \end{pmatrix} \tau g^{-1} \begin{pmatrix} 0 & w_N \\ -w_N & 0 \end{pmatrix} = g \right\}.$$ 

We will let $X \mapsto TX$ denote the corresponding involution on $M_{2N}(F)$:

$$TX = \begin{pmatrix} 0 & -w_N \\ w_N & 0 \end{pmatrix} \tau X \begin{pmatrix} 0 & w_N \\ -w_N & 0 \end{pmatrix}; \quad \tau \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \tau D & -\tau B \\ -\tau C & \tau A \end{pmatrix},$$

where $g \mapsto \tau g$, $g \in \text{GL}_i(F)$, is the transposition relative to the antidiagonal; in other words: $\tau g = \tau^t g = w_i g w_i$.

For any subgroup $H$ of $G$, we put $\overline{H} = H \cap \Gamma$. Let $P$ be the stabilizer of the subspace $\langle e_1, \ldots, e_N \rangle$ in $F^{2N}$, a parabolic subgroup of $G$. Let $U$ be its unipotent radical and let $M$ be the Levi factor of $P$ consisting of matrices stabilizing $\langle e_{N+1}, \ldots, e_{2N} \rangle$. We let $P^-$ be the parabolic subgroup of $G$ opposite of $P$ relative to $M$ and we let $U^-$ be its unipotent radical. We have

$$M = \left\{ \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}; g_1, g_2 \in \text{GL}_N(F) \right\}, \quad U = \left\{ \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}; X \in M_N(F) \right\},$$

$$\overline{M} = \left\{ \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}; g \in \text{GL}_N(F) \right\}, \quad \overline{U} = \left\{ \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}; X \in M_N(F), X = \tau X \right\}.$$

We will accordingly identify $M$ with $\text{GL}_N(F) \times \text{GL}_N(F)$ and $\overline{M}$ with $\text{GL}_N(F)$, the latter through the isomorphism $i$ from $\text{GL}_N(F)$ to $\overline{M}$ defined by

$$i(g) = \begin{pmatrix} g & 0 \\ 0 & \tau g^{-1} \end{pmatrix}, \quad g \in \text{GL}_N(F).$$

If $\mu$ is a representation of a subgroup $H$ of $\text{GL}_N(F)$, $i(\mu)$ will be the representation of $i(H)$ defined by $i(\mu)(i(g)) = \mu(g)$ ($g \in H$).

Let $H$ be a compact open subgroup of $G$ and $\rho$ a smooth irreducible representation of $H$. The $G$-intertwining of $\rho$ is:

$$I_G(\rho) = I_G(\rho, H) = \{ g \in G \mid \text{Hom}_{H \cap H'} (\rho, \rho^g) \neq \{0\} \}.$$
For any \( g \) in \( G \) we define the intertwining space of \( \rho \) at \( g \) to be
\[
I_g(\rho) = I_g(\rho, H) = \text{Hom}_{H \cap H^s}(\rho, \rho^g).
\]

1.2. Some decomposed pairs in \( \text{Sp}_{2N}(F) \)

Let \( \pi \) be a smooth irreducible self-contragredient supercuspidal representation of \( \text{GL}_N(F) \), hence viewed as a representation of \( \overline{M} \); likewise \( \pi \otimes \pi \) is viewed as a representation of \( M \). We want to find types in \( G \) and \( \overline{G} \) for the inertial classes attached to these representations, and we want those types to be a \( G \)-cover and a \( \overline{G} \)-cover respectively, of types attached to \( \pi \otimes \pi \) in \( M \) and to \( i(\pi) \) in \( \overline{M} \). The situation in \( G \) has been settled by Bushnell and Kutzko in [11], as we will recall in Section 2. Indeed we will use the types built in loc. cit. to construct the \( \overline{G} \)-covers we are looking for: the process will involve a suitable conjugation followed by a restriction to \( \text{Sp}_{2N}(F) \).

The basic mechanism is the following:

**Proposition.** – Let \( \Gamma \) be a compact open subgroup of \( \text{GL}_N(F) \) and let \( \gamma \) be a smooth finite-dimensional complex representation of \( \Gamma \). Let \( \gamma^* \) be the representation of \( \check{\Gamma} \) defined by:
\[
\gamma^*(g) = \gamma(\check{g}^{-1}), \quad g \in \check{\Gamma}.
\]

If \((J, \lambda)\) is a decomposed pair in \( G \) relative to \((M, P)\) such that \( J \cap M = \Gamma \times \check{\Gamma} \) and \( \lambda_{|J \cap M} \simeq \gamma \otimes \gamma^* \), then \( J \cap M = i(\Gamma) \) and there exists a unique representation \( \hat{\lambda} \) of \( \overline{J} \) such that \((\overline{J}, \hat{\lambda})\) is a decomposed pair in \( \overline{G} \) relative to \((\overline{M}, \overline{P})\) with
\[
\hat{\lambda}|_{i(\Gamma)} = i(\gamma).
\]

The representation \( \overline{\lambda} = \lambda|_{\overline{J}} \) of \( \overline{J} \) is isomorphic to \( \hat{\lambda} \otimes \check{\lambda} \).

**Proof.** – We recall the following useful fact: let \( x \in U^- \), \( m \in M \) and \( y \in U \) be such that their product \( xmy \) belongs to \( \overline{G} \); then \( x \), \( m \) and \( y \) already belong to \( \overline{G} \). Indeed the involution defining \( \text{Sp}_{2N}(F) \) stabilizes \( U^- \), \( M \) and \( U \).

Hence taking intersections with \( \overline{G} \) provides a decomposed pair \((\overline{J}, \overline{\lambda})\) in \( \overline{G} \) relative to \((\overline{M}, \overline{P})\). We have \( \overline{J} \cap \overline{M} = \overline{J} \cap M = \overline{\Gamma} \times \check{\Gamma} = i(\Gamma) \) and
\[
X \left( \begin{pmatrix} g & 0 \\ 0 & \tau g^{-1} \end{pmatrix} \right) \simeq \gamma(g) \otimes \gamma^*(\check{g}^{-1}) = \gamma(g) \otimes \gamma(g), \quad g \in \overline{\Gamma}.
\]

In particular: \((\overline{J} \cap U)(\overline{J} \cap U^-) \subset (\overline{J} \cap U^-)T(\overline{J} \cap U^-)\), with
\[
T = \left\{ \begin{pmatrix} g & 0 \\ 0 & \tau g^{-1} \end{pmatrix} : g \in \Gamma, \gamma(g) \otimes \gamma(g) = I \right\}.
\]

To go from there to the decomposed pair we are looking for, it is enough [4, Lemme 1] to prove a similar inclusion with \( T \) replaced by
\[
T' = \left\{ \begin{pmatrix} g & 0 \\ 0 & \tau g^{-1} \end{pmatrix} : g \in \Gamma, \gamma(g) = I \right\}.
\]

Indeed, the representation \( \hat{\lambda} \) will then be uniquely defined by the condition \( \hat{\lambda}|_{i(\Gamma)} = i(\gamma) \), plus the fact that it is trivial on \( \overline{J} \cap U \) and \( \overline{J} \cap U^- \).
Now the subgroup of $\overline{J}$ generated by $\overline{J} \cap U$ and $\overline{J} \cap U^-$ is a pro-$p$-group [6] and so is its intersection with $\overline{M}$, hence we can replace $T$ in the above inclusion by a suitable pro-$p$-subgroup of $T$. All we have to show is:

**Lemma.** – Let $γ$ be a finite-dimensional smooth complex representation of a pro-$p$-group $H$, with $p$ odd. If the representation $γ ⊗ γ$ of $H$ is trivial, so is $γ$.

Indeed $γ$ factors through a finite quotient of $H$, so it is unitarisable. In particular each operator $γ(h)$, $h ∈ H$, is diagonalisable, and the triviality of $γ ⊗ γ$ implies that any product of two eigenvalues of $γ(h)$ is equal to 1. Hence $γ(h)$ is a scalar operator, namely $±I$. Now $−I$ is impossible for $p$ odd. □

If the pair $(Γ, γ)$ in the proposition is a maximal simple type in $GL_N(F)$ and the pair $(J, λ)$ is a $G$-cover of $(Γ \times Γ, γ ⊕ γ^*)$, one would like to know whether or not, under relevant conditions on $(J, λ)$, the associated pair $(\overline{J}, \overline{λ})$ is a $G$-cover of $(Γ, γ)$. We do address this question here in the special case of a pair $(Γ, γ)$ attached to the inertial class of a self-contragredient representation; the object of Section 2 is to use Bushnell and Kutzko’s simple types to produce in this context a $G$-cover $(J, λ)$ with suitable properties for that purpose.

2. “Self-contragredient” $GL_{2N}(F)$-covers

2.1. Bushnell and Kutzko’s $GL_{2N}(F)$-covers

All references in this paragraph are to [11], any undefined notion or notation comes from [11]. Let $π$ be an irreducible supercuspidal representation of $GL_N(F)$ and $(Γ, γ)$ a maximal simple type in $GL_N(F)$ attached to the inertial class of $π$. From Definition (5.5.10) – where we do treat (b) as a special case of (a) – and Theorems (6.2.1), (6.2.2), the pair $(Γ, γ) = (J(β, \mathfrak{A}_0), λ(β, \mathfrak{A}_0))$ comes equipped with the following data:

(i) A principal $σ_F$-order $\mathfrak{A}_0$ and a simple stratum $[\mathfrak{A}_0, n_0, 0, β]$ in $M_N(F)$; in particular $E = F[β]$ is a field extension of $F$.

We let $\mathfrak{D}_0$ be the radical of $\mathfrak{A}_0$ and $B_0$ be the commutant of $E$ in $M_N(F)$. Then $\mathfrak{A}_0 = \mathfrak{D}_0 ∩ B_0$ is a maximal $σ_F$-order with radical $\mathfrak{D}_0 = \mathfrak{D}_0 ∩ B_0$.

(ii) A simple character $θ_0 ∈ C(\mathfrak{D}_0, 0, β)$ and a $β$-extension $σ_0$ to $J(β, \mathfrak{A}_0)$ of the unique irreducible representation $σ_0$ of $J^1(β, \mathfrak{A}_0)$ which contains $θ_0$.

(iii) An irreducible cuspidal representation $σ_0$ of $GL(f, k_E)$ inflated to $Γ$ through:

$$J(β, \mathfrak{A}_0)/J^1(β, \mathfrak{A}_0) \cong U(\mathfrak{D}_0)/U^1(\mathfrak{D}_0) \cong GL(f, k_E), \quad f = N/[E : F].$$

We now write $M_{2N}(F)$ as $2 × 2$ block matrices with entries in $M_N(F)$. This amounts to a decomposition of the underlying vector space $V = F^{2N}$, written as column matrices, as a direct sum $V = V^{(1)} ⊕ V^{(2)}$ with $V^{(1)}$ (resp. $V^{(2)}$) the subspace of column matrices having their first (resp. last) $N$ entries equal to 0.

Let $(Λ_{0,j})_{j ∈ Z}$ be the lattice chain in $F^N$ associated to the order $\mathfrak{A}_0$. It determines lattice chains $(Λ_{0,j})_{j ∈ Z}$ in $V^{(j)}$, $j = 1, 2$, under the natural identification of $V^{(j)}$ with $F^{2N}$. Let $(Λ_i)_{i ∈ Z}$ be the lattice chain in $V$ defined by:

$$Λ_{2i} = Λ_{0,i}^{(1)} ⊕ Λ_{0,i}^{(2)}, \quad Λ_{2i+1} = Λ_{0,i+1}^{(1)} ⊕ Λ_{0,i}^{(2)} \quad (i ∈ Z).$$

The corresponding principal order in $M_{2N}(F)$ is $\mathfrak{A} = \langle Λ_{0,0}, Λ_{0,i} \rangle$.
We identify $E$ with its block-diagonal image in $M_{2N}(F)$, hence we also write $\beta$ for $\begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}$. We write $B$ for the commutant of $E$ in $M_{2N}(F)$ and define the $\sigma_E$-order $\mathfrak{B} = \mathfrak{A} \cap B$, with radical $\mathfrak{Q} = \mathfrak{B} \cap B$ (where $\mathfrak{B}$ is the radical of $\mathfrak{A}$).

Since the period of $(\Lambda, i)_{i \in \mathbb{Z}}$ is twice the period of $(\Lambda, 0)_{i \in \mathbb{Z}}$, we get (see 1.2.11, 1.4.13, 1.2.4):

**FACT.** – $[\mathfrak{A}, 2\mathfrak{a}_0, 0, \beta]$ is a simple stratum in $M_{2N}(F)$ and all assumptions in (7.1.11), (7.2.1) are satisfied, with $t = e(\mathfrak{B} | \sigma_E) = 2$.

The following proposition can be regarded as obvious: it is a paraphrase of [11], §7. We state it to fix notations and make references easy, and give a sketch of proof as a matter of conscientiousness. The groups $P$, $U$, $M$ are defined in (7.1.13) or equivalently in Section 1.1 above.

**PROPOSITION.** – *There exists a unique representation $\lambda$ of $J = J(\beta, \mathfrak{A})$ which is a simple type with the following property.*

Let $\lambda_P$ denote the natural representation of $J_P = (J \cap P)H^1(\beta, \mathfrak{A})$ on the space of $J^1(\beta, \mathfrak{A}) \cap U$-fixed vectors in $\lambda$. The pair $(J_P, \lambda_P)$ is a decomposed pair in $G$ relative to $(M, P)$ with:

\[ J_P \cap M = J \cap M = \Gamma \times \Gamma \quad \text{and} \quad (\lambda_P)_{J_P \cap M} = \gamma \otimes \gamma. \]

**Proof.** – Indeed this is Theorem (7.2.17) in [11], except that we want an actual equality between representations instead of an equivalence.

From (7.1.16): $J \cap M = J(\beta, \mathfrak{A}^{(1)}) \times J(\beta, \mathfrak{A}^{(2)})$; but we have arranged $\mathfrak{A}$ so that $\mathfrak{A}^{(1)} = \mathfrak{A}^{(2)} = \mathfrak{A}_0$, hence $J(\beta, \mathfrak{A}^{(i)}) = \Gamma$.

If $\beta$ belongs to $\sigma_F$ we just note that $J = J_P$ and $J \cap M = U(\mathfrak{A}_0) \times U(\mathfrak{A}_0)$; we take the representation $\sigma_0 \otimes \sigma_0$ there.

We now assume that $\beta \notin \sigma_F$. From (7.1.19), the restriction to $H^1 = H^1(\beta, \mathfrak{A})$ of any simple type $\lambda$ is a multiple of a simple character $\theta \in C(\mathfrak{A}, 0, \beta)$ and the pair $(H^1, \theta)$ is a decomposed pair in $G$ relative to $(M, P)$, satisfying: $H^1 \cap M = H^1(\beta, \mathfrak{A}^{(1)}) \times H^1(\beta, \mathfrak{A}^{(2)})$ and $\theta_{H^1 \cap M} = \theta^{(1)} \otimes \theta^{(2)}$, where $\theta^{(i)} \in C(\mathfrak{A}^{(i)}, 0, \beta)$ is the image of $\theta$ under the bijection $C(\mathfrak{A}, 0, \beta) \cong C(\mathfrak{A}, 0, \beta)$ given by Theorem (3.6.14). Since the family of bijections given by (3.6.14) is unique and $\mathfrak{A}^{(1)} = \mathfrak{A}^{(2)} = \mathfrak{A}_0$, we must have $\theta^{(1)} = \theta^{(2)}$, and from (3.6.14) there is a unique $\theta \in C(\mathfrak{A}, 0, \beta)$ such that $\theta_{H^1 \cap M} = \theta_0 \otimes \theta_0$.

With this choice of $\theta$, the next step towards $\lambda$ is the choice of a $\beta$-extension $\kappa$. From (7.2.5), (7.2.15), (7.2.16), it has the form $\kappa = \text{Ind}_{J_P}^{J_P} \kappa_P$ where again the pair $(J_P, \kappa_P)$ is decomposed and $(\kappa_P)_{J_P \cap M} = \kappa^{(1)} \otimes \kappa^{(2)}$, both being $\beta$-extensions of $\eta_0$. Since $(\kappa_P)_{J_P \cap M}$ is normalized by \( \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix} \) (7.2.15) and $\sigma_E$ intertwines $\kappa^{(i)}$, we have $\kappa^{(1)} \simeq \kappa^{(2)} \simeq \kappa_0 \circ \det_B$, in the notation of (5.2.2). Since $\kappa \otimes \chi_1^{-1} \circ \det_B$ is another $\beta$-extension we may pick $\kappa$ in the first place so that $(\kappa_P)_{J_P \cap M} = \kappa_0 \otimes \kappa_0$.

All we have to do now is to tensor $\kappa$ with $\sigma_0 \otimes \sigma_0$ as before (7.2.17). \( \Box \)

**COROLLARY.** – *The pair $(J_P, \lambda_P)$ is a $G$-cover of the pair $(\Gamma \times \Gamma, \gamma \otimes \gamma)$ in $M$. The pair $(J_P, \lambda_P)$ is a type in $G$ attached to the inertial class $[M, \pi \otimes \pi_G]$.*

The first assertion follows from (7.3.2) and the results in [12, §7], the second from Theorem 8.3 in [12]. Note that by symmetry – see (7.1.13) – this also holds for the pair $(J_{P-}, \lambda_{P-})$, with $J_{P-} = (J \cap P^-)H^1$.

Before turning to the case of self-contragredient supercuspidals in the next paragraphs, let us fix some more notations and write down some properties that will be used later on; they all derive from [11, §3.1].
Let
\[ U = \begin{pmatrix} 0 & M_N(F) \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad U^- = \begin{pmatrix} 0 & 0 \\ M_N(F) & 0 \end{pmatrix}. \]

We will identify \( U^- \) with \( M_N(F) \) through \( \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \mapsto X \) and \( U \) with \( M_N(F) \) through
\[ \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \mapsto X. \]

We also use the isomorphisms \( i^+ \) and \( i^- \) from \( M_N(F) \) to \( U = 1 + U \) and \( U^- = 1 + U^- \) respectively, defined by
\[ i^+(X) = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}, \quad i^-(X) = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}. \]

Write \( J^1 = J^1(\beta, \mathfrak{A}), J^1_\beta = J^1(\beta, \mathfrak{A}) \cap \mathbb{U}^- \), \( J^1_+ = J^1 \cap \mathbb{U} \) (both viewed as lattices in \( M_N(F) \)), and similarly for \( H^1, \mathfrak{H}^1 \). From [11, §3.1, 7.1] we have:
\begin{align*}
J^1 &= 1 + \mathfrak{J}^1 = i^- (J^1_\beta) (J^1 \cap M) i^+(J^1_+) , \\
H^1 &= 1 + \mathfrak{H}^1 = i^- (\mathfrak{H}^1_\beta) (H^1 \cap M) i^+(\mathfrak{H}^1_+).
\end{align*}

The lattices \( \mathfrak{J}^1 \) and \( J^1 \) are invariant under conjugation by \( \mathbb{R}(\mathfrak{A}) \), hence by \( \begin{pmatrix} 0 & 1 \nu \\ 1 & 0 \end{pmatrix} \). Hence
\[ J^1_\beta = \varpi_E J^1_\beta = \mathfrak{J}^1_\beta \varpi_E; \quad \mathfrak{H}^1_\beta = \varpi_E \mathfrak{H}^1_\beta = \mathfrak{H}^1_\beta \varpi_E. \]

Those lattices in \( M_N(F) \) also satisfy:
\[ \varpi_E J^1_\beta \subset \mathfrak{J}^1_\beta \subset \mathfrak{J}^1_\beta \varpi_E \subset \mathfrak{H}^1_\beta \subset \mathfrak{H}^1_\beta \varpi_E. \]

2.2. Intertwining properties in the self-contragredient case

Let \((\Gamma, \gamma) = (J(\beta, \mathfrak{A}_0), \lambda(\beta, \mathfrak{A}_0))\) be a maximal simple type in \( \text{GL}_N(F) \) as above, attached to the inertial class of \( \pi \). We want to use the decomposed pair \((J_P, \lambda_P)\) given by Proposition 2.1 to produce a decomposed pair in \( \mathcal{G} \) through the process described in Proposition 1.2. This is easy if \( \Gamma \) is equal to \( \gamma^r \Gamma \) and \( \gamma^r \) equivalent to \( \gamma \), which implies that \( \pi \) and its contragredient representation belong to the same inertial class. We actually want to show that the converse is true up to conjugacy.

PROPOSITION. – Let \((\Gamma, \gamma)\) be a maximal simple type in \( \text{GL}_N(F) \) such that the corresponding inertial class of irreducible supercuspidal representations of \( \text{GL}_N(F) \) contains a self-contragredient representation \( \pi \). We keep the notation in Section 2.1 and assume the order \( \mathfrak{A}_0 \) is standard.

(i) There exists \( \sigma \in \mathbb{U}(\mathfrak{A}_0) \) such that \( \Gamma \) is stable under \( \tilde{\sigma} : x \mapsto \sigma^T x^{-1} \sigma^{-1} \), and \( \gamma \) is equivalent to \( \gamma \circ \tilde{\sigma} \).

(ii) Such an element \( \sigma \) is unique up to left multiplication by \( \Gamma \). It satisfies:
(a) \( \sigma^T \sigma^{-1} \in \Gamma \) and \( \varpi_E^{-1} \sigma^T \varpi_E \varpi_E^{-1} \sigma^{-1} \in \Gamma \).
(b) The map \( \sigma \) stabilizes \( H^1(\beta, \mathfrak{A}_0) \) and \( J^1(\beta, \mathfrak{A}_0) \) and we have: \( \theta_0 = \theta_0 \circ \tilde{\sigma} \).
(c) The lattices \( J^1_\beta \), \( \mathfrak{J}^1_\beta \), \( \mathfrak{H}^1_\beta \), and \( \mathfrak{J}^1_\beta \) in \( M_N(F) \) defined in Section 2.1 are stable under \( \chi \mapsto \chi^T X^{-1} \).
Example. – Assume $N = 2$ and $E$ is a quadratic extension of $F$ generated by an element of matrix form $\left(\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right)$. Then we may take $\sigma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$; here conjugation by $\sigma$ realizes the conjugation of $E$ over $F$ and $\sigma$ plays an explicit part in the construction of $\overline{G}$-covers in [2, 3].

Proof. – We call here standard an order whose matrix form is given by [11], (2.5.1) or [7], (1.9). Note that we can always conjugate the lattices in $\mathcal{M}$ under $\tau$, the property we need here is that the determinant of $\pi_\sigma$ is conjugate under $\tau$, being standard and principal, is stable as in [2,3].

(i) By a theorem of Gelfand and Kazhdan [13, Theorem 2], the contragredient representation of $\pi_\sigma$ is equivalent to the representation $\pi^{\ast}$ defined by $\pi^{\ast}(x) = \pi(\tau x^{-1})$. Since $\mathcal{A}_0$ is stable under $\tau$, the automorphism $x \mapsto \tau x^{-1}$ transforms the pair $(\Gamma, \gamma) = (J(\beta, \mathcal{A}_0), \lambda(\beta, \mathcal{A}_0))$ into $(\Gamma^\ast, \gamma^\ast) = (J(-\tau \beta, \mathcal{A}_0), \lambda(-\tau \beta, \mathcal{A}_0))$, a maximal simple type underlaid by the simple stratum $[\mathcal{A}_0, n_0, \mathcal{A}_0]^{-\tau}$. Since $\pi^\ast$ is equivalent to $\pi$, those two types, $(\Gamma, \gamma)$ and $(\Gamma^\ast, \gamma^\ast)$, intertwine in $GL_N(F)$ and we may use [11, Theorem (5.7.1)] to derive that they are conjugate in $GL_N(F)$. Since the two types are associated to the same order $\mathcal{A}_0$, the proof of loc. cit. actually tells us more: indeed it says that $\sigma$, the contragredient representation of $\pi_\sigma$, conjugates the lattices in $\mathcal{M}$ under $\tau$, and that, eventually, that same element $\sigma$ conjugates $(\Gamma, \gamma)$ into $(\Gamma^\ast, \gamma^\ast)$ where $\gamma^\ast$ is equivalent to $\gamma^\ast$ (if $\gamma$ has level 0, it says that $\gamma$ and $\gamma^\ast$ are equivalent).

(ii) Let $\sigma_1$ be another such element. The automorphism $\tilde{\sigma}_1 \circ \sigma$ normalizes $(\Gamma, \gamma)$ and is a conjugation by $\sigma_1^{-1} \circ \sigma^{-1}$ that must therefore belong to $E^\times \Gamma \cap U(\mathcal{A}_0) = \Gamma$ ([11], (6.2.2)). Then the first part in (a) follows whence $\sigma_1 \in \Gamma \sigma = \Gamma \sigma$. For the last part in (a), we use the same argument: the element $\sigma_1 = \varpi_{E,0}^{-1}$ also satisfies the conditions in (i) – except that it does not belong to $U(\mathcal{A}_0)$ – because $\varpi_{E,0}$ normalizes $\Gamma$, and the determinant of $\sigma_1^{-1}$ is 1, whence the result.

Now, for (b) we only have to note that the element $\sigma$ produced in the proof of (i) satisfies the required properties.

The first step in the proof of (c) is the description of $\mathcal{M}^1(\beta, \mathcal{A})$ and $\mathcal{M}^1(\beta, \mathcal{A})$ in terms of the lattices in $\mathcal{M}_N(F)$ attached to $[\mathcal{A}_0, n_0, \mathcal{A}_0]$, as in [11], (7.1.12). Since the given lattices in $\mathcal{M}_N(F)$ are the direct sums of their intersections with the $\text{Hom}_E(V^{(i)}, V^{(j)})$ we use the corresponding block-matrix notation.

**Lemma.** – For any non-negative integer $k$, we have:

\[
\mathcal{M}^1(\beta, \mathcal{A}) = \begin{pmatrix}
\mathcal{M}^{k}[1](\beta, \mathcal{A}_0) \\
\varpi_{E,0}^{k}[2](\beta, \mathcal{A}_0)
\end{pmatrix} \quad \text{and} \quad \mathcal{M}^1(\beta, \mathcal{A}) = \begin{pmatrix}
\mathcal{M}^{k-1}[1](\beta, \mathcal{A}_0) \\
\varpi_{E}^{-1}[k]+1(\beta, \mathcal{A}_0)
\end{pmatrix}.
\]

Proof. – The diagonal blocks are already described in [11], Proposition (7.1.12), (iii). Since the lattices $\mathcal{M}^1(\beta, \mathcal{A}), \mathcal{M}^1(\beta, \mathcal{A})$ are invariant under conjugation by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, all we have to show is $\mathcal{M}^1(\beta, \mathcal{A}) \cap \mathbb{N} = \mathcal{M}^{k}[2](\beta, \mathcal{A}_0)$ and $\mathcal{M}^1(\beta, \mathcal{A}) \cap \mathbb{N} = \mathcal{M}^{k}[2]+1(\beta, \mathcal{A}_0)$. The proof of this goes exactly as in loc. cit.: the equality is first checked for $\beta$ a minimal element, then obtained by induction along $\beta$.

Since $\mathcal{M}^1(\beta, \mathcal{A}) = \varpi_{E,0}^{-1}[1]$ and $\mathcal{M}^1(\beta, \mathcal{A}) = \varpi_{E}[1]$ it is easy, using property (a), to check that $\mathcal{M}^1(\beta, \mathcal{A})$ and $\mathcal{M}^1(\beta, \mathcal{A})$ are stable under the map $\Sigma: X \mapsto \sigma^{-1} X \sigma^{-1}$ if $\mathcal{M}^1(\beta, \mathcal{A})$ and $\mathcal{M}^1(\beta, \mathcal{A})$. From the lemma we have $\mathcal{M}^1(\beta, \mathcal{A}) = \mathcal{M}^1(\beta, \mathcal{A})$ and $\mathcal{M}^1(\beta, \mathcal{A}) = \mathcal{M}^1(\beta, \mathcal{A})$.

We know from (b) that $\Sigma$ stabilizes $H^1(\beta, \mathcal{A}_0) = 1 + \mathcal{M}^1(\beta, \mathcal{A}_0)$ hence $\mathcal{M}^1(\beta, \mathcal{A}_0)$. 

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We have \( \Theta(\beta, \mathfrak{A}_0) = \mathcal{B}_0 + \mathcal{I}_1(\beta, \mathfrak{A}_0) \) ([11] (3.1.8)) and \( \Theta_1(\beta, \mathfrak{A}_0) \) is stable under \( \Sigma \) by (b), so we have to show that \( \Sigma(\mathfrak{S}_0) \) is contained in \( \Theta(\beta, \mathfrak{A}_0) \). We know that \( \Sigma(\mathfrak{S}_0) = \Sigma(\mathfrak{S}_0 \cap \mathfrak{A}_0) \) is contained in \( \Theta_1(\beta, \mathfrak{A}_0) \). Since \( J(\beta, \mathfrak{A}_0) = \Gamma \) is invariant under \( \sigma \) we also know that \( \Sigma(\mathfrak{S}_0) \) is contained in \( J(\beta, \mathfrak{A}_0) \). Our claim then follows from the fact that \( \mathfrak{S}_0 \) is the \( \sigma_E \)-linear span of \( \mathfrak{S}_0 : \mathfrak{S}_0 = \sigma_E[\mathfrak{S}_0] \), as asserted in [7] on page 190 (recall \( p \geq 3 \)). □

Let \( S = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \); it belongs to \( U(\mathfrak{A}) \). Define \( J^S = S^{-1}JS \) and \( \lambda^S(x) = \lambda(SxS^{-1}), x \in J^S \), where \((J, \lambda)\) is the simple type given by Proposition 2.1. Recall that the restriction of \( \lambda \) to \( H^1 = H^1(\beta, \mathfrak{A}) \) is a multiple of a simple character \( \theta \in \mathcal{C}(\mathfrak{A}, 0, \beta) \) such that \((H^1, \theta)\) is a decomposed pair in \( G \) relative to \((M, P)\) with \( H^1 \cap M = H^1(\beta, \mathfrak{A}_0) \times H^1(\beta, \mathfrak{A}_0) \) and \( \theta|_{H^1 \cap M} = \theta_0 \otimes \theta_0 \).

**Corollary.** – The representation \( \lambda^S \) of \( J^S = J(S^{-1}\beta S, \mathfrak{A}) \) is a simple type with the following properties:

(i) The pair \((J^S, \lambda^S)\) is a decomposed pair in \( G \) relative to \((M, P)\) with:

\[
J^S \cap M = J^S \cap M = \Gamma \times \Gamma \quad \text{and} \quad (\lambda^S)|_{J^S \cap M} \simeq \gamma \otimes \gamma^*.
\]

The same holds for the pair \((J^S_{\beta}, \lambda^S_{\beta})\).

(ii) The groups \( J^S, J^S_{\beta} \) and \( J^S_{\beta} \) are invariant under the involution \( X \mapsto T^TX^{-1} \).

(iii) The group \( H^1(\beta, \mathfrak{A})^S \) is invariant under the involution \( X \mapsto T^TX^{-1} \) and so is the simple character \( \theta^S \). We have \( \theta^S|_{H^1 \cap M} = \theta_0 \otimes \theta_0^\ast \).

Indeed, using Iwahori decompositions of those groups, one checks easily that the invariance of their intersections with \( M, U \) and \( U^- \) derives from the properties in the proposition.

### 2.3. A block-diagonal skew-simple stratum

The simple character \( \theta^S \in \mathcal{C}(\mathfrak{A}, 0, S^{-1}\beta S) \) in the above corollary is fixed under the involution \( X \mapsto T^TX^{-1} \); it follows from [22], Theorem 6.3, that this character can be viewed as a simple character attached to a skew simple stratum, namely:

There exists a simple stratum \([\mathfrak{A}, 2\gamma_0, 0, \delta]\) in \( \text{End}_F(V) \) satisfying \( \delta = -T^\delta \) such that \( \theta^S \) belongs to \( \mathcal{C}(\mathfrak{A}, 0, \delta) \).

In our situation we want, though, to work with a field extension both stable under the involution and contained in \( M \). We will thus derive a number of properties of the above stratum that will lead us to that goal: we will first study the hermitian structure of \( V \) over the field \( L = F[\delta] \) and show that we can conjugate \( \delta \) into a block diagonal element \( g^{-1}\delta g \) (Lemma 1); then we will use the very strong intertwining properties of simple types (Lemma 2) to show that we can pick \( g \) in \( J^S \).

If \( \delta \) is equal to 0, then \( L = F \) is already contained in \( M \) anyway. We thus assume that \( \delta \neq 0 \). The involution \( X \mapsto TX \) induces on \( L \) an automorphism of order 2; let \( L_0 \) be the fixed field of this automorphism. Then \( L \) is a separable extension of \( L_0 \) (recall \( p \neq 2 \)) with:

\[
\text{Tr}_{L/L_0}(X) = X + T^X \quad (X \in L).
\]

Let \( \phi \) be any non-zero \( F \)-linear form on \( L_0 \); then \( \phi \circ \text{Tr}_{L/L_0} \) is a non-zero \( F \)-linear form on \( L \), invariant under the involution \( T \). We define a non-degenerate \( L \)-anti-hermitian form \( b_\phi \) on \( V \) through:

\[
\forall a \in L, \quad \forall x \in V, \quad \forall y \in V, \quad \langle ax, y \rangle = \phi \circ \text{Tr}_{L/L_0} \left( ab_\phi(x, y) \right).
\]

The intersection of \( \text{End}_L(V) \) with \( \text{Sp}(V) \) is the unitary group \( U(V, b_\phi) \) relative to that form.
We first want to find a decomposition of the symplectic $F$-space $V$ as a direct sum of maximal isotropic subspaces stable under $L$; it amounts to showing that the anisotropic part of the (anti-)hermitian space $(V, b_o)$ is equal to $\{0\}$. To show this we will use lattice duality, and it will actually be easier, as Shaun Stevens pointed out to us, to work with an hermitian form.

We then fix an element $u$ in $L$ satisfying $\text{Tr}_{L/L_0}(u) = 0$ and such that $\text{val}_L u = 0$ if $L/L_0$ is unramified, $\text{val}_L u = 1$ if $L/L_0$ is ramified, and we define: $d_\phi(x, y) = ub_\phi(x, y)$ ($x, y \in V$). This is an hermitian form.

Recall ([11], (3.5.1)) that the field extensions $E = F[\beta]$ and $L = F[\delta]$ have the same ramification index and residual degree over $F$ (hence $[E : F]$ is even). In particular, the self-dual lattice chain $(\Lambda_i)_{i \in \mathbb{Z}}$ attached to $\mathfrak{A}$ (2.1) has period 2 over $L$ (by definition of a stratum, those lattices are $\sigma_L$-lattices). For $Y$ an $\sigma_L$-lattice in $V$ we put:

\[ Y^2 = \{ v \in V \mid \langle v, Y \rangle \subseteq \sigma_F \}; \quad Y^2 = \{ v \in V \mid d_\phi(v, Y) \subseteq \sigma_L \}. \]

We fix $\phi$ such that $\phi(\sigma_{L_0}) = \sigma_F$ and $\phi(p_i^{-1}) = p_i^{-1}$; we then have $Y^2 = Y^2$. Since $\mathfrak{A}_0$ is standard, we can number the lattice chain $(\Lambda_{0,i})_{i \in \mathbb{Z}}$ in such a way that

\[ \Lambda_{0,0} = \begin{pmatrix} \sigma_F \\ \sigma_F \end{pmatrix}. \]

We get the sequence $\Lambda^2_1 = \Lambda_{-1} \supset \Lambda_0 = \Lambda^2_0 \supset \Lambda_1 = \varpi_L \Lambda_{-1}$, that reads:

\[ \Lambda^2_1 = \Lambda_{-1} \supset \Lambda_0 = \Lambda^2_0 \supset \Lambda_1 = \varpi_L \Lambda_{-1}. \]

This is the self-dual slice of the lattice chain in the sense of Morris [17]. Propositions 1.7, 1.10 in [17] tell us that we can find a decomposition of the hermitian space $(V, d_o)$ into a direct orthogonal sum $V = V_H \oplus V_a$, where $V_a$ is anisotropic and the anisotropic part of $V_H$ is null, such that, for all $i \in \mathbb{Z}$: $\Lambda_i = \Lambda_i \cap V_H \oplus \Lambda_i \cap V_a$. We now use the following fact:

Let $(W, b)$ be an anisotropic hermitian space over $L$. Assume there is an $\sigma_L$-lattice $Y$ in $W$ satisfying:

- $Y = Y^2$ if $L/L_0$ is unramified;
- $Y = \varpi_L Y^2$ if $L/L_0$ is ramified.

Then $W$ has dimension 0 or 1 over $L$.

Remark. – The first case is a remark in [16], 5.1.1. Both cases rely on the classification of anisotropic hermitian spaces (see, e.g., [16], 1.1.4, or [17], 1.8); indeed such a configuration cannot occur in two-dimensional spaces.

We can now conclude, since $V$ has even dimension over $L$ as over $E$, that the anisotropic part of $V$ is null. Again, from Propositions 1.7, 1.10 in [17], we can find a decomposition $V = W_1 \oplus W_2$ into a direct sum of maximal $b_o$-isotropic $L$-subspaces such that:

- for all $i \in \mathbb{Z}$, $\Lambda_i = \Lambda_i \cap W_1 \oplus \Lambda_i \cap W_2$;
- the induced lattice chains on $W_1$ and $W_2$ have period 1 over $L$;
- $\Lambda_1 = \Lambda_0 \cap W_1 \oplus \varpi_L \Lambda_0 \cap W_2$.

Let $(f_1, \ldots, f_N)$ be an $\sigma_F$-basis for $(\Lambda_i \cap W_1)_{i \in \mathbb{Z}}$ (see definition (1.1.7) in [11]) such that $\Lambda_0 \cap W_1 = \sigma_F f_1 + \cdots + \sigma_F f_N$; one checks easily that the basis $(f_{N+1}, \ldots, f_{2N})$ of $W_2$ defined by $\langle f_k, f_{2N-k+1} \rangle = -1$ for $1 \leq k \leq N$ and $\langle f_j, f_{2N-k+1} \rangle = 0$ for $1 \leq k \leq N$, $1 \leq j \leq N$ and $j \neq k$, is an $\sigma_F$-basis for $(\Lambda_i \cap W_2)_{i \in \mathbb{Z}}$.
Let $g$ be the element of $\text{Sp}(V)$ that sends the canonical basis $(e_1, \ldots, e_{2N})$ on $(f_1, \ldots, f_{2N})$. We started (see 2.1) with a decomposition $V = V^{(2)} \otimes V^{(1)}$ having the same properties with respect to $E$ as the above decomposition with respect to $L$, and $(e_1, \ldots, e_N)$ is an $\sigma_F$-basis for $(\Lambda_i \cap V^{(2)})_{i \in \mathbb{Z}}$. We thus have $g(\Lambda_i \cap V^{(2)}) = \Lambda_i \cap W_1$ for all $i \in \mathbb{Z}$, hence, using duals, $g(\Lambda_i) = \Lambda_i$, so $g$ belongs to $\mathbb{A}^\times$. We sum up what we have just proved:

**Lemma 1.** – There exists an element $g$ in $\text{Sp}_{2N}(F) \cap \mathbb{A}^\times$ such that $g^{-1} \delta g$ is block diagonal, namely

$$g^{-1} \delta g = \begin{pmatrix} \delta_0 & 0 \\ 0 & -\tau \delta_0 \end{pmatrix}.$$

Let us come back now to the simple type $(J^S, \lambda^S)$ in Corollary 2.2, related to the simple character $\theta^S$ in $C(\mathbb{A}, 0, S^{-1} \beta S) = C(\mathbb{A}, 0, \delta)$. Its conjugate $(J^{S\bar{s}}, \lambda^{S\bar{s}})$ is related to the simple character $\theta^{S\bar{s}}$ in $C(g^{-1} \mathfrak{A}_g, 0, g^{-1} \delta g)$; note that $g^{-1} \mathfrak{A}_g = \mathfrak{A}$ since $g$ belongs to $\mathbb{A}^\times$. Since $g^{-1} \delta g$ is block diagonal, the machinery of [11], §7.1 and 7.2 applies: $(J^{S\bar{s}}, \lambda^{S\bar{s}})$ determines equivalent maximal simple types $\rho^{(1)}$ and $\rho^{(2)}$, attached respectively to the strata $[\mathfrak{A}_0, n_0, 0, \delta_0]$ and $[\mathfrak{A}_1, n_0, 0, \delta - \tau \delta_0]$; from [11], Theorem 7.2.17, we have: $(\lambda^{S\bar{s}})_U = \rho^{(1)} \otimes \rho^{(2)}$ (see also terminology 7.2.18, (iii)).

We now recall [11], Corollary 7.3.12. Let $\pi'$ be any smooth irreducible representation of $G$ containing $\lambda$. Its supercuspidal support consists of unramified twists of an irreducible supercuspidal representation $\pi$ of $GL_N(F)$ containing $\gamma$, the maximal simple type we started with in 2.2: $(J^1, \gamma) = (J(\beta, \mathfrak{A}_0), \lambda(\beta, \mathfrak{A}_0))$.

But $\pi'$ contains $\lambda$ if and only if it contains $\lambda^{S\bar{s}}$, hence $\pi$ also contains the maximal simple type $\rho^{(1)}$. Since the maximal simple types $\lambda(\beta, \mathfrak{A}_0)$ and $\rho^{(1)}$ intertwine in $\mathbb{GL}_N(F)$ and are associated to the same order $\mathfrak{A}_0$, they are conjugate in $\mathbb{A}^\times_0$ ([11], Theorem 5.7.1 and its proof). We sum up:

**Lemma 2.** – Let $(J(\delta_0, \mathfrak{A}_0), \rho^{(1)})$ be the maximal simple type associated to $(J^{S\bar{s}}, \lambda^{S\bar{s}})$. There exists an element $a$ in $\mathfrak{A}^0_0$ such that $J(\delta_0, \mathfrak{A}_0) = a^{-1} J(\beta, \mathfrak{A}_0) a$ and $\rho^{(1)} \simeq [\lambda(\beta, \mathfrak{A}_0)]^a$.

Now the element $a$ above is related as follows to the element $g$ in Lemma 1:

**Proposition.** – Let $c = (\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix})$ be the element of $\text{Sp}_{2N}(F) \cap \mathbb{A}^\times$ associated to $a$. Then $g$ belongs to the coset $c J^{S\bar{s}}$ in $\mathbb{A}^\times$.

**Proof.** – To simplify notation here we let $H = J^{S\bar{s}} = J(g^{-1} \delta g, \mathfrak{A})$ and $\mu = \lambda^{S\bar{s}}$. We use [11], Theorem 7.2.17, to produce two decomposed pairs $(H_1, \mu_1)$ and $(H_2, \mu_2)$ in $H$ such that

$$\mu_1 = \text{Ind}_{H_1}^H \mu_1 = \text{Ind}_{H_2}^H \mu_2.$$

For the first one we let $\mu_1$ be the natural action of $H_1 = (H \cap P)H_1^H(g^{-1} \delta g, \mathfrak{A})$ on the space of $(H \cap U)$-fixed vectors in $\mu$; indeed $H_1 = (J^{S\bar{s}})_P$, $\mu_1 = (\lambda^{S\bar{s}})_P$. We obtain a decomposed pair $(H_1, \mu_1)$ relative to $(M, P)$ with $\mu_1 = \text{Ind}_{H_1}^H \mu_1$.

For the second one, we let $H_2 = (J^P) S^\beta_1, \mu_2 = (\lambda^P)_S$ and obtain a decomposed pair relative to $(g^{-1} MG, g^{-1} PG)$ with $\mu_2 = \text{Ind}_{H_2}^H \mu_2$.

We now apply Mackey’s theorem [15] to the irreducible representation $\mu$. The intertwining of $\mu$ in $H$ is one-dimensional, hence there exists a unique double coset $H_2 \cap H_1$ in $H$ such that the restrictions of $\mu_1$ and $\mu_2$ to their common domain $H_1 \cap z^{-1} H_2$ intertwine.

Let us look at the induced representation $\text{Ind}^{\mathbb{A}^\times_0}_{H_1} \mu_1$. It is irreducible – indeed the intertwining of $\mu$ in $\mathbb{A}^\times_0$ is contained in the intersection with $\mathbb{A}^\times$ of the intertwining of the simple character $\theta^{S\bar{s}}$, hence in $(HD^X) \cap \mathbb{A}^\times = H(D \cap \mathbb{A}) \cap \mathbb{A}^\times = H$, where $D$ is the commutant algebra of $g^{-1} \delta g$.

Applying the same theorem to $\text{Ind}^{\mathbb{A}^\times_0}_{H_2} \mu_2$ produces a unique double coset $H_2 \cap H_1$ in $\mathbb{A}^\times_0$ with the previous properties. We must have $H_2 \cap H_1 = H_2 \cap H_1$, hence the proposition will follow from:
CLAIM. \( g^{-1}c \) intertwines \( \mu_1 \) and \( \mu_2 \).

First note that the pairs \((H_1, \mu_1)\) and \((H_2, \mu_2)\) are both invariant (up to equivalence of the representations) under the involution \( x \mapsto \tau x^{-1} \); this follows from Corollary 2.2, since \( g \) belongs to \( \text{Sp}(V) \).

We have \( e^{-1}g H_2 g^{-1}e = e^{-1}J \hat{S}\delta e \), so the pairs \((H_1, \mu_1)\) and \((H_2^{-1}e, \mu_2^{-1}e)\) are both decomposed with respect to \((M, P)\) and the representations \( \mu_1 \) and \( \mu_2^{-1}e \) intertwine if and only if their restrictions to \( H_1 \cap M \) and \( H_2^{-1}c \cap M \) intertwine.

Now \( H_1 \cap M = \mathcal{J}(g^{-1} \delta g, \mathfrak{A}) \cap M = \mathcal{J}(\delta_0, \mathfrak{A}_0) \times \tau \mathcal{J}(\delta_0, \mathfrak{A}_0) \) and the restriction of \( \mu_1 \) there is \( \rho^{(1)} \otimes (\rho^{(2)})^*\). On the other hand:

\[
H_2^{-1}c \cap M = e^{-1}(J \cap M) \mathcal{S} e = (a^{-1}J(\beta, \mathfrak{A}_0)a) \times \tau (a^{-1}J(\beta, \mathfrak{A}_0)a)
\]

and the restriction of \( \mu_2^{-1}e \) there is isomorphic to \([\lambda(\beta, \mathfrak{A}_0)]^\alpha \otimes ([\lambda(\beta, \mathfrak{A}_0)]^\alpha)^*\). We now conclude with Lemma 2. \( \square \)

Since \( e^{-1}g \) belongs to \( J \hat{S}g \), then \( e^{-1} \) belongs to \( g^{-1} J \mathcal{S} \) so we can write \( g = hc \) with \( h \in J \mathcal{S} \); note that \( h \) belongs to \( \widetilde{G} \) since \( g \) and \( c \) do. Since the elements \( g^{-1} \delta g = c^{-1} h^{-1} \mathcal{S} hc \) and \( c \) belong to \( M \), so does \( h^{-1} \mathcal{S} h \). Furthermore, since \( h \) belongs to \( J \mathcal{S} \), it stabilizes the simple character \( \theta \mathcal{S} \).

We have finally proved that, given any simple stratum \([\mathfrak{A}, 2n_0, 0, \delta]\) in \( M_{2N}(F) \) satisfying \( \delta = -\tau \delta \) and \( \theta \mathcal{S} \in \mathcal{C}(\mathfrak{A}, 0, \alpha) \), there exists an element \( h \in J \mathcal{S} \cap \widetilde{G} \) such that \( h^{-1} \mathcal{S} h \) belongs to \( M \). We conclude:

THEOREM. Let \( \theta \mathcal{S} \) be the simple character in Corollary 2.2. There exists a simple stratum \([\mathfrak{A}, 2n_0, 0, \alpha]\) in \( M_{2N}(F) \), satisfying \( \alpha \in M \) and \( \alpha = -\tau \alpha \), such that \( \theta \mathcal{S} \) belongs to \( \mathcal{C}(\mathfrak{A}, 0, \alpha) \).

Now write \( \alpha = (\alpha^0, 0, -\tau \alpha_0) \) and note that such an element generates a field over \( F \) if and only if \( \alpha_0 = 0 \) (case ruled out from the start) or the field \( K = F[\alpha_0] \) is a quadratic extension of \( K_0 = F[\alpha_0^0] \).

Recall that the pair \((H^1(\alpha, \mathfrak{A}), \theta \mathcal{S})\) is a decomposed pair above \((H^1(\alpha, \mathfrak{A}_0)) \times H^1(-\tau \alpha_0, \mathfrak{A}_0),\theta_0 \otimes \theta_0^\alpha)\) and is \( T \)-stable, i.e.:

\[
\begin{cases}
H^1(\alpha, \mathfrak{A}_0) \times H^1(-\tau \alpha_0, \mathfrak{A}_0) = H^1(\alpha, \mathfrak{A}_0) \times \tau H^1(\alpha, \mathfrak{A}_0), \\
\theta_0 \circ (g_0 \circ g_0^{-1}) = \theta_0(g_1) \circ \theta_0(\tau g_1^{-1}).
\end{cases}
\]

From [11], Proposition 7.1.19, we conclude that the character \( g \mapsto \theta_0(\tau g^{-1}) \) on \( H^1(-\tau \alpha_0, \mathfrak{A}_0) \) is the image of \( \theta_0 \) under the canonical transfer of simple characters from [11], Theorem 3.6.14:

\[
\mathcal{C}(\mathfrak{A}_0, 0, \alpha_0) \overset{\sim}{\longrightarrow} \mathcal{C}(\mathfrak{A}_0, 0, -\tau \alpha_0).
\]

It is difficult here to use the original notations to denote the canonical map; indeed \( \mathcal{C}(\mathfrak{A}_0, 0, -\tau \alpha_0) \) is still a set of simple characters attached to \( \alpha_0 \), but we change the action of \( K \) on the underlying vector space by composing it with \( \tau \) and with the Galois conjugacy over \( K_0 \), denoted by \( x \mapsto \tau x \).

We now use the following

FACT. Let \( \psi_1, \psi_2 \) be two \( F \)-embeddings of \( K \) into \( M_{2N}(F) \) such that \( \psi_1(K^\times) \) and \( \psi_2(K^\times) \) both normalize \( \mathfrak{A}_0 \). There exists \( u \in U(\mathfrak{A}_0) \) such that, for all \( x \) in \( K \), we have \( \psi_2(x) = u^{-1} \psi_1(x) u \). The canonical transfer map between the set of simple characters \( \mathcal{C}(\mathfrak{A}_0, m, \psi_1(\alpha_0)) \) and \( \mathcal{C}(\mathfrak{A}_0, m, \psi_2(\alpha_0)) \) (\( m \) in \( \mathbb{N} \)) is then given by \( \theta \mapsto \theta^u \).
(The first assertion above is Lemma 1.6 in [9]. The second is so tautological that it is implicit in [11]. In any case properties 3.6.13 in [11] are easily checked.)

We may then choose an element \( \sigma \) in \( U(\mathfrak{A}_0) \) such that \( \sigma^{-1} x \sigma = \tau x \) for all \( x \) in \( K \). The canonical transfer map from \( C(\mathfrak{A}_0, 0, \alpha_0) \) to \( C(\mathfrak{A}_0, 0, -\tau \alpha_0) \) hence transforms a simple character \( \mu \) into the simple character \( x \mapsto \mu(\sigma x \sigma^{-1}) \). We get:

**Corollary.** – Let \((\Gamma, \gamma)\) be a maximal simple type in \( GL_N(F) \) such that the corresponding inertial class of irreducible supercuspidal representations of \( GL_N(F) \) contains a self-contragredient representation \( \pi \), and assume the corresponding principal order \( \mathfrak{A}_0 \) is standard. Then either \((\Gamma, \gamma)\) has level zero, or the simple character \( \theta_0 \) underlying it can be attached to a simple stratum \([\mathfrak{A}_0, n_0, 0, \alpha_0] \) in \( M_N(F) \) with the following properties.

1. The field \( F[\alpha_0] \) is a quadratic extension of \( F[\alpha_0^2] \) – in particular \( N \) is even.
2. Let \( x \mapsto \pi \) denote the Galois conjugation of \( F[\alpha_0] \) over \( F[\alpha_0^2] \). There is an element \( \sigma \) in \( U(\mathfrak{A}_0) \) such that \( \sigma^{-1} x = \pi x \) for all \( x \) in \( F[\alpha_0] \). The simple character \( \theta_0 \) then satisfies:

\[
\theta_0(\sigma^{-1} x \sigma^{-1}) = \theta_0(x^{-1}) \quad (x \in H^1(\alpha_0, \mathfrak{A}_0)).
\]

**Remark 1.** – The element \( \sigma \) above satisfies all assumptions in Proposition 2.2 (see the proof of 2.2). It is unique up to left multiplication by \( U(\mathfrak{B}_0) \), and \( \sigma^{-1} \sigma \) belongs to \( U(\mathfrak{B}_0) \).

**Remark 2.** – We can apply the above fact to the embedding \( x \mapsto \tau x \) of \( K \) into \( M_N(F) \), and get \( u \) in \( U(\mathfrak{A}_0) \) such that \( u^{-1} x u = \tau x \) for all \( x \) in \( K \). The transfer map between \( C(\mathfrak{A}_0, m, \alpha_0) \) and \( C(\mathfrak{A}_0, m, \tau \alpha_0) \) is then given by \( \theta \mapsto \theta^u \). Since it is also given by \( \theta \mapsto \theta \circ \tau \), any simple character \( \theta \) in \( C(\mathfrak{A}_0, m, \alpha_0) \) satisfies \( \theta(\gamma) = \theta(u^{-1} \gamma u) \). We let \( \nu = \sigma \tau u^{-1} \) and combine this with the above corollary: we have \( \nu^{-1} x \nu = \pi \) for \( x \) in \( K \) and \( \theta_0(\nu^{-1} x \nu^{-1}) = \theta_0(x^{-1}) \) for \( x \) in \( H^1(\alpha_0, \mathfrak{A}_0) \). This is the formulation given in the introduction; it is conjugacy-invariant, hence the assumption that \( \mathfrak{A}_0 \) is standard can be removed there.

**Remark 3.** – One can actually go further along the same lines and show that \( f = N/[E: F] \) is either even or equal to 1; hence either \([E: F]\) is equal to \( N \), or \( N \) is a multiple of 4.

### 3. A sequence of \( \text{Sp}_{2N}(F) \)-covers

#### 3.1. Construction of the sequence

We do not need in this paragraph the results obtained in Section 2.3; their use would not help. Henceforth we keep the notations and assumptions in Section 2.2 – in particular \((\Gamma, \gamma)\) is a maximal simple type in \( GL_N(F) \), attached to the inertial class of a self-contragredient supercuspidal representation \( \pi \), and \( \mathfrak{A}_0 \) is standard – and start with a sequence \((J_i, \lambda_i)_{0 \leq i \leq 4}\) of \( G \)-covers of the pair \((\Gamma \times \Gamma, \gamma \otimes \gamma)\) as obtained from Proposition 2.1:

- \((J_3, \lambda_3) = (J_P, \lambda_P); \quad (J_2, \lambda_2) = (J_{P^*}, \lambda_{P^*})\);
- \((J_1, \lambda_1) = (J_2^*, \lambda_2^*); \quad (J_0, \lambda_0) = (J_3^*, \lambda_3^*)\) with \( s = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \);
- \((J_4, \lambda_4) = (J_0, \lambda_0^*)\) with \( c = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \).

Note that the elements \( s \) and \( c \) normalize \((\Gamma \times \Gamma, \gamma \otimes \gamma)\). Let us write down the Iwahori decompositions of the \( J_i \)’s to visualize them:
\[
J_0 = i^- (\mathfrak{t}^1_+)(J \cap M)i^+(\mathfrak{h}^-_1), \quad J_1 = i^- (\mathfrak{h}^+_1)(J \cap M)i^+(\mathfrak{t}^1_-), \\
J_2 = i^- (\mathfrak{t}^1_+)(J \cap M)i^+(\mathfrak{h}^+_1), \quad J_3 = i^- (\mathfrak{h}^-_1)(J \cap M)i^+(\mathfrak{t}^1_+), \\
J_4 = i^- (\mathfrak{h}^-_1)(J \cap M)i^+(\mathfrak{t}^1_+).
\]

The process in Proposition 1.2, applied to the conjugates \(J^S_i\) of Corollary 2.2, provides us with a corresponding sequence of decomposed pairs \((J^S_i, \hat{X}^S_i)\) in \(\mathcal{G}\), that we will denote by \((\Omega_i, \omega_i)\) to simplify notations; namely:

\[
(\Omega_i, \omega_i)_{0 \leq i \leq 4} \quad \text{with} \quad \Omega_i = J^S_i \cap \mathcal{G}, \quad \Omega_i \cap \mathcal{M} = i(\Gamma) \quad \text{and} \quad (\omega_i)_{\Omega_i \cap \mathcal{M}} = i(\gamma).
\]

We have:

\[
J^S_i = \begin{pmatrix} \mathcal{W}^{-1} & 0 & 0 & 0 \\
0 & \sigma^{-1}\mathcal{W}E & 0 & 0 \\
0 & 0 & \mathcal{W}^{-1} & 0 \\
0 & 0 & 0 & \sigma^{-1}\mathcal{W}E\sigma \end{pmatrix}
\]

and we know from Proposition 2.2 that \(\tau\mathcal{W}^{-1}\sigma^{-1}\mathcal{W}E\sigma\) belongs to \(\sigma^{-1}\mathcal{M} = \tau\Gamma\). Hence \(\Omega_i\) is equal to \(\mathcal{W}^{-1}\omega_i\mathcal{W}\) where \(z = (\mathcal{W}^{-1}\mathcal{W}E\sigma, 0, \mathcal{W}^{-1}\mathcal{W}E\sigma)\) belongs to \(\mathcal{T}\). We can thus derive from the above an infinite sequence of decomposed pairs in \(\mathcal{G}\) through:

\[
\Omega_{i+4j} = z^{-j}\Omega_i z^j, \quad i \in \{0, 1, 2, 3\}, \quad j \in \mathbb{N}.
\]

**Theorem.** - The pairs \((\Omega_i, \omega_i), \ i \in \mathbb{N},\) are \(\mathcal{G}\)-covers of the pair \((i(\Gamma), i(\gamma))\).

**Remark.** - Let \((\Gamma', \gamma')\) be another maximal simple type attached to the inertia class of \(\pi\). From [11], (6.2.4), we can find \(a \in \text{GL}_N(F)\) such that \(\Gamma' = \Gamma^a, \gamma' = \gamma^a\). The conjugates of the subgroups in Corollary 2.2 by the element \(A = i(a)\) in \(\text{Sp}_{2N}(F)\) satisfy analogous properties with respect to \((\Gamma', \gamma')\). The process in Proposition 1.2 then gives us decomposed pairs above \((i(\Gamma'), i(\gamma'))\) which are \(\mathcal{G}\)-covers of \((i(\Gamma), i(\gamma))\), as \(A\)-conjugates of the above. Hence Theorem 3.1 actually provides us with a construction of a \(\mathcal{G}\)-cover of \((i(\Gamma), i(\gamma))\) whether or not \(\mathfrak{A}_0\) is standard.

The proof of this theorem will occupy the remainder of this paper; it is organised as follows. From the properties recalled in Section 2.1 we know that the sequences \(\Omega_i \cap \mathcal{U}\) and \(\Omega_i \cap \mathcal{U}\), \(i \in \mathbb{N}\), are respectively increasing and decreasing, with \(\bigcup_{i \in \mathbb{N}} \Omega_i = \mathcal{U}\). We can then use [2], Theorem I.3.4: to show that the sequence of decomposed pairs \((\Omega_i, \omega_i)\) is actually a sequence of covers, it is enough to show that each couple of consecutive pairs \((\Omega_i, \omega_i), (\Omega_{i+1}, \omega_{i+1})\), \(i \in \mathbb{N}\), satisfies one of three criteria. In the present paragraph we will prove a convenient periodicity lemma, allowing us to reduce this checking of criteria to the cases \(i = 0\) to \(3\). For \(i = 0\) or \(2\), criterion 1 in loc. cit. is satisfied (3.2). For \(i = 1\) or \(3\), criterion 2 is used, but the proof in 3.3 takes for granted an intertwining property, property \((*)\). Section 4 is then devoted to proving property \((*)\), or rather Proposition 4.1 which implies the former; for this we will need Theorem 2.3, i.e. the stability of the underlying field extension under the involution \(T\).

Since an appropriate power of \(\mathcal{W}E\) belongs to \(\mathcal{W}F\), the sequence \((\Omega_i, \omega_i)\) is periodic in the sense of [5], Lemma 1, with period \(4e(E/F)\). Since we would rather restrict the checking of criteria to the smallest possible number of cases, we have to generalize this lemma to the case of our element \(z\), which does not lie in the center of \(\mathcal{M}\). Note that although we state the periodicity lemma below in our present context, it actually holds in the more general situation of [5], Lemma 1.
LEMMA. – Let $z$ be an element of $\overline{M}$ which normalizes $\Omega_{\tau} \cap \overline{M} = i(\Gamma)$ and such that $\gamma$ and $\gamma z$ are equivalent representations of $\Gamma$. Let $i, k \in \mathbb{N}$ such that $\Omega_{i+k} = z^{-1} \Omega_{i} z$ and $\Omega_{i+k+1} = z^{-1} \Omega_{i+1} z$. Let $(\tau, W)$ be a smooth representation of $\overline{G}$ and define, for $w \in W^{\omega_i}$:

$$\mathcal{R}_{i}^\tau(w) = \int_{\Omega_{i} \cap \overline{U}} \tau(y) \, dy \int_{\Omega_{i+1} \cap \overline{U}} \tau(n) \, dw.$$

We have $\mathcal{R}_{i+k}^\tau = \tau(z^{-1}) \mathcal{R}_{i}^\tau \tau(z)$, hence $\mathcal{R}_{i+k}^\tau$ is injective on $W^{\omega_{i+k}}$ if and only if $\mathcal{R}_{i}^\tau$ is injective on $W^{\omega_{i}}$.

Proof. – Since $z$ belongs to $\overline{M}$ we have: $\Omega_{i+k} \cap \overline{U} = z^{-1} (\Omega_{i} \cap \overline{U}) z$ and $\Omega_{i+k+1} \cap \overline{U} = z^{-1} (\Omega_{i+1} \cap \overline{U}) z$. From a change of variables $(y, n) \mapsto (z^{-1} y z, z^{-1} n z)$ in the integral defining $\mathcal{R}_{i+k}^\tau$, we get $\mathcal{R}_{i+k}^\tau = \tau(z^{-1}) \mathcal{R}_{i}^\tau \tau(z)$ (indeed the moduli of the action of $z$ on $\overline{U}$ and $\overline{U}^\tau$ are mutually inverse). The consequence on injectivity relies on the equality $\tau(z) W^{\omega_{i+k}} = W^{\omega_{i}}$, due to the fact that, since $\gamma$ and $\gamma z$ are equivalent, the representations $\omega_i$ and $\tau \omega_{i+k}$ of $\Omega_{i}$ are equivalent. □

Since our present element $z = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ satisfies the assumptions in the lemma, showing injectivity of the operators $\mathcal{R}_{i}^\tau$ for $i \in \mathbb{N}$ amounts to showing it for $i = 0$ to $3$. In other words (see [5], Proposition 1), we only need to check the criteria in [2], Theorem 1.3.4, for $i = 0$ to $3$.

3.2. Injectivity of $\mathcal{R}_{i}^\tau$ for $i = 0$ or $2$

We start with the case $i = 2$ and will prove that criterion I is satisfied, namely:

for any $y$ in $\Omega_2 \cap \overline{U}$, $y \notin \Omega_2$, there is a closed subgroup $X$ of $\Omega_2 \cap \overline{U}$ such that $y^{-1} X y$ is contained in $\Omega_2$ and has no non-zero fixed vectors under $\omega_2$.

We use the groups $H^1 = H^1(\beta, \mathbb{A})$ and $J^1 = J^1(\beta, \mathbb{A})$ from Section 2. From our definitions of $J_2$ and $J_3$, both groups contain $H^1$ and the restrictions of $\lambda_2$ and $\lambda_3$ to $H^1$ are a multiple of the simple character $\theta$ which satisfies $([11], (7.2.3))$: F

FACT. – For $x$ in $J^1 \cap U^-$ and $y$ in $J^1 \cap U$, the commutator $[x, y] = x y a^{-1} y^{-1}$ belongs to $H^1$ and the map:

$$(J^1 \cap U^- / H^1 \cap U^-) \times (J^1 \cap U / H^1 \cap U) \longrightarrow \mathbb{C}^\times,$$

$$(x, y) \longmapsto \theta([x, y])$$

is a perfect duality between those two groups.

We have by definition $J_2 \cap U^- = J^1 \cap U^-$, $J_3 \cap U^- = H^1 \cap U^-$, $J_3 \cap U = J^1 \cap U$ and $J_2 \cap U = H^1 \cap U$. Conjugating by $S$ then gives us a perfect duality:

$$(J^S_2 \cap U^- / J^S_3 \cap U^-) \times (J^S_3 \cap U / J^S_2 \cap U) \longrightarrow \mathbb{C}^\times,$$

$$(x, y) \longmapsto \theta^S([x, y]).$$

Corollary 2.2 states that the involution $X \mapsto T X^{-1}$ on $G$ preserves $J^S_2$, $J^S_3$ and the above duality given by $\theta^S$. From Stevens’s remark in [22, §4], we conclude that by restriction to $Sp_{2N}(F)$ we still have a perfect duality:

$$(\Omega_2 \cap U^- / \Omega_3 \cap U^-) \times (\Omega_3 \cap U / \Omega_2 \cap U) \longrightarrow \mathbb{C}^\times,$$

$$\theta^S([x, y]).$$

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Now for $x$ in $\Omega_2 \cap U^\sim$ and $y$ in $\Omega_3 \cap U$, the commutator $[x, y]$ belongs to

$$(H^1 \cap U^\sim) i(g)(H^1 \cap U)$$

for some $g$ in $H^1(\beta, \mathfrak{A}_0)$, and we have $\theta^S([x, y]) = \theta_0(g)^2$. On the other hand $\omega_2([x, y])$ is a multiple of $\theta_0(g)$. Since $H^1$ is a $p$-group with $p$ odd, the perfect duality above implies that, if $y \notin \Omega_2$, the subgroup $g^{-1}(\Omega_2 \cap U^\sim)y$ acts in $\omega_2$ through a non-trivial character. \(\square\)

The case $i = 0$ is entirely similar: indeed $J_1 = sJ_2s^{-1}$ and $J_0 = sJ_3s^{-1}$.

### 3.3. Injectivity of $\Omega_i^\tau$ for $i = 1$ or 3

Those steps are more involved than the previous ones – indeed, in cases when $\Omega_1^\tau = \bar{\Omega}_1^\tau$, we have $\Omega_2 = \Omega_3$. We start with $i = 1$ and want to show that criterion 2 is satisfied, i.e.:

- there is a compact subgroup $K$ of $\mathcal{G}$, containing $\Omega_1$, such that the Hecke algebra $\mathcal{H}(K, \Omega_1, \omega_1)$ is supported on $\Omega_1 \cup \Omega_2 \Omega_1$ for some $t$ in $K$ satisfying:

$$t^{-1}i(\Gamma)t = i(\Gamma), \quad t^{-1}(\Omega_1 \cap U)t = \Omega_2 \cap U^\sim, \quad t^{-1}(\Omega_1 \cap U^\sim)t = \Omega_2 \cap U.$$

We certainly have $\Omega_1 = t\Omega_2t^{-1}$ with

$$t = \left( \begin{array}{cc} 0 & \sigma \\ -\sigma^{-1} & 0 \end{array} \right) = S^{-1} \left( \begin{array}{cc} 0 & I \\ -\sigma^{-1} & 0 \end{array} \right) S \quad (t \in \mathcal{G}).$$

Note that $\left( \begin{array}{cc} 0 & \sigma^{-1} \\ 0 & 1 \end{array} \right)$ belongs to the coset $s(\Gamma \times \Gamma)$ from Proposition 2.2(a); hence $t$ normalizes $(\Omega_2 \cap M, i(\gamma))$ and intertwines $\omega_2$.

Since $J_1 = sJ_2s^{-1}$ and $J_2$, $s$ and $S$ are contained in $K = \text{GL}_{2\!\!N}(\sigma)$ (recall $\mathfrak{A}_0$ is standard), the subgroup generated by $\Omega_1$ and $\Omega_2$ is contained in the maximal compact subgroup $K = \text{Sp}_{2\!\!N}(\sigma)$.

Note that working with $\Omega_2$ or $\Omega_1$ here amounts to the same since the element $t^{-1}$ belongs to $i(\Gamma)$. The support of $\mathcal{H}(\mathcal{G}, \Omega_2, \omega_2)$ is the $\mathcal{G}$-intertwining of $\omega_2$, criterion 2 hence amounts to showing:

**Proposition.** We have: $I_G(\omega_2) \cap K \subset \Omega_2 \cup \Omega_2 \Omega_2$.

**Proof.** – From Proposition 1.2 we know that $I_G(\omega_2)$ is contained in $I_G(\lambda_2^\circ)$, itself contained in $I_G(\theta^S)$, since the restriction of $\lambda_2^\circ$ to $H' = S^{-1}H \cap S$ is a multiple of $\theta^S$.

We must now make an essential use of Shaun Stevens’s results in [22]. Indeed the character $\theta^S$ of $H'$ is fixed under the involution $x \mapsto \tau x^{-1}$. From [22], Theorem 6.3, it follows that $\theta^S$ can be viewed as a simple character attached to a skew simple stratum, hence satisfies the properties shown in [22], §3. In particular we have by [22], Theorem 3.7:

$$I_G(\theta^S) = I_G(\theta^S) \cap \mathcal{G}.$$

We thus have the following information on the support we are looking for:

$$I_G(\omega_2) \cap K \subset I_G(\theta^S) \cap K.$$

From [11], (5.1.1) and (5.5.11), the intertwining of $\theta$ is equal to $JB^\times J = J\tilde{W}J$, where $\tilde{W}$ is the affine Weyl group of $B^\times$ relative to the basis given in loc. cit. Assume for a moment that $[E : F] = N$. Since $B^\times$ is isomorphic to $\text{GL}_i(E)$ with $i[E : F] = 2N$, we are considering in this
case the affine Weyl group of $GL_2(E)$, whose intersection with a maximal compact subgroup has at most two elements, so $I_G(0) \cap K$ consists of the two double classes $J$ and $J'sJ$ and we get:

\[(\ast)\]  
$I_{\overline{W}}(\omega_2) \cap K \subset (J^S \cup J^S \tau J^S) \cap \overline{K}$.

We now drop our assumption on $[E:F]$ and get on with our proof assuming that $(\ast)$ holds.

Since $J = i^-(\Omega_1)(J \cap M) i^+(\Omega_1) = (J_2 \cap U^{-})(J_2 \cap M)(J_3 \cap U) = J_2J_3$, the group $\Omega = J^S \cap \overline{G}$ satisfies $\Omega = (\Omega_2 \cap \overline{U})(\Omega_2 \cap M)(\Omega_3 \cap \overline{U}) = \Omega_2\Omega_3 = \Omega_2(\Omega_3 \cap \overline{U})$. Furthermore we have:

**Lemma.** $J^S t J^S \cap \overline{G} = \Omega \cap \overline{G}$.

**Proof.** Since $t(J^S \cap U)^{-1} t^{-1}$ is contained in $J^S \cap U$, we have

$J^StJ^S = (J^S \cap U)^{-1} t(J^S \cap M)(J^S \cap U)$.

We apply [23], Theorem 2.3, to the automorphism $x \mapsto T x^{-1}$ of $G$, the pro-$p$-subgroup $J^S \cap U$, and the subgroup $H = (J^S \cap M) \cup t(J^S \cap M)$. Condition (2.1) in [23] is easily checked, hence

$J^StJ^S \cap \overline{G} = (\Omega \cap \overline{U}) t((\Omega \cap M)(\Omega \cap \overline{U}) = \Omega \cap \overline{G}$.

At this point we know: $I_{\overline{W}}(\omega_2) \cap K \subset \Omega \subset \Omega \cap \overline{G}$. Now note that the above argument can be applied in exactly the same way to the representation $\omega_3$ of $\Omega_3$; indeed $J^S_3$ also contains $H'$. We thus get $I_{\overline{W}}(\omega_3) \cap K \subset \Omega \subset \Omega \cap \overline{G}$, with the pleasant feature that $\Omega \cap \overline{G} = \Omega_3 \cap \overline{G}$ since $\Omega \cap \overline{U} = \Omega_3 \cap \overline{U}$.

We now use [11], (4.1.5); for $i = 2, 3$, the dimension of the subspace of the Hecke algebra $H(G, \Omega_i, \omega_i)$ supported on $\Omega$ (resp. $\Omega \cap \overline{G}$) is equal to the dimension of the subspace of the Hecke algebra $H(G, \Omega_i, \text{Ind}_{\Omega_i}^{\Omega} \omega_i)$ supported on $\Omega$ (resp. $\Omega \cap \overline{G}$). But the argument in Section 3.2 shows that the induced representation of $\omega_i$ to $\Omega$ is irreducible, hence the first dimension – dimension of the subspaces supported on $\Omega$ – is equal to 1. Furthermore the two induced representations $\text{Ind}_{\Omega_i}^{\Omega} \omega_i$, $i = 2, 3$, are isomorphic (use for instance Mackey restriction formula, plus the irreducibility and the fact that the representations $\omega_2$ and $\omega_3$ coincide on $\Omega_2 \cap \Omega_3$), so the second dimension – dimension of the subspaces supported on $\Omega \cap \overline{G}$ – is the same for $i = 2$ and $i = 3$. For $i = 3$ it is equal to 1, because any intertwining between $\omega_3$ and $\omega_3^\tau$ must intertwine the irreducible representation $i(\gamma)$, then it is also equal to 1 for $i = 2$, and since $t$ does intertwine $\omega_2$ we get the required property. \[\square\]

The last case left, $i = 3$, is dealt with exactly in the same manner, after observing that $\Omega_3 = q \Omega_3 q^{-1}$ with:

\[q = \begin{pmatrix} 0 & -\tau \sigma^{-1} \omega E^{-1} \\ \sigma^{-1} \omega E & 0 \end{pmatrix} = S^{-1} \begin{pmatrix} -\tau \sigma^{-1} \omega E^{-1} \sigma^{-1} \omega E & 0 \\ 0 & I \end{pmatrix} \text{wS}, \quad w = \begin{pmatrix} 0 & \omega E^{-1} \\ \omega E & 0 \end{pmatrix}.

Again we have $J_4 = wJ_3w^{-1}$. Furthermore, let $y = \begin{pmatrix} 0 & 1 \\ \omega E & 0 \end{pmatrix}$; then:

$yJ_1y^{-1} = J_4, \quad yJ_2y^{-1} = J_3, \quad yJ_3y^{-1} = J_2$ and $yJ_4y^{-1} = w$.

So we can repeat the previous argument, and we obtain a formula analogous to $(\ast)$, where we replace $K$ by $(K^W)^S$, $\omega_2$ by $\omega_3$, $t$ by $q$. The lemma becomes $J^S q J^S \cap \overline{G} = \Omega \cap \overline{G}$, with the same proof except that the roles of $U$ and $U^{-}$ are exchanged. Since $\Omega$ is contained in $(K^W)^S$, the last
part of the argument also follows through after exchanging the roles of \( \omega_2 \) and \( \omega_3 \). The proof of Theorem 3.1 is now complete, provided we prove (\( \ast \)) in Section 4 below.

**Remark.** – Along the lines of the above proof we might get on to show that the Hecke algebra of the pair \((\Omega_2, \omega_2)\) for instance, in \( G \), is the algebra with generators \( T_t \) and \( T_q \) (elements with support the double coset respectively of \( t \) and \( q \)) and relations the quadratic relations satisfied by \( T_t \) and \( T_q \) (they belong to a two-dimensional subalgebra). We cannot expect though that these quadratic relations be the same as the relations satisfied by \( T_t \) and \( T_q \) (they belong to a two-dimensional subalgebra). We cannot expect though that these quadratic relations be the same as the relations satisfied by \( T_t \) and \( T_q \) (they belong to a two-dimensional subalgebra).

We previously called \((\beta, \gamma, \alpha)\) the representations involved and use the argument in [11], Proposition 5.3.2, to reduce the proof (compare [11], Proposition 5.5.11). To prove this proposition, we will detail in 4.2 properties of \( \beta \) and call \( \gamma \) in 3.3, as well as the analogous property needed in the last case (\( \gamma \)). The proof of Theorem 3.1 is now complete, provided we prove (\( \ast \)) in Section 4 below.

4. Glauberman’s correspondence and intertwining

We must in this last part complete the proof of Theorem 3.1, that is, establish the property (\( \ast \)) in 3.3, as well as the analogous property needed in the last case (\( i = 3 \)) in 3.3. We will first show that these properties follow from a bound on intertwining, namely Proposition 4.1 (compare [11], Proposition 5.5.11). To prove this proposition, we will detail in 4.2 properties of the representations involved and use the argument in [11], Proposition 5.3.2, to reduce the proof to a very precise intertwining assertion: Proposition 4.3. At last we will establish that assertion using Glauberman’s correspondence together with arguments from [11], §5.1 and 5.2.

4.1. Intertwining and Weyl group

We must now use the full content of Theorem 2.3, so we change notations in this last part, both to simplify them and to stick to the notations in [11]. We call \((J, \lambda, \theta, \beta, \gamma, \zeta)\) the representations involved and use the argument in [11], Proposition 5.3.2, to reduce the proof to a very precise intertwining assertion: Proposition 4.3. At last we will establish that assertion using Glauberman’s correspondence together with arguments from [11], §5.1 and 5.2.

Let \( E \) be the field \( F[\beta] \) in \( M_N(F) \) and \( B_0 \) be its commutant; we still call \( E \) the field embedding \( F[\beta] \) in \( M_{2N}(F) \) and call \( B \) its commutant. The crucial fact is that the embedding \( F[\beta] \) of \( E \) in \( M_{2N}(F) \) is stable under the involution \( T \) and \( \overline{B}^\times \) is a unitary group (Section 2.3).

Let \( W \) be the affine Weyl group of \( \overline{B}^\times \) relative to the subgroup of diagonal matrices. We have \( \overline{B}^\times = \overline{T}(\mathfrak{B})W\overline{T}(\mathfrak{B}) \), since \( \overline{T}(\mathfrak{B}) \) contains a standard Iwahori subgroup of \( \overline{B} \). As in [11], §5.5, we let \( \mathfrak{M}(\mathfrak{B})^\times \) be the intersection with \( M \) of \( \mathfrak{B}^\times \).

Let \( I(\lambda) \) be the representation of \( J \) defined by \( I(\lambda) = \text{Ind}_{\mathfrak{M}(\mathfrak{B})}^{\mathfrak{M}(\mathfrak{B})^\times} \lambda \) (notation defined in 1.2). We already know (3.3) that its intertwining is contained in \( J\mathfrak{B}^\times J = \mathfrak{J}^1\mathfrak{B}^\times J^1 \) (recall \( J = U(\mathfrak{B})J^1 \)). Since \( B \) is now stable under \( T \) we can use fully [22], Theorem 3.7, to get: \( \mathfrak{J}^1\mathfrak{B}^\times J^1 = \mathfrak{J}^1\mathfrak{B}^\times J^1 \).

We will prove in the next paragraphs the following proposition:

**Proposition.** – The intertwining of \( I(\lambda) \) is contained in \( \mathfrak{J}^1\mathfrak{N}\mathfrak{W}(\mathfrak{M}(\mathfrak{B})^\times)\mathfrak{J}^1 \), equal to \( \mathfrak{J}^1\mathfrak{N}(\mathfrak{W}(\mathfrak{M}(\mathfrak{B})^\times)\mathfrak{J}^1) \).
To derive property \((*)\) we need only note that the normalizer of \(\mathfrak{N}(\mathfrak{B})^\times\) in \(W\) is equal to 
\((\mathcal{U}(\mathfrak{B}) \cap W)W_2(\mathcal{U}(\mathfrak{B}) \cap W)\) with

\[
W_2 = \left\{ \begin{pmatrix} \omega_E & 0 \\ 0 & \omega_E^{-1} \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & \omega_E \sigma^{-1} \\ - \omega_E^{-1} \sigma & 0 \end{pmatrix} \right\}, \quad i \in \mathbb{Z},
\]

where \(\sigma\) is an element in \(U(\mathfrak{A}_0)\) satisfying the property in Corollary 2.3 (or \(\sigma = I\) in the level zero case).

The intersection of \(W_2\) with any compact subgroup has at most two elements. It follows that the intersection with any compact subgroup of the intertwining of \(I(\lambda)\) contains at most two \(J\)-double classes.

4.2. A one-dimensional intertwining space

To prove the above proposition we have to collect informations on the representation \(I(\lambda)\). We need more notation. We let \(J_- = J \cap U^{-} = J^1 \cap U^{-}, J_+ = J \cap U = J^1 \cap U, J_M = J \cap M, J^1_M = J^1 \cap M, \)
\(H_- = H^1 \cap U^{-}, H_+ = H^1 \cap U, H^1_M = H^1 \cap M,\) and so on. We define the auxiliary subgroups \(J^1_P = H_- J^1_M J_+\), and \(K = H_- H^1_M J_+\). We will move around the following diagram (where arrows mean inclusion):

\[
\begin{array}{ccc}
\widetilde{i}(\kappa_0) & \xrightarrow{\text{induction}} & I(\kappa) \\
J_P = \overline{\mathcal{P}}_- J_M J_+ & \xrightarrow{\text{extension}} & \bar{J}_P = \overline{\mathcal{P}}_- \bar{J}_M \bar{J}_+ \\
& \xrightarrow{\text{induction}} & \eta_P = \overline{i}(\eta_0) \\
J^1_P = \overline{\mathcal{P}}_- J^1_M J_+ & \xrightarrow{\text{Heisenberg}} & \eta_P = \overline{i}(\eta_0) \\
& \xrightarrow{\text{extension}} & \overline{\mathcal{P}}^1 = \overline{\mathcal{P}}_- \overline{\mathcal{P}}^1_M \overline{\mathcal{P}}^1 \\
\end{array}
\]

Here, for any subgroup \(N\) of \(\overline{\mathcal{G}}\) admitting an Iwahori decomposition with respect to \((M, P)\), we denote by \(i(\mu)\) the representation of \(N\) trivial on \(N \cap U^{-}\) and \(N \cap U\) and with restriction \(i(\mu)\) to \(N \cap M\), whenever it makes sense. We define \(I(\kappa) = \text{Ind}_{J_P}^J i(\kappa_0)\) and \(I(\eta) = \text{Ind}_{J^1_P}^J i(\eta_0)\). We let \(\eta_P\) be the representation \(\eta_{J_P}^J\) of \(J^1_P\) in the space of \(J^1_-\)-fixed vectors in \(\eta\), as in [11], §7.2.

By definition of \(\tilde{\eta}_P\) (Proposition 1.2) we have \(\overline{i}(\eta_0) = \tilde{\eta}_P\). Furthermore, since the representation \(i(\sigma_0)\) of \(J\) is trivial on \(J^1\), we have:

\[
\tilde{\lambda}_P = i(\kappa_0 \otimes \sigma_0) = i(\kappa_0) \otimes i(\sigma_0)
\]

hence

\[
I(\lambda) = \text{Ind}_{J_P}^J \tilde{\lambda}_P \simeq \left( \text{Ind}_{J_P}^J i(\kappa_0) \right) \otimes i(\sigma_0) = I(\kappa) \otimes i(\sigma_0).
\]

We need the following properties:

**Proposition.**

(i) The representations \(I(\kappa)\) and \(I(\eta)\) are irreducible.

(ii) The restriction of \(I(\kappa)\) to \(J^1\) is isomorphic to \(I(\eta)\).
(iii) The representation $I(\eta)$ is the Heisenberg representation above $(\mathbb{H}^1, \overline{i(\theta_0)})$.

(iv) The intertwining of $I(\eta)$ is equal to $J^1\overline{B}^\times J^1$ and for any $g$ in $J^1\overline{B}^\times J^1$ the dimension of the intertwining space $I_g(I(\eta), J_1)$ is equal to 1.

Proof. – The irreducibility of $I(\eta)$ is a consequence of the fact in 3.2 and (ii) follows from Frobenius reciprocity. For (iii) the argument is in [22, §4].

(iv) is more intricate but the proof is entirely in [11]. We recall the main points. First of all we already know that the intertwining is contained in $J^1\overline{B}^\times J^1$ so we may assume that $g$ belongs to $\overline{B}^\times$. Since $I(\eta)$ is a Heisenberg representation, the argument in [11], 5.1.8, 5.1.9, reduces us to proving that

$$[\overline{J}^1 : J^1 \cap (J^1)^g] = [\overline{\mathbb{H}}^1 : \mathbb{H}^1 \cap (\mathbb{H}^1)^g]$$

namely Lemma 5.1.10 in [11], but for $\overline{G}$ instead of $G$. Using the Cayley transform $x \mapsto (1 + x/2)(1 - x/2)^{-1}$, which is defined on $(\mathbb{Z}^1)^- = \{X \in \mathbb{Z}^1 \mid TX = X\}$ and establishes bijections between $(\mathbb{Z}^1)^-$ and $J^1$, $(\bar{J}^1)^-$ and $\overline{\mathbb{H}}^1$, and so on (see [24]), we replace the equality to be proved by

$$[(\mathbb{Z}^1)^- : (\mathbb{Z}^1)^- \cap ((\mathbb{Z}^1)^-)^g] = [(\bar{J}^1)^- : (\bar{J}^1)^- \cap ((\bar{J}^1)^-)^g]$$

(the Cayley transform on $\mathcal{P}$ is easily seen to preserve subgroup indices).

Now the proof of loc. cit. applies mutatis mutandis: all exact sequences there remain exact after replacing each lattice involved, say $Z$, by $Z^- = \{X \in Z \mid TX = X\}$. Indeed, since $g$ belongs to $B^\times$ and satisfies $g = Tg^{-1}$, all the lattices involved are $T$-invariant; furthermore, the map $\alpha \beta$ is easily seen to commute with the involution $T$ and from [24], Lemma 2.1.1, we may (and must here) choose a corestriction $s$ that also commutes with $T$. □

We are now in a position to work out the intertwining of $I(\lambda)$. Let $g$ belong to $W$ and intertwine $I(\lambda)$; we have to show that $g$ normalizes $\mathfrak{M}(\mathcal{B})^\times$. Since $I(\lambda)$ is isomorphic to $I(\kappa) \otimes \overline{i(\sigma_0)}$ and the following two facts hold:

- $i(\sigma_0)$ is trivial on $J^1$;
- dim $I_g(I(\eta), J_1) = 1$;

we can imitate the proof of [11], Proposition 5.3.2, to get that any non-zero intertwining operator in $I_g(I(\eta), J_1)$ has the form $S \otimes R$ with $S \in I_g(I(\eta), J_1)$ and $R$ an endomorphism in the space of $i(\sigma_0)$.

Let us use Proposition 4.3 below: for any $T$-stable minimal $\sigma_E$-order $\mathcal{B}_m$ contained in $\mathcal{B}$, the operator $S$ also intertwines the restriction of $I(\kappa)$ to the subgroup $U^1(\mathcal{B}_m, J^1)$ (use one-dimensionality for $I(\eta)$). Again as in loc. cit., this implies that $R$ belongs to $I_g(i(\sigma_0), U^1(\mathcal{B}_m, J^1))$.

Proposition 4.1 now follows from:

**Lemma**. – Let $g \in W$ intertwine the restriction of $i(\sigma_0)$ to $U^1(\mathcal{B}_m, J^1)$ for any $T$-stable minimal $\sigma_E$-order $\mathcal{B}_m$ contained in $\mathcal{B}$. Then $g$ normalizes the group $\mathfrak{M}(\mathcal{B})^\times$.

**Proof**. – Indeed we almost recognize [11], Proposition 5.5.5, that again we will imitate. The sequence of lemmas there holds unchanged, so we assume $g$ does not normalize $\mathfrak{M}(\mathcal{B})^\times$ and produce an hereditary order $\mathcal{B}_0$, with radical $\Omega_0$, and a parabolic subgroup $Q$ of $\text{GL}_N(F)$, with unipotent radical $N = 1 + \mathfrak{N}$, such that:

(i) $\mathcal{B}_0 \subset \mathcal{B}_0$; $\mathcal{B}_0 \cap \mathfrak{N} = \Omega_0 \cap \mathfrak{N} \not\subset \Omega_0$; $g^{-1}(\Omega_0 \cap \mathfrak{N})g \subset \Omega_0$;
4.3. Glauberman’s correspondence

We now pick a $\sigma$. This relation holds in particular for the original references. Furthermore, since $\sigma$ belongs to $B^i$, we can as well assume here that $B^i = B^i$.

We put $\mathcal{B}' = \mathcal{B}_0 \oplus \mathcal{B}_0'$; hence $(\mathcal{B}')^\infty$ is contained in $\mathcal{M}$ and equal to $U(\mathcal{B}_0) \times ^t U(\mathcal{B}_0)$. We put $\Omega' = \Omega_0' \oplus \mathcal{B}_0'$.

This last paragraph will be devoted to the proof of the proposition below.

(ii) the image of $1 + \Omega'_0 \cap \mathfrak{g}$ in $U(\mathcal{B}_0)/U^1(\mathcal{B}_0)$ is the unipotent radical of a proper parabolic subgroup of $U(\mathcal{B}_0)/U^1(\mathcal{B}_0)$.

Indeed, in the notations of loc. cit., $\mathcal{B}_0'$ is $\mathfrak{T}$, contained in some $B^i$, and one can decompose $V^i$ into a direct sum $V^i = W^1 \oplus W^2$ of $E$-vector spaces such that $L_1 \cap W^1 = L_0 \cap W^1$ and $L_1 \cap W^2 = \pi_E L_0 \cap W^2$; then $\mathfrak{H} = \mathrm{Hom}_E(W^2, W^1) \subset \mathrm{End}_E V^i$ satisfies the assumptions.

Looking at the diagram and proposition in 4.2, we find that

$$I(\mathfrak{g})_{U^1(\mathcal{B}_0), J^i} \simeq \text{Ind}_{i(U^1(\mathcal{B}_0), J^i)}^{U^1(\mathcal{B}_0), J^i} i(\kappa_0')$$

where $\kappa_0' = k_0^{U^1(\mathcal{B}_0), J^i}$.

It is enough to show that the representations $i(\kappa_0')$ and $i(\eta_0)$ have the same intertwining. Indeed, by [11], 4.1.5, we have $I_\mathfrak{T}(I(\mathfrak{g}), J^i_{\mathcal{T}_i(\mathcal{B}_0), J^i}) = (J^i)_{\mathcal{T}_i(\mathcal{B}_0), J^i}$ and $I_\mathfrak{T}(\mathcal{B}_0), J^i = J^i$. Glauberman’s correspondence is the tool we need here. We recall briefly what it is in our setting; more general and precise statements can be found in [22], §2, or [10], §A2, as well as the original references.

Let $\epsilon$ be the involution $x \mapsto x^{-1}$ on $G$, with fixed points $\mathcal{C}$. For any open compact pro-$p$-subgroup $H$ of $G$ which is $\epsilon$-stable, Glauberman’s correspondence gives us a unique
bijection \( g: \rho \mapsto g(\rho) \), between the set \( \Irr(H) \) of \( \epsilon \)-stable equivalence classes of smooth irreducible representations of \( H \) and the set \( \Irr(\overline{T}) \) of equivalence classes of smooth irreducible representations of \( \overline{T} \), characterized by the property that \( g(\rho) \) occurs in \( \rho \overline{T} \) with odd multiplicity.

Let \( K \) be a subgroup of \( H \) satisfying the same assumptions as \( H \), let \( \sigma \in \Irr(K) \) and \( \rho \in \Irr(H) \). Then \( \rho \simeq \Ind_{\rho}^\rho \sigma \) implies \( g(\rho) \simeq \Ind_{\rho \overline{T}}^\rho \sigma \), and \( \rho_{\overline{K}} \sim \sigma \) implies \( g(\rho)_{\overline{K}} \sim g(\sigma) \).

A crucial property of this correspondence is the following:

**FACT** (Stevens [22], Lemma 2.4). – Let \( \rho \in \Irr(H \times G) \). The dimension of \( I_\theta(\rho, H) \) is odd if and only if the dimension of \( I_\theta(g(\rho), \overline{T}) \) is odd.

We now proceed to find the inverse images under Glauberman’s correspondence of the representations \( i(\psi_0) \) and \( i(\eta_0) \). We start a series of lemmas. The first one is valid for any simple stratum in \( M_N(F) \).

**Lemma 1.** –
(i) Let \( [\mathfrak{A}, n, 0, \beta] \) be a simple stratum in \( M_N(F) \) and let \( \alpha \in \mathfrak{c}_F^n \). Then, for any defining sequence \( [\mathfrak{A}, n, r_i, \gamma_i] \) for \( [\mathfrak{A}, n, 0, \beta] \) (see [11], 2.4.2), the sequence \( [\mathfrak{A}, n, r_i, \alpha \gamma_i] \) is a defining sequence for \( [\mathfrak{A}, n, 0, \alpha \beta] \). In particular we have \( \mathcal{H}^1(\beta, \mathfrak{A}) = \mathcal{H}^1(\alpha \beta, \mathfrak{A}) \) and \( \mathcal{H}^2(\beta, \mathfrak{A}) = \mathcal{H}^2(\alpha \beta, \mathfrak{A}) \).

(ii) The map \( \theta \mapsto \theta^2 \) is a bijection from \( C(\mathfrak{A}, g, ^{1/2} \beta) \) onto \( C(\mathfrak{A}, g, \beta) \) \((m \in \mathbb{N})\), which is compatible with the canonical bijections of [11], §3.6. We will denote the inverse bijection by \( \theta \mapsto \theta^{1/2} \).

**Proof.** – (i) is simple checking. Since the groups \( H^{m+1}(\beta, \mathfrak{A}) \) are \( p \)-groups with \( p \) odd, (ii) is easily checked by induction along a defining sequence for \( \beta \).

Let again \( \theta \) be the simple character that underlies our simple type \( (J, \lambda) \) and let \( \theta^{1/2} \) in \( C(\mathfrak{A}, 0, ^{1/2} \beta) \) be its inverse image under the square map. To the simple character \( \theta^{1/2} \) we attach \( \eta^{1/2} \), \( \kappa^{1/2} \), and \( \eta_{\overline{P}}^{1/2} \), \( \kappa_{\overline{P}}^{1/2} \) in the usual way of [11], §§5 and 7. For instance, \( \kappa^{1/2} \) is a representation of \( J(\overline{1}, \beta, \mathfrak{A}) \) which is a beta-extension of \( \eta^{1/2} \); note that we have to choose one here, while \( \eta^{1/2} \) is completely determined by \( \theta^{1/2} \).

**Lemma 2.** – We still write \( \theta^{1/2} \) for the extension of \( \theta^{1/2} \) to \( K \) trivial on \( J^+ \). We have \( \bar{g}(\theta^{1/2}) \simeq i(\psi_0) \) (on \( K, \overline{K} \)) and \( \bar{g}(\eta_{\overline{P}}^{1/2}) \simeq i(\eta_0) \) (on \( J_{\overline{P}}, J_{\overline{P}} \)).

**Proof.** – The character \( \theta^{1/2} \) on \( K \) is trivial on \( J^+ \) and \( H^- \); on \( H_1^M \) it is given by \( \theta^{1/2}(g) = \theta_{\overline{0}}^{1/2}(g_1) \theta_{\overline{0}}^{1/2}(g_2^{-1}) \) for \( g = g_1 \overline{g_2} \) in \( H^1(\beta, \mathfrak{A}_0) \) ([11], 7.1.19, and Corollary 2.2.2).

We thus have: \( \theta^{1/2}(g) = \theta_{\overline{0}}^{1/2}(g) \theta_{\overline{0}}^{1/2}(g) = \theta_0(g) \), for \( g \in H^1(\beta, \mathfrak{A}_0) \), which proves the first assertion. The restriction of \( \eta_{\overline{P}}^{1/2} \) to \( K \) is a multiple of \( \theta^{1/2} \) so its restriction to \( \overline{K} \) is a multiple of \( i(\psi_0) \); unicity of Heisenberg representations says that the restriction of \( \eta_{\overline{P}}^{1/2} \) to \( J_{\overline{P}} \) is then a multiple of \( i(\eta_0) \).

Of course Glauberman’s correspondence does not apply to \( J(\overline{1}, \beta, \mathfrak{A}) \) which is not a \( p \)-group, nor to \( J_P \). But it does apply to the following group, intermediate between \( J_{\overline{P}} \) and \( J_P \):

\[
L = \left[ U^1(\mathfrak{B}_{0, m}) \times \tau U^1(\mathfrak{B}_{0, m}) \right] J_{\overline{P}}
= H \cdot \left[ \left[ U^1(\mathfrak{B}_{0, m}) J^1(\beta, \mathfrak{A}_0) \right] \times \tau \left( U^1(\mathfrak{B}_{0, m}) J^1(\beta, \mathfrak{A}_0) \right) \right] J_+.
\]

The subgroup \( L \) is certainly stable under \( x \mapsto \tau x^{-1} \); let us check that the restriction to \( L \) of the representation \( \kappa_{\overline{P}}^{1/2} \) is also stable by this involution, up to isomorphism. From [11], §7.2, \( \kappa_{\overline{P}}^{1/2} \) is
trivial on $L_-$ and $L_+$ and its restriction to $L \cap M$ has the form

$$
\kappa_P^{1/2} \begin{pmatrix}
g_1 & 0 \\
0 & g_2
\end{pmatrix} \simeq \kappa_1(g_1) \otimes \kappa_2(g_2^{-1}),
$$

where $\kappa_1$ and $\kappa_2$ are both beta-extensions of the Heisenberg representation $\eta_0^{1/2}$ of $J^1(\beta, \mathfrak{A}_0)$ attached to $\theta_0^{1/2}$. Hence $\kappa_1$ and $\kappa_2$ differ from a character $\chi \circ \det_{B_0}$, where $\chi$ is a character of $\mathfrak{A}_E / 1 + pE$ ([11], 5.2.2). This implies that $\kappa_1$ and $\kappa_2$ agree on $U^1(\mathfrak{B}_{0,m})J^1$ whence the stability of $(\kappa_P^{1/2})_L$ under $\epsilon$.

**Lemma 3.** – We have $g((\kappa_P^{1/2})_L) \simeq \tilde{i}(\kappa_0')$ (on $L$, $\tilde{L}$).

Proof. – Both representations have trivial restrictions to $L_-$ and $\tilde{L}_+$ and irreducible restrictions to $L \cap M$. So what we have to show is: $g((\kappa_P^{1/2})_{L \cap M}) \simeq i(\kappa_0')$.

Let $\kappa_M^{1/2} = (\kappa_P^{1/2})_{L \cap M}$; this is a representation of

$$
L \cap M = \left( U^1(\mathfrak{B}_{0,m})J^1(\beta, \mathfrak{A}_0) \right) \times \left( U^1(\mathfrak{B}_{0,m})J^1(\beta, \mathfrak{A}_0) \right).
$$

Denote by $\mathfrak{A}_{0,m}$ the unique hereditary $\mathfrak{A}_E$-order in $\mathfrak{A}_0$ stable under conjugation by $E^\times$ such that $\mathfrak{A}_{0,m} \cap B_0 = \mathfrak{B}_{0,m}$. Let $\theta_0^{1/2}$ be the image of $\theta_0^{1/2}$ under the canonical transfer map: $C(\mathfrak{A}_0, 0, 1/2) \rightarrow C(\mathfrak{A}_{0,m}, 0, 1/2)$, and let $\eta_0^{1/2}$ be the unique irreducible representation of $J^1(\beta, \mathfrak{A}_{0,m})$ containing $\theta_0^{1/2}$. Let $\mu_0^{1/2}$ be the unique extension of $\eta_0^{1/2}$ to

$$
L_0 = U^1(\mathfrak{B}_{0,m})J^1(\beta, \mathfrak{A}_0)
$$

satisfying

$$
\text{Ind}_{\mathfrak{L}_{0,m}}^{U^1(\mathfrak{A}_{0,m})} \mu_0^{1/2} \simeq \text{Ind}_{J^1(\beta, \mathfrak{A}_{0,m})}^{U^1(\mathfrak{A}_{0,m})} \eta_0^{1/2}.\]

From [11], 5.2.6 and 5.1.15 (where $\mu_0^{1/2}$ is denoted by $\eta_0'$ or in our context $\eta_0^{1/2}$), we have $\kappa_M^{1/2} \simeq \mu_0^{1/2} \otimes \mu_0^{1/2}$. The induced representations above are irreducible and loc. cit. implies:

$$
\begin{align*}
\text{Ind}_{L \cap M}^{U^1(\mathfrak{B}_{0,m})} \kappa_M^{1/2} & \simeq \text{Ind}_{\mathfrak{L}_{0,m}}^{U^1(\mathfrak{A}_{0,m})} \mu_0^{1/2} \otimes \left[ \text{Ind}_{\mathfrak{L}_{0,m}}^{U^1(\mathfrak{A}_{0,m})} \mu_0^{1/2} \right]^* \\
& \simeq \text{Ind}_{J^1(\beta, \mathfrak{A}_{0,m})}^{U^1(\mathfrak{A}_{0,m})} \eta_0^{1/2} \otimes \left[ \text{Ind}_{J^1(\beta, \mathfrak{A}_{0,m})}^{U^1(\mathfrak{A}_{0,m})} \eta_0^{1/2} \right]^*.
\end{align*}
$$

Since the representations involved are irreducible, one gets through Glauberman’s correspondence an isomorphism:

$$
\text{Ind}_{i(L_0)}^{i(U^1(\mathfrak{A}_{0,m}))} \tilde{i}(\kappa_0') \simeq \text{Ind}_{i(J^1(\beta, \mathfrak{A}_{0,m}))}^{i(U^1(\mathfrak{A}_{0,m}))} \tilde{i}(\eta_0^{1/2} \otimes [\eta_0^{1/2}]).
$$

We already know that $g(\eta_0^{1/2})$ is isomorphic to $\tilde{i}(\eta_0)$ and that $i(\kappa_0')$ extends $i(\eta_0)$. Again from loc. cit., the representation $i(\kappa_0')$ is the unique irreducible representation of $i(L_0)$ extending $i(\eta_0)$ and satisfying:

$$
\text{Ind}_{i(L_0)}^{i(U^1(\mathfrak{A}_{0,m}))} i(\kappa_0') \simeq \text{Ind}_{i(J^1(\beta, \mathfrak{A}_{0,m}))}^{i(U^1(\mathfrak{A}_{0,m}))} i(\eta_0^{1/2}).
$$
Hence it is enough to show that $\mathfrak{g}(\eta_{0,m}^{1/2} \otimes [\eta_{0,m}^{1/2}]) \simeq i(\eta_{0,m})$. But the restriction of $\eta_{0,m}^{1/2} \otimes [\eta_{0,m}^{1/2}]^*$ to $i(H^1(\beta, \mathfrak{A}_{0,m}))$ is a multiple of

$$i(g) \mapsto \theta_{\beta_{0,m}}^{1/2}(g) \otimes [\theta_{\beta_{0,m}}^{1/2}]^* \left( \gamma g^{-1} \right) = [\theta_{\beta_{0,m}}^{1/2}(g)]^2 = \theta_{0,m}(g),$$

so its image is the Heisenberg representation above $i(\theta_{0,m})$ namely $i(\eta_{0,m})$. □

End of proof of Proposition 4.3. – The intertwining of $i(\kappa_0)$ contains the intertwining of $i(\kappa_0')$ since the second representation restricts to the first. Let us now take $g$ in $I_{\mathcal{G}}(i(\kappa_0))$ and show that $g$ belongs to $I_{\mathcal{G}}(i(\kappa_0'))$. The above fact about intertwining spaces and Glauber's correspondence, combined with the lemmas, gives us:

– $g$ intertwines $\eta_{P}^{1/2}$ (Fact and Lemma 2). Indeed $\dim I_g(i(\kappa_0))$ is equal to 1, from Proposition 4.2 and [11], 4.1.5.

– $g$ intertwines $\kappa_{P}^{1/2}$. Indeed, from [11], §7.2, $g$ intertwines $\eta_{P}^{1/2}$ (induced from $\eta_{P}^{1/2}$), hence $g$ intertwines $\kappa_{P}^{1/2}$ (that has the same intertwining as $\eta_{P}^{1/2}$); furthermore $J_P g J_P^\perp$ is the unique $J_P^\perp$-double coset in $J^1 g J^1$ that intertwines $\eta_{P}^{1/2}$ ([11], 4.1.5 and 5.1.8). Similarly, in $J g J$ there is a unique $J_P^\perp$-double coset $J_P b J_P$ that intertwines $\kappa_{P}^{1/2}$. Since $J = J_P J_-$ we may assume that $b$ belongs to $J_- g J_-$, hence to $J^1 g J^1$. But then, since $b$ also intertwines $\eta_{P}^{1/2}$, we must have $b \in J_P^\perp g J_P^\perp$; so $b \in J_P^\perp g J_P$ whence the result.

– the dimension of $I_g(\eta_{P}^{1/2})$ is equal to 1, so is the dimension of $I_g(\kappa_{P}^{1/2})$ (again the second representation restricts to the first, by [11], §7.2). Hence $g$ intertwines the image of this representation by the Glauber correspondence, namely $i(\kappa_0')$ (Fact and Lemma 3). □

Remark. – Of course Proposition 4.3 says something about the restriction of $I(\kappa)$ to a suitable subgroup being a $\beta$-extension of $I(\eta)$. To prove Theorem 3.1, we actually do not need to know whether or not $I(\kappa)$ itself is a $\beta$-extension of $I(\eta)$. It should follow from the study of the Hecke algebra of the $G$-cover that the bound on intertwining given by Proposition 4.1 is actually an equality, i.e. the intertwining of $I(\lambda)$ is equal to $\mathcal{J}_N \mathfrak{W}(\mathfrak{M}(\mathfrak{B})^\times) \mathcal{J}$ – see the remark at the end of 3.3.

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