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TWISTED K-THEORY OF DIFFERENTIABLE STACKS

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ABSTRACT. – In this paper, we develop twisted K-theory for stacks, where the twisted class is given by an S^1-gerbe over the stack. General properties, including the Mayer–Vietoris property, Bott periodicity, and the product structure $K_\alpha^i \otimes K_\beta^j \to K_{\alpha+\beta}^{i+j}$ are derived. Our approach provides a uniform framework for studying various twisted K-theories including the usual twisted K-theory of topological spaces, twisted equivariant K-theory, and the twisted K-theory of orbifolds. We also present a Fredholm picture, and discuss the conditions under which twisted K-groups can be expressed by so-called "twisted vector bundles".

Our approach is to work on presentations of stacks, namely *groupoids*, and relies heavily on the machinery of K-theory (KK-theory) of C^*-algebras.

RÉSUMÉ. – Dans cet article, nous développons la K-théorie tordue pour les champs différentiables, où la torsion s'effectue par une S^1-gerbe sur le champ en question. Nous en établissons les propriétés générales telles que les suites exactes de Mayer–Vietoris, la périodicité de Bott, et le produit $K_\alpha^i \otimes K_\beta^j \to K_{\alpha+\beta}^{i+j}$. Notre approche fournit un cadre général permettant d'étudier diverses K-théories tordues, en particulier la K-théorie tordue usuelle des espaces topologiques, la K-théorie tordue équivariante, et la K-théorie tordue des orbifolds. Nous donnons également une définition équivalente utilisant des opérateurs de Fredholm, et nous discutons les conditions sous lesquelles les groupes de K-théorie tordue peuvent être réalisés à partir de "fibrés vectoriels tordus".

Notre approche consiste à travailler sur les réalisations concrètes des champs, à savoir les *groupoïdes*, et s'appuie de façon importante sur les techniques de K-théorie (KK-théorie) des C^*-algèbres.

Contents

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1. Introduction

Recently, motivated by D-branes in string theory, there has been a great deal of interest in the study of twisted K-theory [15,49,48,75]. The K-theory of a topological space M twisted by a torsion class in $H^3(M,\mathbb{Z})$ was first studied by Donovan–Karoubi [23] in the early 1970s, and, in the 1980s, Rosenberg [63] introduced K-theory twisted by a general element of $H^3(M,\mathbb{Z})$. More recently, twisted K-theory has enjoyed renewed vigor due to the discovery of its close connection with string theory [75,76]. See also [4,65,16] and references therein.

A very natural problem which arises is the development of other types of twisted K-theory. In particular, twisted equivariant K-theory and twisted K-theory for orbifolds should be developed. Indeed, various definitions of such theories have been offered. For instance, Adem–Ruan introduced a version of twisted K-theory of an orbifold by a discrete torsion element [1]. Others, for example [45,46], offer various related (but unsupported) definitions. We also remark that Freed–Hopkins–Teleman announced [27] the amazing result that the twisted equivariant K-theory of a compact Lie group is related to the Verlinde algebra.

It is important that twisted K-theory is a cohomology theory and, in particular, satisfies the Mayer–Vietoris property. One also expects that, like ordinary K-theory, it should satisfy Bott periodicity. The purpose of this paper is to develop a twisted K-theory for stacks, the idea being that this is general enough to include all the above cases, including twisted equivariant K-theory and twisted K-theory of orbifolds. As far as we know, except for the special case of manifolds, there has been no twisted K-theory for general stacks for which all such properties have been established (as far as we know, this is the case even for twisted equivariant K-theory).

Rather than working directly with stacks, we will work on presentations of stacks, namely groupoids. Indeed there is a dictionary in which a stack corresponds to a *Morita equivalence* *class* of groupoids [9,10]. In this paper, we will deal with differentiable stacks which are more relevant to string theory. They correspond to Lie groupoids.

An advantage of working with Lie groupoids, for a differential geometer, is that one can still do differential geometry even though the spaces they represent do not usually allow such a possibility.

The notion of a *groupoid* is a standard generalization of the concepts of *spaces* and *groups*. In the theory of groupoids, spaces and groups are treated on equal footing. Simplifying somewhat, one could say that a groupoid is a mixture of a space and a group; it has space-like and group-like properties that interact in a delicate way. In a certain sense, groupoids provide a uniform framework for many different geometric objects. For instance, when a Lie group acts on a manifold properly, the corresponding equivariant cohomology theories, including K-theory, can be treated using the transformation groupoid $G \times M \rightrightarrows M$. On the other hand, an orbifold can be represented by an étale groupoid [54,52].

The problem of computing the K-theory of groupoids has been studied by many authors. For instance, given a locally compact groupoid Γ, the Baum–Connes map

$$\mu_r : K_*^{\mathrm{top}}(\Gamma) \to K_*\big(C_r^*(\Gamma)\big)$$

can be used to study the K-theory groups of $C_r^*(\Gamma)$. The above map generalizes both the assembly map for groups [7] and the coarse assembly map [67]. Of course, many techniques used to study the Baum–Connes conjecture for groups [35] can be extended to groupoids such as foliation groupoids [68,69]. However, recent counterexamples [36] show that other ways of attacking the problem need to be discovered. Applications of the K-theory of groupoids include: tilings and gap labeling (see for instance [40]), index theorems, and pseudodifferential calculi [50,57].

By twisted K-theory, in this paper we mean K-theory twisted by an S^1-gerbe. All S^1-gerbes over a groupoid Γ (or more precisely a stack $\mathfrak{X}_\Gamma$) form an abelian group which can be identified with $H^2(\mathfrak{X}_\Gamma, \mathcal{S}^1)$ [9,10]. Unlike the manifold case, this is not always isomorphic to the third integer cohomology group $H^3(\mathfrak{X}_\Gamma, \mathbb{Z})$. Indeed, this fails to be the case even when Γ is a non-compact group. Not enough attention seems to have been paid to this in the literature. However, for a proper Lie groupoid Γ, these two groups are always isomorphic, and it therefore makes sense to talk about its K-theory twisted by an integer class in $H^3(\mathfrak{X}_\Gamma, \mathbb{Z})$. In particular, when a Lie group G acts on a smooth manifold M properly, one can define the equivariant K-theory twisted by an element in $H^3_G(M, \mathbb{Z})$. The same situation applies to orbifolds since their corresponding groupoids are always proper.

Our approach in developing twisted K-theory is to utilize operator algebras, where many sophisticated K-theoretic techniques have been developed. An S^1-central extension $S^1 \to R \to \Gamma \rightrightarrows M$ of groupoids gives rise to an S^1-gerbe $\mathfrak{R}$ over the differentiable stack $\mathfrak{X}_\Gamma$ associated to the groupoid Γ [9], and Morita equivalent S^1-central extensions correspond to isomorphic gerbes. Therefore, given a Lie groupoid $\Gamma \rightrightarrows M$, one may identify an S^1-gerbe over the stack $\mathfrak{X}_\Gamma$ as a Morita equivalence class of S^1-central extensions $S^1 \to R' \to \Gamma' \rightrightarrows M'$, where $\Gamma' \rightrightarrows M'$ is a Lie groupoid Morita equivalent to $\Gamma \rightrightarrows M$. Given an S^1-central extension of Lie groupoids $S^1 \to R \to \Gamma \rightrightarrows M$, its associated complex line bundle $L = R \times_{S^1} \mathbb{C}$ can be considered as a Fell bundle of C^*-algebras over the groupoid $\Gamma \rightrightarrows M$. Therefore, from this one can construct the reduced C^*-algebra $C_r^*(\Gamma, R)$. The K-groups are simply defined to be the K-groups of this C^*-algebra.

This definition yields several advantages. First, since it is standard that Morita equivalent central extensions yield Morita equivalent C^*-algebras, the K-groups indeed only depend on the stack and the S^1-gerbe, instead of on any particular groupoid S^1-central extension. Such a viewpoint is quite interesting already, even when dealing with untwisted K-theory. For instance, some classical results of Segal on equivariant K-theory [65] may be reinterpreted as a consequence of the fact that equivariant K-theory only depends on the stack M/G, i.e. the Morita equivalence class of the transformation groupoid $G \times M \rightrightarrows M$. Secondly, important properties of K-theory, such as the Mayer–Vietoris property and Bott periodicity, are immediate consequences of this definition.

A drawback of this definition, however, is that it is too abstract and algebraic. Our second goal in this paper is to connect it with the usual topological approach of K-theory in terms of Fredholm bundles [3,66]. As in the manifold case, an S^1-central extension of a groupoid naturally gives rise to a canonical principal $PU(\mathbb{H})$-bundle over the groupoid, which in turn induces associated Fredholm bundles over the groupoid. We show that the K-groups can be interpreted as homotopy classes of invariant sections of these Fredholm bundles (assuming a certain appropriate continuity). This picture fits with the usual definition of twisted K-theory [4] when the groupoid reduces to a space.

Geometrically, it is always desirable to describe K-groups in terms of vector bundles. For twisted K-groups, a natural candidate is to use twisted vector bundles over the groupoid. This is a natural generalization, in the context of groupoids, of projective representations of a group. More precisely, given an S^1-central extension of Lie groupoids $S^1 \to R \xrightarrow{\pi} \Gamma \rightrightarrows M$, a twisted vector bundle is a vector bundle E over the groupoid R where $\ker \pi \cong M \times S^1$ acts on E by scalar multiplication. When Γ is a groupoid Morita equivalent to a manifold, they correspond to the so-called bundle gerbe modules in [16]. However, note that twisted vector bundles do not always exist. In fact, a necessary condition for their existence is that the twisted class $\alpha \in H^2(\Gamma^\bullet, \mathcal{S}^1)$ must be a torsion. Another main theme of this paper is to explore the conditions under which the twisted K_0-group is isomorphic to the Grothendieck group of twisted vector bundles.

As is already the case for manifolds, twisted K-groups no longer admit a ring structure [4]. It is expected, however, that there exists a bilinear product $K^i_\alpha \otimes K^j_\beta \to K^{i+j}_{\alpha+\beta}$. For twisted vector bundles, such a product is obvious and corresponds to the tensor product of vector bundles. However, in general, twisted K-groups cannot be expressed by twisted vector bundles as discussed above. The main difficulty in constructing such a product using the Fredholm picture of twisted K-theory is obtaining a Fredholm operator T out of two Fredholm operators T_1 and T_2. This is very similar to the situation of the Kasparov product in KK-theory where a non-constructive method must be used. Motivated by KK-theory, our approach is to develop a generalized version of Le Gall's groupoid equivariant KK-theory and interpret our twisted K-groups as such KK-groups, which allows us to obtain such a product.

The paper is organized as follows. Section 2 is devoted to the basic theory of S^1-gerbes over stacks in terms of the groupoid picture; related cohomology theory and characteristic classes are reviewed briefly. In Section 3, we introduce the definition of twisted K-groups and outline some basic properties. Section 4 is devoted to the study of the K-groups of the C^*-algebra of a Fell bundle over a proper groupoid, which includes our C^*-algebras of groupoid S^1-central extensions as a special case. In particular, we give the Fredholm picture of the K-groups. In Section 5, we investigate the conditions under which the twisted K_0-group can be expressed in terms of twisted vector bundles. In Section 6, we discuss the construction of the K-group product as outlined above. In Appendix A, we review some basic material concerning Fell bundles over groupoids which we use frequently in the paper.

We would like to point out that there are many interesting and important questions that we are not able to address in this paper. One of them is the study of the Chern character in twisted K-theory, in which Connes' noncommutative differential geometry [17] will play a prominent role due to the nature of our algebraic definition. This subject will be discussed in a separate paper.

Finally, we note that after our paper was submitted a paper by Atiyah and Segal [5] appeared, in which twisted equivariant K-theory (for a compact group acting on a space) was introduced independently using a different method. It is not hard to check that at least in our case of interest, that of a compact Lie group acting on a manifold, our twisted K-theory coincides with theirs (using the remark in Appendix A.1 of [5] that, in the metrizable case, the compact-open topology is the same as the strong topology).

Notations. Finally, we list the notation used throughout the paper. Γ will denote a groupoid (all groupoids considered are Hausdorff, locally compact, and second countable). We denote by s and t the source and target maps of Γ, respectively. $\Gamma^{(0)}$ will denote the unit space of Γ, and $\Gamma^{(n)}$ will denote the set of strings of length n

$$g_1 \leftarrow g_2 \leftarrow \cdots \leftarrow g_n,$$

i.e., the set of n-tuples $(g_1,\ldots,g_n)\in\Gamma\times\cdots\times\Gamma$ such that $s(g_i)=t(g_{i+1})$ for all $i=1,\ldots,n-1$.

We will commonly use the expression "Let $\Gamma \rightrightarrows M$ be a Lie groupoid ..." to indicate that Γ is a Lie groupoid and $\Gamma^{(0)}=M$.

For all K, $L\subset\Gamma^{(0)}$, we let $\Gamma_K=s^{-1}(K)$, $\Gamma^L=t^{-1}(L)$, and $\Gamma_K^L=\Gamma_K\cap\Gamma^L$. If $K=\{x\}$ and $L=\{y\}$, we will use the notation Γ_x, Γ^y, and Γ_x^y, respectively.

If Y is a space, then $Y\times Y$ will be endowed with the pair groupoid structure: $(Y\times Y)^{(0)}=Y$, $s(y_1,y_2)=y_2$, $t(y_1,y_2)=y_1$, and $(y_1,y_2)(y_2,y_3)=(y_1,y_3)$.

If Y is a space and $f:Y\to\Gamma^{(0)}$ is a map, we denote by $\Gamma[Y]$ the subgroupoid of $(Y\times Y)\times\Gamma$ consisting of $\{(y_1,y_2,\gamma)\mid\gamma\in\Gamma_{f(y_2)}^{f(y_1)}\}$. Then $\Gamma[Y]$ is called the pullback of the groupoid Γ by f.

In particular, if $\mathcal{U}=(U_i)$ is an open cover of $\Gamma^{(0)}$, then the pullback of Γ by the canonical map $\coprod U_i\to\Gamma^{(0)}$ is denoted either by $\Gamma[\mathcal{U}]$ or by $\Gamma[U_i]$.

If $\Gamma\rightrightarrows M$ is a locally compact groupoid (resp., a Lie groupoid), a Haar system for Γ will usually be denoted by $\lambda=(\lambda^x)_{x\in M}$, where λ^x is a measure with support Γ^x such that for all $f\in C_c(\Gamma)$ (resp., $C_c^\infty(\Gamma)$), $x\mapsto\int_{g\in\Gamma^x}f(g)\,\lambda^x(dg)$ is continuous (resp., smooth).

Let $\mathbb{H}$ be the separable Hilbert space. We denote by $\mathcal{K}(\mathbb{H})$, or even $\mathcal{K}$, the algebra of compact operators on $\mathbb{H}$; we denote by $\mathcal{L}(\mathbb{H})$ the algebra of linear bounded operators.

For a C^*-algebra A, $M(A)$ denotes its multiplier algebra [58, Section 3.12]. Recall that $M(A)$ is a unital C^*-algebra containing A as an essential ideal, and, moreover, if a C^*-algebra B also contains A as an essential ideal, then $A\subset B\subset M(A)$. For instance, if X is a locally compact space and $A=C_0(X)$, then $M(A)=C_b(X)$ is the space of continuous bounded functions on X. On the other hand, if $A=\mathcal{K}$, then $M(A)=\mathcal{L}(\mathbb{H})$.

For a Hilbert C^*-module $\mathcal{E}$ over A (see [72]), we denote by $\mathcal{L}(\mathcal{E})$ the algebra of (A-linear bounded) adjointable operators on $\mathcal{E}$. For all $\xi,\eta\in\mathcal{E}$, let $T_{\xi,\eta}$ be the operator $T_{\xi,\eta}(\zeta)=\xi\langle\eta,\zeta\rangle$. Then $T_{\xi,\eta}$ is called a rank-one operator. The closed linear span of rank-one operators is called the algebra of compact operators on $\mathcal{E}$ and will be denoted by $\mathcal{K}(\mathcal{E})$; this is an ideal of $\mathcal{L}(\mathcal{E})$.

We gather below the most frequently used notations and terminology:

$\mathcal{C}(\mathcal{E})$ .... Eq. (31)
$C_r^*(\Gamma)$ [reduced C^*-algebra of a groupoid] .... Ref. [61]
$C_r^*(\Gamma;E)$ [reduced C^*-algebra of a Fell bundle] .... Section A.3
$C_r^*(\Gamma;R)$ [C^*-algebra of an S^1-central extension] .... Definition 3.1
$C_r^*(R)^{S^1}$ .... Eq. (24)
S^1-Central extension .... Definition 2.7
G-bundle over a groupoid .... Definition 2.33
Γ-space .... Definition 2.32
Generalized homomorphism .... Definition 2.1

2. S^1-gerbes and central extensions of groupoids

2.1. Generalized homomorphisms

In this subsection, we will review some basic facts concerning generalized homomorphisms. Here groupoids are assumed to be Lie groupoids although most of the discussions can be easily adapted to general locally compact groupoids. Let us recall the definition below [33,38,51].

DEFINITION 2.1. – A generalized groupoid homomorphism from Γ to G is given by a manifold Z, two smooth maps

$$\Gamma^{(0)} \xleftarrow{\tau} Z \xrightarrow{\sigma} G^{(0)},$$

a left action of Γ with respect to τ, a right action of G with respect to σ, such that the two actions commute, and Z is a locally trivial G-principal bundle over $\Gamma^{(0)} \xleftarrow{\tau} Z$.

To explain the terminology, if $f : \Gamma \to G$ is a strict homomorphism (i.e. a smooth map satisfying $f(gh) = f(g)f(h)$) then $Z_f = \Gamma^{(0)} \times_{f, G^{(0)}, t} G$, with $\tau(x, g) = x$, $\sigma(x, g) = s(g)$, and the actions $\gamma \cdot (x, g) = (t(\gamma), f(\gamma)g)$ and $(x, g) \cdot g' = (x, gg')$, is a generalized homomorphism from Γ to G.

Generalized homomorphisms can be composed just like the usual groupoid homomorphisms.

PROPOSITION 2.2. – *Let Z and Z' be generalized homomorphisms from $\Gamma \rightrightarrows \Gamma^{(0)}$ to $G \rightrightarrows G^{(0)}$, and from $G \rightrightarrows G^{(0)}$ to $H \rightrightarrows H^{(0)}$ respectively. Then*

$$Z'' = Z \times_G Z' := (Z \times_{\sigma, G^{(0)}, \tau'} Z')/_{(z, z') \sim (zg, g^{-1}z')}$$

is a generalized groupoid homomorphism from $\Gamma \rightrightarrows \Gamma^{(0)}$ to $H \rightrightarrows H^{(0)}$. Moreover, the composition of generalized homomorphisms is associative, and thus there is a category $\mathcal{G}$ whose objects are Lie groupoids and morphisms are isomorphism classes of generalized homomorphisms. [2] *There is a functor*

$$\mathcal{G}_s \to \mathcal{G}$$

where $\mathcal{G}_s$ is the category of Lie groupoids with strict homomorphisms given by $f \mapsto Z_f$ as described above.

Proof. – All the assertions are easy to check. For instance, to show that $\Gamma^{(0)} \leftarrow Z''$ is a locally trivial H-principal bundle, note that Z and Z' are locally of the form $Y \times_{G^{(0)}} G$ and $Y' \times_{H^{(0)}} H$ respectively. Therefore Z'' is locally of the form $Y'' \times_{G^{(0)}} H$ where $Y'' = Y \times_{H^{(0)}} Y'$. $\square$

[2] Two generalized homomorphisms Z_1 and Z_2 are isomorphic whenever they are Γ, G-equivariantly diffeomorphic.

Note that isomorphism in the category $\mathcal{G}$ is just Morita equivalence [56,77].

PROPOSITION 2.3 (see [38, Definition 1.1]). – *Any generalized homomorphism*

$$\Gamma^{(0)} \xleftarrow{\tau} Z \xrightarrow{\sigma} G^{(0)}$$

is obtained by composition of the canonical Morita equivalence between Γ *and* $\Gamma[U_i]$, *where* (U_i) *is an open cover of* $\Gamma^{(0)}$, *with a strict homomorphism* $\Gamma[U_i] \to G$.

Consequently, giving a generalized homomorphism $\Gamma \to G$ is equivalent to giving a Morita equivalence $\Gamma \sim_{morita} \Gamma'$ together with a strict homomorphism $\Gamma' \to G$.

Proof. – Denoting by $\Gamma[Z]$ the pull-back of Γ via the surjective submersion $Z \xrightarrow{\tau} \Gamma^{(0)}$, i.e. the groupoid $Z \times_{\Gamma^{(0)},t} \Gamma \times_{\Gamma^{(0)},s} Z$ with multiplication law $(z_1, g, z_2)(z_2, h, z_3) = (z_1, gh, z_3)$. Then the canonical strict homomorphism $\Gamma[Z] \to \Gamma$ is a Morita equivalence.

Moreover, it is not hard to check that

$$\Gamma[Z] \cong \big\{(z, z', \gamma, g) \in (Z \times Z) \times \Gamma \times G \mid \gamma z' = zg\big\}.$$

Thus there is a strict homomorphism $f : \Gamma[Z] \to G$ given by the fourth projection. One can then verify that the following diagram is commutative (in the category $\mathcal{G}$):

$$\begin{tikzcd} \Gamma \arrow[dr, "Z"'] & \Gamma[Z] \arrow[l, "\cong"'] \arrow[d, "f"] \\ & G \end{tikzcd}$$

Now, since $Z \to \Gamma^{(0)}$ is a submersion, it admits local sections. Hence there exists an open cover (U_i) of $\Gamma^{(0)}$ and maps $s_i : U_i \to Z$ such that $\tau \circ s_i = \mathrm{Id}$, and therefore a map $\widetilde{s} : \Gamma[U_i] \to \Gamma[Z]$ such that the composition $\Gamma[U_i] \to \Gamma[Z] \to \Gamma$ is the canonical map. Then, $f \circ \widetilde{s} : \Gamma[U_i] \to G$ is the desired strict homomorphism. $\square$

LEMMA 2.4. – *Let* f_1, $f_2 : \Gamma \to G$ *be two strict homomorphisms. Then* f_1 *and* f_2 *define the isomorphic generalized homomorphisms if and only if there exists a smooth map* $\varphi : \Gamma^{(0)} \to G$ *such that* $f_2(\gamma) = \varphi(t(\gamma)) f_1(\gamma) \varphi(s(\gamma))^{-1}$.

Proof. – Suppose that there exists a smooth Γ, G-equivariant map $Z_{f_1} \to Z_{f_2}$. Then it is necessarily of the form $(x, g) \mapsto (x, \varphi(x) g)$. Using Γ-equivariance, we get

$$\big(t(\gamma), \varphi\big(t(\gamma)\big) f_1(\gamma)\big) = \big(t(\gamma), f_2(\gamma) \varphi\big(s(\gamma)\big)\big).$$

The converse is proved by working backwards. $\square$

The following result is useful when dealing with generalized homomorphisms (see also [29]).

PROPOSITION 2.5. – *Let* $\mathcal{C}$ *be a category, and* $\Phi : \mathcal{G}_s \to \mathcal{C}$ *be a functor. The following are equivalent*:

(i) *For every smooth groupoid* Γ *and every open cover* (U_i), $\Phi(\pi)$ *is an isomorphism, where* π *is the canonical map* $\Gamma[U_i] \to \Gamma$.

(ii) *The functor* Φ *factors through the category* $\mathcal{G}$ (*and thus* $\Phi(G) \cong \Phi(H)$ *if* G *and* H *are Morita equivalent*).

Proof. – The only non-trivial implication is (i) $\Rightarrow$ (ii). Let $\Gamma^{(0)} \leftarrow Z \rightarrow G^{(0)}$ be a generalized homomorphism. From Proposition 2.3, there exists a strict homomorphism $f:\Gamma[U_i] \rightarrow G$ which is the same morphism in the category $\mathcal{G}$. We define $\Phi(Z):\Phi(\Gamma) \rightarrow \Phi(G)$ to be the composition

$$\Phi(\Gamma) \overset{\cong}{\leftarrow} \Phi\big(\Gamma[U_i]\big) \overset{\Phi(f)}{\rightarrow} \Phi(G).$$

To check that this is well-defined, suppose that $f_1:\Gamma[U_i] \rightarrow G$ and $f_2:\Gamma[V_j] \rightarrow G$ define the same generalized homomorphism. We need to show that $\Phi(f_1) = \Phi(f_2)$ via the identification $\Phi(\Gamma[U_i]) = \Phi(\Gamma[V_j])$. Using the cover $(U_i \cap V_j)$, we may assume that $(U_i) = (V_j)$, and f_1, f_2 are strict homomorphisms from Γ to G.

From Lemma 2.4, there exists $\varphi:\Gamma^{(0)} \rightarrow \Gamma$ such that

$$f_2(\gamma) = \varphi\big(t(\gamma)\big) f_1(\gamma) \varphi\big(s(\gamma)\big)^{-1}.$$

Let $\widetilde{\Gamma} = \{1,2\}^2 \times \Gamma$ and let $\widetilde{f}:\widetilde{\Gamma} \rightarrow G$ be the morphism

$$\begin{aligned}
&(1,1,\gamma) \mapsto f_1(\gamma),\\
&(1,2,\gamma) \mapsto f_1(\gamma)\varphi\big(s(\gamma)\big)^{-1},\\
&(2,1,\gamma) \mapsto \varphi\big(t(\gamma)\big) f_1(\gamma),\\
&(2,2,\gamma) \mapsto \varphi\big(t(\gamma)\big) f_1(\gamma)\varphi\big(s(\gamma)\big)^{-1}.
\end{aligned}$$

Let $i_j:\Gamma \rightarrow \widetilde{\Gamma}$ be defined by $i_j(\gamma) = (j,j,\gamma)$ and $\pi:\widetilde{\Gamma} \rightarrow \Gamma$ be the map $\pi(i,j,\gamma) = \gamma$. Since $\widetilde{\Gamma} = \Gamma[W_k]$ with $W_1 = W_2 = \Gamma^{(0)}$, $\Phi(\pi)$ is an isomorphism. Now, from $\pi \circ i_1 = \pi \circ i_2$, we get $\Phi(i_1) = \Phi(\pi)^{-1} = \Phi(i_2)$, and therefore $\Phi(f_1) = \Phi(\widetilde{f} \circ i_1) = \Phi(\widetilde{f} \circ i_2) = \Phi(f_2)$. $\square$

Remark 2.6. – Given two Lie groupoids $\Gamma_i \rightrightarrows \Gamma_i^{(0)}$, $i = 1,2$, a generalized homomorphism from $\Gamma_1 \rightrightarrows \Gamma_1^{(0)}$ to $\Gamma_2 \rightrightarrows \Gamma_2^{(0)}$ induces a morphism between their associated differential stacks $\mathfrak{X}_1 \rightarrow \mathfrak{X}_2$, and vice versa. And a generalized isomorphism, i.e. a Morita equivalence, corresponds to an isomorphism of stacks. Therefore the category $\mathcal{G}$ and the category of differentiable stacks are equivalent categories (see [10] for details).

2.2. S^1-central extensions of groupoids

DEFINITION 2.7. – Let $\Gamma \rightrightarrows M$ be a Lie groupoid. An *S^1-central extension* (*or "twist"*) of $\Gamma \rightrightarrows M$ consists of

(1) a Lie groupoid $R \rightrightarrows M$, together with a morphism of Lie groupoids

$$(\pi, \mathrm{id}):[R \rightrightarrows M] \rightarrow [\Gamma \rightrightarrows M]$$

which restricts to the identity on M,

(2) a left S^1-action on R, making $\pi:R \rightarrow \Gamma$ a (left) principal S^1-bundle. These two structures are compatible in the sense that $(\lambda_1 \cdot x)(\lambda_2 \cdot y) = \lambda_1\lambda_2 \cdot (xy)$, for all $\lambda_1, \lambda_2 \in S^1$ and $(x,y) \in R^{(2)} = R \times_{s,M,t} R$.

We denote by $Tw^{sm}(\Gamma)$ the set of S^1-central extensions of Γ (the superscript "*sm*" stands for "smooth").

Note that R being restricted to $\epsilon_0(M)$ is a trivial S^1-bundle, where $\epsilon_0:M \rightarrow \Gamma$ is the unit map. In fact, it admits a canonical trivialization since $R|_{\epsilon_0(M)}$ admits a smooth section, namely, the base space of the groupoid R. By $\ker\pi$, we denote this trivial bundle $R|_{\epsilon_0(M)}$, i.e., $\ker\pi\cong M\times S^1$. It is clear that $\ker\pi$, as a bundle of groups, is a normal subgroupoid of $R\rightrightarrows M$, and lies in the center. Indeed its quotient groupoid is isomorphic to $\Gamma\rightrightarrows M$. This coincides with the usual definition of Lie group S^1-central extensions.

When $\pi:R\to\Gamma$ is topologically trivial (for instance, this is true if as a space the 2nd cohomology of Γ vanishes), then $R\cong\Gamma\times S^1$ and the central extension is determined by a groupoid 2-cocycle of Γ valued in S^1, i.e., a smooth map

$$c:\Gamma^{(2)}=\big\{(x,y)\mid s(x)=t(y),x,y\in\Gamma\big\}\to S^1$$

satisfying the cocycle condition:

$$c(x,y)c(xy,z)c(x,yz)^{-1}c(y,z)^{-1}=1,\quad \forall(x,y,z)\in\Gamma^{(3)}. \tag{1}$$

The groupoid structure on R is given by

$$(x,\lambda_1)\cdot(y,\lambda_2)=\big(xy,\lambda_1\lambda_2c(x,y)\big),\quad \forall(x,y)\in\Gamma^{(2)}. \tag{2}$$

For every locally compact groupoid Γ with a Haar system (thus for every Lie groupoid), Kumjian, Muhly, Renault and Williams [43] constructed a group, called the Brauer group of Γ. Some of the constructions below is an easy adaptation of their results to our context, so we will omit most of proofs.

Note that $Tw^{sm}(\Gamma)$ admits an abelian group structure in a canonical way: if $S^1\to R\xrightarrow{\pi}\Gamma\rightrightarrows M$ and $S^1\to R'\xrightarrow{\pi'}\Gamma\rightrightarrows M$ are S^1-central extensions, then the addition of R and R', called the *tensor of R and R'* and denoted by $R\otimes R'$, is

$$(R\times_\Gamma R')/S^1:=\big\{(r,r')\in R\times_\Gamma R'\big\}/_{(r,r')\sim(\lambda r,\lambda^{-1}r')} \tag{3}$$

$(\lambda\in S^1)$, and the inverse of R is $\bar R$ (where the action of S^1 on $\bar R$ is $\lambda\bar r=\overline{\bar\lambda r}$ and $\bar r\in\bar R$ denotes the same element $r\in R$).

The zero element is the strictly trivial extension, i.e., the extension satisfying the following equivalent conditions.

PROPOSITION 2.8. – *Let $S^1\to R\xrightarrow{\pi}\Gamma\rightrightarrows M$ be an S^1-central extension. The following are equivalent*:

(i) *there exists a groupoid homomorphism $\sigma:\Gamma\to R$ such that $\pi\circ\sigma=\mathrm{Id}$*;
(ii) *there exists an S^1-equivariant groupoid homomorphism $\varphi:R\to S^1$*;
(iii) $R\cong\Gamma\times S^1$ (*as a product of groupoids*).

Proof. – (i) ⇒ (ii): take $\varphi(r)=r(\sigma\circ\pi(r))^{-1}$.
(ii) ⇒ (iii): the map $r\mapsto(\pi(r),\varphi(r))$ is a groupoid isomorphism from R to $\Gamma\times S^1$.
(iii) ⇒ (i): obvious. □

The set of S^1-central extensions of Γ of the form $R=(t^*\Lambda\times\overline{s^*\Lambda})/S^1\rightrightarrows M$, where Λ is an S^1-principal bundle on M, is a subgroup of $Tw^{sm}(\Gamma)$. The quotient of $Tw^{sm}(\Gamma)$ by this subgroup is denoted by $\mathcal{E}^{sm}(\Gamma)$.

We now introduce the definition of Morita equivalence of S^1-central extensions, and define an abelian group structure on the set of Morita equivalence classes of extensions $S^1\to R'\rightrightarrows\Gamma'$, with Γ' Morita equivalent to Γ.

DEFINITION 2.9. – Let $S^1 \to R \to \Gamma \rightrightarrows M$ and $S^1 \to R' \to \Gamma' \rightrightarrows M'$ be S^1-central extensions. We say that a generalized homomorphism $M \leftarrow Z \to M'$ from R to R' is S^1-equivariant if Z is endowed with an action of S^1 such that

$$(\lambda r) \cdot z \cdot r' = r \cdot (\lambda z) \cdot r' = r \cdot z \cdot (\lambda r')$$

whenever $(\lambda, r, r', z) \in S^1 \times R \times R' \times Z$ and the products make sense.

LEMMA 2.10. – *Let $S^1 \to R \to \Gamma \rightrightarrows M$ and $S^1 \to R' \to \Gamma' \rightrightarrows M'$ be S^1-central extensions, and $M \leftarrow Z \to M'$ an S^1-equivariant generalized homomorphism from R to R'. Then the S^1-action on Z is free and $M \leftarrow Z/S^1 \to M'$ defines a generalized homomorphism from Γ to Γ'.*

Proof. – Assume that $\lambda z = z$ for $\lambda \in S^1$ and $z \in Z$. From the compatibility condition $(\lambda z) \cdot r' = z \cdot (\lambda r')$ and the fact that the R'-action on Z is free, we obtain that $\lambda r' = r'$. Hence $\lambda = 1$ since the S^1-action on R' is free. The rest of the assertion follows immediately from the compatibility condition again. □

DEFINITION 2.11. – Two S^1-central extensions $S^1 \to R \to \Gamma \rightrightarrows M$ and $S^1 \to R' \to \Gamma' \rightrightarrows M'$ are called *Morita equivalent* if there is a generalized S^1-equivariant isomorphism $M \leftarrow Z \to M'$. In this case, Z is called an *equivalence bimodule*.

As an immediate consequence of Lemma 2.10, in particular, if $S^1 \to R \to \Gamma \rightrightarrows M$ and $S^1 \to R' \to \Gamma' \rightrightarrows M'$ are Morita equivalent S^1-central extensions, then Γ and Γ' must be Morita equivalent groupoids.

The following result gives a useful construction of S^1-equivariant generalized homomorphism, and in particular shows that for two Morita equivalent S^1-central extensions, one may recover one from the other in terms of the equivalence bimodule.

Let $S^1 \to R \to \Gamma \rightrightarrows M$ be an S^1-central extension, and $\tau : Z \to M$ a left principal R-bundle over $M' := Z/R$. Then Z admits an S^1-action defined as follows: for all $\lambda \in S^1$ and $z \in Z$, denote by $\lambda_z \in R$ the element $\lambda\tau(z)$, where $\tau(z) \in R^{(0)}$ is considered as an element of R. We let

$$\lambda \cdot z = \lambda_z \cdot z.$$

It follows from the properties of S^1-central extensions that this indeed defines an S^1-action. Moreover, by assumption, this action is free and therefore Z/S^1 is a smooth manifold, which is denoted by X. It is simple to see that the following identity holds:

$$(\lambda r) \cdot z = r \cdot (\lambda z) \quad \forall (\lambda, r, z) \in S^1 \times R \times Z \text{ with } s(r) = \tau(z).$$

PROPOSITION 2.12. – *As above, assume that $S^1 \to R \to \Gamma \rightrightarrows M$ is an S^1-central extension, and $\tau : Z \to M$ a principal R-bundle over $M' := Z/R \cong X/\Gamma$, where $X = Z/S^1$. Let $\Gamma' = X \times_\Gamma X$ and $R' = Z \times_R Z$. Then*

(i) *$R' \to \Gamma' \rightrightarrows M'$ is an S^1-central extension of groupoids, and $M' \leftarrow Z \to M$ with the natural actions defines an S^1-equivariant generalized homomorphism from R' to R.*

(ii) *If moreover $\tau : Z \to M$ is a surjective submersion, then R' and R are Morita equivalent S^1-central extensions.*

As a consequence, if Z is an S^1-equivariant Morita equivalence bimodule from R to R'', then $R'' \cong R'$.

Proof. – The proof is a straightforward verification, and is similar to [77, Theorem 3.2-3.3]. □

Given a Lie groupoid $\Gamma \rightrightarrows M$, there is a natural abelian group structure on the set of Morita equivalence classes of S^1-central extensions $S^1 \to R' \to \Gamma' \rightrightarrows M'$, where $\Gamma' \rightrightarrows M'$ is a Lie groupoid Morita equivalent to $\Gamma \rightrightarrows M$. To see this, assume that $S^1 \to R_i \to \Gamma_i \rightrightarrows M_i$, $i = 1, 2$, are two such extensions. Since $\Gamma_i \rightrightarrows M_i$, $i = 1, 2$, are Morita equivalent, there exists a generalized isomorphism $M_1 \leftarrow X \to M_2$. By pulling back using the above maps, one obtains two S^1-central extensions over the groupoid $\Gamma_1[X] \cong \Gamma_2[X] \rightrightarrows X$, namely $R_1[X] \rightrightarrows X$ and $R_2[X] \rightrightarrows X$. Thus one may define $[R_1] + [R_2]$ to be the class of the S^1-central extension $R_1[X] \otimes R_2[X]$ (see Eq. (3)). It is simple to check that this operation is well-defined. The inverse is defined by $\overline{[R]} = [\bar{R}]$. Let us denote by $\mathrm{Ext}^{sm}(\Gamma, S^1)$ the group thus obtained.

The zero element in $\mathrm{Ext}^{sm}(\Gamma, S^1)$ is characterized by the following

PROPOSITION 2.13. – *Consider an S^1-central extension of Lie groupoids $S^1 \to R \to \Gamma \rightrightarrows M$. The following are equivalent*:

(i) *there exists an S^1-equivariant generalized homomorphism $R \to S^1$;*

(ii) *there exists a cover (U_i) of M such that the extension $S^1 \to R[U_i] \to \Gamma[U_i] \rightrightarrows \coprod U_i$ is strictly trivial*;

(iii) *the extension is Morita equivalent to a strictly trivial S^1-central extension $0 \to S^1 \to \Gamma' \times S^1 \to \Gamma' \rightrightarrows M'$;*

(iii)′ *the class of the extension in* $\mathrm{Ext}^{sm}(\Gamma, S^1)$ *is* 0;

(iv) *the extension is Morita equivalent to the strictly trivial S^1-central extension $0 \to S^1 \to \Gamma \times S^1 \to \Gamma \rightrightarrows M$;*

(iv)′ *the class of the extension in* $\mathcal{E}^{sm}(\Gamma)$ *is* 0.

Proof. – (ii) ⇒ (iii) ⇒ (iii)′ ⇒ (i) and (iv)′ ⇒ (iv) ⇒ (iii) are obvious. (i) ⇒ (ii) is a consequence of Propositions 2.8 and 2.3. To show (iii) ⇒ (iv), let Z be an equivalence bimodule between Γ and Γ', then $Z \times S^1$ is obviously an equivalence bimodule between the trivial central extensions $S^1 \to \Gamma \times S^1 \to \Gamma \rightrightarrows M$ and $S^1 \to \Gamma' \times S^1 \to \Gamma' \rightrightarrows M'$. □

S^1-extensions which satisfy any of the conditions in the previous proposition are said to be trivial. Therefore, $\mathcal{E}^{sm}(\Gamma)$ is the quotient $Tw^{sm}(\Gamma)$ by trivial extensions. Thus two S^1-central extensions $S^1 \to R_i \to \Gamma \rightrightarrows M$ are equal in $\mathcal{E}^{sm}(\Gamma)$ if and only if they are Morita equivalent.

The groups $\mathcal{E}^{sm}(\Gamma')$, where Γ' is a groupoid Morita equivalent to Γ, form an inductive system. It follows from Proposition 2.13 that

$$\mathrm{Ext}^{sm}(\Gamma, S^1) \cong \varinjlim_{\Gamma' \sim \Gamma} \mathcal{E}^{sm}(\Gamma') \cong \varinjlim_{\mathcal{U}} \mathcal{E}^{sm}\big(\Gamma[\mathcal{U}]\big),$$

where $\mathcal{U}$ runs over open covers of M.

Remark 2.14. – An S^1-central extension $S^1 \to R \to \Gamma \rightrightarrows M$ gives rise to an S^1-gerbe $\mathfrak{R}$ over the differentiable stack $\mathfrak{X}_\Gamma$ associated to the groupoid $\Gamma \rightrightarrows M$ [9,10], and Morita equivalent S^1-central extensions correspond to isomorphic gerbes.

Conversely, given an S^1-gerbe $\mathfrak{R} \xrightarrow{\pi} \mathfrak{X}$ over a differential stack $\mathfrak{X}$, if $R \rightrightarrows M$ and $R' \rightrightarrows M'$ are the Lie groupoids corresponding to the presentations $M \to \mathfrak{R}$ and $M' \to \mathfrak{R}$ of $\mathfrak{R}$ respectively, and $\Gamma \rightrightarrows M$ and $\Gamma' \rightrightarrows M'$ are the Lie groupoids corresponding to the induced presentations $M \to \mathfrak{X}$ and $M' \to \mathfrak{X}$ of $\mathfrak{X}$ respectively, then the S^1-central extensions $S^1 \to R \to \Gamma \rightrightarrows M$ and $S^1 \to R' \to \Gamma' \rightrightarrows M'$ are Morita equivalent [9,10]. The equivalence bimodule is $Z = M \times_{\mathfrak{R}} M'$, which is a principal S^1-bundle over $M \times_{\mathfrak{X}} M'$.

Therefore, given a Lie groupoid $\Gamma \rightrightarrows M$, one may identify an S^1-gerbe over the stack $\mathfrak{X}_\Gamma$ as an element in $\mathrm{Ext}^{sm}(\Gamma, S^1)$, i.e. a Morita equivalence class of S^1-central extensions $S^1 \to R' \to \Gamma' \rightrightarrows M'$, where $\Gamma' \rightrightarrows M'$ is a Lie groupoid Morita equivalent to $\Gamma \rightrightarrows M$. We will call such a Morita equivalence class an isomorphism class of S^1-gerbes by abuse of notations. Moreover, the group structure on $\mathrm{Ext}^{sm}(\Gamma, S^1)$ corresponds to the abelian group structure on the S^1-gerbes over $\mathfrak{X}_\Gamma$. Therefore, one may simply identify these two groups.

2.3. Cohomology and characteristic classes

In this subsection, we briefly review some basic cohomology theory of groupoids, which will be needed later in the paper.

There exist many equivalent ways of introducing cohomology groups associated to a Lie groupoid $\Gamma \rightrightarrows \Gamma^{(0)}$ [9,10,20]. A simple and geometric way is to consider the simplicial manifold canonically associated to the groupoid and apply the usual cohomology theory. More precisely, let $\Gamma \rightrightarrows \Gamma^{(0)}$ be a Lie groupoid. Define for all $p \geqslant 0$

$$\Gamma^{(p)} = \underbrace{\Gamma \times_{\Gamma^{(0)}} \cdots \times_{\Gamma^{(0)}} \Gamma}_{p \text{ times}},$$

i.e., $\Gamma^{(p)}$ is the manifold of composable sequences of p arrows in the groupoid $\Gamma \rightrightarrows \Gamma^{(0)}$. We have $p+1$ canonical maps $\Gamma^{(p)} \to \Gamma^{(p-1)}$ giving rise to a diagram

$$\cdots \Gamma^{(2)} \rightrightarrows\!\!\!\!\to \Gamma^{(1)} \rightrightarrows \Gamma^{(0)}. \tag{4}$$

In fact, $\Gamma^\bullet$ is a simplicial manifold, so one can introduce (singular) cohomology groups $H^k(\Gamma^\bullet, \mathbb{Z})$, $H^k(\Gamma^\bullet, \mathbb{R})$ and $H^k(\Gamma^\bullet, \mathbb{R}/\mathbb{Z})$. We refer the reader to [24] for the detailed study of cohomology of simplicial manifolds. In fact, for any abelian sheaf F on the category of differentiable manifolds, we have the cohomology groups $H^k(\Gamma^\bullet, F)$ [2,8,10]. One way to define them is by choosing for every p an injective resolution $F^p \to I^{p\bullet}$ of sheaves on $\Gamma^{(p)}$, where F^p is the small sheaf induced by F on $\Gamma^{(p)}$; then choosing homomorphisms $f^* I^{p-1\bullet} \to I^{p\bullet}$ for every map $f : \Gamma^{(p)} \to \Gamma^{(p-1)}$ in (4). This gives rise to a double complex $I^\bullet(\Gamma^\bullet)$, whose total cohomology groups are the $H^k(\Gamma^\bullet, F)$. Examples of abelian sheaves on the category of manifolds are: $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$, Ω^k, $\mathcal{R}$ and $\mathcal{S}^1$. The first three are sheaves of locally constant functions, $\mathcal{R}$ and $\mathcal{S}^1$ are the sheaves of differentiable $\mathbb{R}$-valued and S^1-valued functions, respectively (see [8–10]). With respect to the first three, the notation $H^k(\Gamma^\bullet, F)$ does not conflict with the notation introduced before. Note that the cohomology groups $H^k(\Gamma^\bullet, F)$ satisfy the functorial property with respect to generalized homomorphisms according to Proposition 2.5.

Another cohomology, which is relevant to us, is the De Rham cohomology. Consider the double complex $\Omega^\bullet(\Gamma^\bullet)$:

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\Omega^1(\Gamma^{(0)}) & \xrightarrow{\partial} & \Omega^1(\Gamma^{(1)}) & \xrightarrow{\partial} & \Omega^1(\Gamma^{(2)}) & \xrightarrow{\partial} & \cdots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\Omega^0(\Gamma^{(0)}) & \xrightarrow{\partial} & \Omega^0(\Gamma^{(1)}) & \xrightarrow{\partial} & \Omega^0(\Gamma^{(2)}) & \xrightarrow{\partial} & \cdots
\end{array} \tag{5}$$

Its boundary maps are $d:\Omega^k(\Gamma^{(p)})\to\Omega^{k+1}(\Gamma^{(p)})$, the usual exterior derivative of differentiable forms and $\partial:\Omega^k(\Gamma^{(p)})\to\Omega^k(\Gamma^{(p+1)})$, the alternating sum of the pull-back maps of (4). We denote the total differential by $\delta=(-1)^p d+\partial$. The cohomology groups of the total complex $C^\bullet(\Gamma^\bullet)$:

$$H^k_{DR}(\Gamma^\bullet)=H^k\big(\Omega^\bullet(\Gamma^\bullet)\big)$$

are called the *De Rham cohomology* groups of $\Gamma\rightrightarrows\Gamma^{(0)}$.

The following proposition lists some well-known properties regarding De Rham cohomology groups of a Lie groupoid.

PROPOSITION 2.15 [8–10,20,33]. –

(1) *For any Lie groupoid* $\Gamma\rightrightarrows\Gamma^{(0)}$*, we have*

$$H^k_{DR}(\Gamma^\bullet)\xrightarrow{\sim}H^k(\Gamma^\bullet,\mathbb{R}); \tag{6}$$

(2) *if* $\Gamma\rightrightarrows\Gamma^{(0)}$ *and* $G\rightrightarrows G^{(0)}$ *are Morita equivalent, then*

$$H^k_{DR}(\Gamma^\bullet)\xrightarrow{\sim}H^k_{DR}(G^\bullet),\quad \textit{and}\quad H^k(\Gamma^\bullet,\mathcal{S}^1)\xrightarrow{\sim}H^k(G^\bullet,\mathcal{S}^1).$$

We call a De Rham k-cocycle an *integer cocycle*, if it maps under (6) into the image of the canonical map $H^k(\Gamma^\bullet,\mathbb{Z})\to H^k(\Gamma^\bullet,\mathbb{R})$.

Example 2.16. – (1) When Γ is a manifold M, it is clear that $H^k(\Gamma^\bullet,\mathbb{Z})$ (or $H^k(\Gamma^\bullet,\mathbb{R})$) reduces to the usual cohomology $H^k(M,\mathbb{Z})$ (or $H^k(M,\mathbb{R})$ respectively). If $\{U_i\}$ is an open covering of M and $X=\coprod_i U_i\to M$ is the étale map, then $\Gamma:=X\times_M X\rightrightarrows X$, which is $\coprod_{ij}U_i\cap U_j\rightrightarrows\coprod_i U_i$, is Morita equivalent to $M\rightrightarrows M$. Hence $H^k(\Gamma^\bullet,\mathbb{Z})$ (or $H^k(\Gamma^\bullet,\mathbb{R})$ respectively) is isomorphic to $H^k(M,\mathbb{Z})$ (or $H^k(M,\mathbb{R})$ respectively). The double complex (5), when $\{U_i\}$ is a nice covering, is the one used by Weil in his proof of De Rham theorem [73].

(2) When Γ is a transformation groupoid $G\times M\rightrightarrows M$, $H^k(\Gamma^\bullet,\mathbb{Z})$ (or $H^k(\Gamma^\bullet,\mathbb{R})$ respectively) is the G-equivariant cohomology group $H^k_G(M,\mathbb{Z})$ (or $H^k_G(M,\mathbb{R})$ respectively). If G is compact, $H^k_G(M,\mathbb{R})$ can be alternatively computed by either Cartan model or Weil model (see [32] for more details).

(3) On the other hand, if $\Gamma\rightrightarrows M$ is an étale groupoid representing an orbifold [52] and $\Lambda(\Gamma)\rightrightarrows\Gamma$ its associated inertia groupoid, then $H^k(\Lambda(\Gamma)^\bullet,\mathbb{R})$ is the orbifold cohomology.

It is known that $H^2(\Gamma^\bullet,\mathcal{S}^1)$ classifies S^1-gerbes over the stack $\mathfrak{X}_\Gamma$ [30]. As a consequence (see Remark 2.14), we have

PROPOSITION 2.17. – *For a Lie groupoid* $\Gamma\rightrightarrows M$*, we have*

$$\mathrm{Ext}^{sm}(\Gamma,S^1)\cong H^2(\Gamma^\bullet,\mathcal{S}^1).$$

For instance, when Γ is a manifold M, by Example 2.26.1 below, $\mathrm{Ext}^{sm}(M,S^1)$ is isomorphic to the Čech cohomology group $\check{H}^2(M,\mathcal{S}^1)=H^2(M,\mathcal{S}^1)$.

The exponential sequence $0\to\mathbb{Z}\to\mathcal{R}\to\mathcal{S}^1\to 0$ gives rise to a long exact sequence:

$$\begin{aligned}\cdots\to H^2(\Gamma^\bullet,\mathbb{Z})&\xrightarrow{\psi_2}H^2(\Gamma^\bullet,\mathcal{R})\to H^2(\Gamma^\bullet,\mathcal{S}^1)\xrightarrow{\phi}H^3(\Gamma^\bullet,\mathbb{Z})\\ &\xrightarrow{\psi_3}H^3(\Gamma^\bullet,\mathcal{R})\to\cdots.\end{aligned} \tag{7}$$

LEMMA 2.18. –

$$H^k(\Gamma^\bullet,\mathcal{R})\cong H^k(\Gamma,\mathbb{R}),$$

where $H^k(\Gamma,\mathbb{R})$ denotes the (smooth) groupoid cohomology with the trivial coefficients $\mathbb{R}$, i.e. the cohomology of the complex $(C^\infty(\Gamma^{(n)},\mathbb{R}))_{n\in\mathbb{N}}$ with the differential

$$(dc)(g_1,\ldots,g_{n+1}) = c(g_2,\ldots,g_{n+1}) + \sum_{k=1}^{n}(-1)^k c(g_1,\ldots,g_k g_{k+1},\ldots,g_{n+1}) \\ + (-1)^{n+1} c(g_1,\ldots,g_n).$$

Proof. – There is a spectral sequence

$$E_1^{p,q} = H^q(\Gamma^{(p)},\mathcal{R}) \Rightarrow H^{p+q}(\Gamma^\bullet,\mathcal{R}).$$

Since $\Gamma^{(p)}$ is a manifold and the sheaf $\mathcal{R}|\Gamma^{(p)}$ is soft, $H^q(\Gamma^{(p)},\mathcal{R}) = 0$ for $q > 0$. Therefore the spectral sequence degenerates. It follows that $H^*(\Gamma^\bullet,\mathcal{R})$ can be calculated using the complex $H^0(\Gamma^{(p)},\mathcal{R}) = C^\infty(\Gamma^{(p)})$. $\square$

By identifying the groups $H^k(\Gamma^\bullet,\mathcal{R})$ with $H^k(\Gamma,\mathbb{R})$, the homomorphism

$$\psi_k : H^k(\Gamma^\bullet,\mathbb{Z}) \to H^k(\Gamma^\bullet,\mathcal{R})$$

in the exact sequence (7) is the composition of the following sequences of morphisms:

$$H^k(\Gamma^\bullet,\mathbb{Z}) \to H^k(\Gamma^\bullet,\mathbb{R}) \xrightarrow{\sim} H^k_{DR}(\Gamma^\bullet) \xrightarrow{\mathrm{pr}} H^k(\Gamma,\mathbb{R}) \xrightarrow{\sim} H^k(\Gamma^\bullet,\mathcal{R}),$$

where $\mathrm{pr} : H^k_{DR}(\Gamma^\bullet) \to H^k(\Gamma,\mathbb{R})$ is given, on the cochain level, by the projection

$$\bigoplus_{i+j=k} \Omega^i(\Gamma_j) \to \Omega^0(\Gamma_k).$$

See [19,74] for details on (smooth) groupoid cohomology.

Note that in general $\phi : H^2(\Gamma^\bullet,\mathcal{S}^1) \to H^3(\Gamma^\bullet,\mathbb{Z})$ is neither surjective nor injective. Write

$$H^3_{gerbe}(\Gamma^\bullet,\mathbb{Z}) = \phi\big(H^2(\Gamma^\bullet,\mathcal{S}^1)\big).$$

PROPOSITION 2.19. – (i) *$H^3_{gerbe}(\Gamma^\bullet,\mathbb{Z})$ is a subgroup of $H^3(\Gamma^\bullet,\mathbb{Z})$ consisting of those elements whose image in $H^3_{DR}(\Gamma^\bullet)$ projects to zero under* $\mathrm{pr} : H^3_{DR}(\Gamma^\bullet) \to H^3(\Gamma,\mathbb{R})$.
(ii) *The kernel of ϕ is isomorphic to $H^2(\Gamma,\mathbb{R})/\psi_2(H^2(\Gamma^\bullet,\mathbb{Z}))$.*

For an S^1-central extension $R \to \Gamma \rightrightarrows M$, let $[R] \in H^2(\Gamma^\bullet,\mathcal{S}^1)$ denote its class. The image of $[R]$ in $H^3(\Gamma^\bullet,\mathbb{Z})$ under the homomorphism ϕ is called the *Dixmier–Douady class* of R [9,10]. The Dixmier–Douady class behaves well with respect to the pull-back and the tensor operation. Unlike the manifold case, in general the Dixmier–Douady class does not completely determine an S^1-gerbe. However this is true when $\Gamma \rightrightarrows M$ is a proper groupoid. Let us recall its definition below.

DEFINITION 2.20. – Let $\Gamma \rightrightarrows M$ be a locally compact groupoid. Then Γ is said to be proper if any of the following equivalent conditions is satisfied:
(i) the map $(s,t) : \Gamma \to M \times M$ is proper;
(ii) for every $K \subset M$ compact, Γ_K^K is compact.

For instance, compact groupoids are of course proper; a transformation groupoid $G \times M \rightrightarrows M$ is proper if and only if the action is proper.

LEMMA 2.21. – (1) *The notion of properness is invariant by Morita equivalence*;

(2) *For a proper groupoid* $\Gamma \rightrightarrows M$*, the orbit space* M/Γ *is a Hausdorff topological space, and is invariant by Morita equivalence.*

Proof. – Suppose that $f: Y \to M$ is a surjective submersion. If Γ is proper, then for every $K \subset Y$ compact, $(\Gamma[Y])_K^K$ is a closed subset of $K \times K \times \Gamma_{f(K)}^{f(K)}$, and therefore it is compact. Hence, $\Gamma[Y]$ is proper.

Conversely, if $\Gamma[Y]$ is proper, then for every $L \subset M$ compact, there exists $K \subset Y$ compact such that $f(K) = L$ (since f is open surjective). Now, Γ_L^L is a continuous image of the compact set $(\Gamma[Y])_K^K$, and thus is compact. It follows that Γ is proper. This proves (1).

The first assertion in (2) is proved for instance in [68, Proposition 6.3]. For the second one, it is clear that if $f: Y \to M$ is a surjective submersion, then f induces a homeomorphism $Y/(\Gamma[Y]) \cong M/\Gamma$. □

When $\Gamma \rightrightarrows M$ is a proper Lie groupoid, since the smooth groupoid cohomology $H^k(\Gamma, \mathbb{R})$ vanishes when $k \geqslant 1$ according to Crainic [19], we see that ϕ is an isomorphism.

PROPOSITION 2.22. – *If* $\Gamma \rightrightarrows M$ *is a proper Lie groupoid, then*

$$\phi: H^2(\Gamma^\bullet, \mathcal{S}^1) \to H^3(\Gamma^\bullet, \mathbb{Z})$$

is an isomorphism.

As a consequence, we have

COROLLARY 2.23. – (1) *If a Lie group* G *acts on a smooth manifold* M *properly, then the equivariant cohomology* $H_G^3(M, \mathbb{Z})$ *is isomorphic to the abelian group of* S^1*-gerbes over the stack* M/G *associated to the transformation groupoid* $G \times M \rightrightarrows M$.

(2) *If* $\Gamma \rightrightarrows M$ *is an étale groupoid corresponding to an orbifold* $\mathbf{X}$*, then* $H^3(\Gamma^\bullet, \mathbb{Z})$ *is isomorphic to the abelian group of* S^1*-gerbes over the orbifold* $\mathbf{X}$.

In other words, in both cases above, the third integer cohomology classes can be geometrically described by Morita equivalent classes of groupoid S^1-central extensions. In particular, for a smooth manifold M, since $M \rightrightarrows M$ is a special case when $G = 1$, then $H^3(M, \mathbb{Z})$ characterizes S^1-gerbes over the manifold M [14]. However the Dixmier–Douady class does not completely characterize S^1-gerbes even when Γ is a non-compact group as we see below.

Example 2.24 [70]. – Consider the abelian group $\mathbb{R}^2$ as a groupoid $\mathbb{R}^2 \rightrightarrows \cdot$. It is clear that $H^k(\mathbb{R}^{2^\bullet}, \mathbb{Z}) \cong H^k(B\mathbb{R}^2, \mathbb{Z}) = 0$ since $\mathbb{R}^2$ is contractible. Therefore the kernel of ϕ is isomorphic to the 2nd group cohomology of $\mathbb{R}^2$, which is in turn isomorphic to the 2nd Lie algebra cohomology with trivial coefficients since $\mathbb{R}^2$ is simply connected. The latter is isomorphic to the invariant De Rham cohomology of $\mathbb{R}^2$ under the translation, and therefore is one-dimensional as a $\mathbb{R}$-vector space. More explicitly, the group 2-cocycle is given by

$$\sigma\big((x,y),(x',y')\big) = \frac{1}{2}(x'y - xy').$$

In other words, the group 2-cocycle $\exp(2\pi i \sigma)$ defines a non-trivial S^1-central extension of $\mathbb{R}^2$ (hence a non-trivial S^1-gerbe over $\mathfrak{X}_{\mathbb{R}^2}$) with the trivial Dixmier–Douady class.

It is often useful to use differential forms to describe the Dixmier–Douady class as in the manifold case. Recall that a pseudo-connection is $\theta + B \in \Omega^1(R) \oplus \Omega^2(M)$ such that θ is a connection one-form of the principal S^1-bundle $R \xrightarrow{\pi} \Gamma$ [9]. Its pseudo-curvature

$$\eta + \omega + \Omega \in \Omega^1(\Gamma^{(2)}) \oplus \Omega^2(\Gamma) \oplus \Omega^3(M) \subset C^3(\Gamma^\bullet)$$

is defined by

$$\delta(\theta + B) = \pi^*(\eta + \omega + \Omega).$$

Then we have the following [9]:

THEOREM 2.25. – (1) $[\eta + \omega + \Omega]$ *is independent of the pseudo-connection and defines an integer class in* $H^3_{DR}(\Gamma^\bullet)$. *Under the canonical homomorphism* $H^3(\Gamma^\bullet, \mathbb{Z}) \to H^3_{DR}(\Gamma^\bullet)$, *the Dixmier–Douady class of* R *maps to* $[\eta + \omega + \Omega]$.

(2) *Assume that* $\Gamma \rightrightarrows M$ *is proper. Given any integer* 3*-cocycle* $\eta + \omega + \Omega$ *as above, by passing to a Morita equivalent groupoid* $\Gamma' \rightrightarrows M'$ *if necessary, there is an* S^1*-central extension* $R \to \Gamma$ *with a pseudo-connection whose pseudo-curvature equals* $\eta + \omega + \Omega$.

In conclusion, for a proper Lie groupoid $\Gamma \rightrightarrows M$, if $H^3(\Gamma^\bullet, \mathbb{Z})$ has no torsion, then $H^3(\Gamma^\bullet, \mathbb{Z}) \to H^3(\Gamma^\bullet, \mathbb{R})$ is injective by the universal coefficient theorem. Hence any integer class in $H^3_{DR}(\Gamma^\bullet)$ can be represented uniquely by an S^1-gerbe over $\mathfrak{X}_\Gamma$ and vice-versa. In this case, one can define K-theory twisted by such a class $[\eta + \omega + \Omega]$. However, in general, our twisted K-theory is only defined for twisting a class in $H^2(\Gamma^\bullet, \mathcal{S}^1)$ not for an integer 3rd De Rham class $[\eta + \omega + \Omega]$. This is an essential difference when dealing with general groupoids.

Let us end this subsection by some examples, which have been studied extensively in the literature.

Example 2.26. – (1) Let M be a manifold and $\alpha \in H^3(M, \mathbb{Z})$, and let $\{U_i\}$ be a good covering of M. Then the groupoid $\coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i$, where $U_{ij} = U_i \cap U_j$ is Morita equivalent to $M \rightrightarrows M$. See Example 2.16. Then the S^1-gerbe corresponding to the class α can be realized as an S^1-central extension of groupoids $\coprod_{ij} R_{ij} \to \coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i$, where R_{ij} are S^1-bundles over U_{ij}, and the groupoid multiplication is defined as follows: taking a trivialization $R_{ij} \cong U_{ij} \times S^1$, then

$$(x_{ij}, \lambda_1)(x_{jk}, \lambda_2) = (x_{ik}, \lambda_1\lambda_2 c_{ijk}), \tag{8}$$

where $x_{ij},\ x_{jk}, x_{ik}$ are the same point x in the three-intersection U_{ijk} considered as elements in the two-intersections, and $c_{ijk} : U_{ijk} \to S^1$ is a 2-cocycle which represents the Cech class in $H^2(M, \mathcal{S}^1)$ corresponding to α. Note that c_{ijk} can also be considered as an S^1-valued groupoid 2-cocycle of the groupoid $\coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i$, and Eq. (8) above is a special case of Eq. (2). See [14,39] for details.

(2) Let Γ be a transformation groupoid $G \times M \rightrightarrows M$, where G acts on M properly. By Corollary 2.23, we have $H^3_G(M, \mathbb{Z}) \cong H^2(\Gamma^\bullet, \mathcal{S}^1)$. Assume that there exists a G-invariant good cover $\{U_i\}$, then $\Gamma \rightrightarrows M$ is Morita equivalent to $\coprod_{ij} G \times U_{ij} \rightrightarrows \coprod_i U_i$, where the groupoid structure is given by $s(g, x_{ij}) = x_j,\ t(g, x_{ij}) = gx_i$, and

$$(g, x_{ij}) \cdot (h, y_{jk}) = (gh, z_{ik})$$

where $x = hy$ and $y = z$. Then the S^1-gerbe corresponding to the class α can be realized as an S^1-central extension of groupoids $S^1 \to \coprod_{ij} R_{ij} \to \coprod_{ij} G \times U_{ij} \rightrightarrows \coprod_i U_i$, where R_{ij} are S^1-bundles over $G \times U_{ij}$. For all i, j, take an open cover $(V_\alpha)_{\alpha \in I_{ij}}$ of G such that the restriction $R_{ij\alpha}$ of R_{ij} over $V_\alpha \times U_{ij}$ is isomorphic to the trivial bundle $V_\alpha \times U_{ij} \times S^1$. The product

$$R_{ij\alpha} \times_{U_j} R_{jk\beta} \to R_{ik\gamma}$$

has the form

$$(i,j,\alpha,g,x,\lambda)(j,k,\beta,h,y,\mu)=\big(i,k,\gamma,gh,y,\lambda\mu c_{ijk;\alpha\beta,\gamma}(g,x,h,y)\big), \tag{9}$$

where $c_{ijk;\alpha\beta,\gamma}:\{(g,x,h,y)\in V_\alpha\times U_{ij}\times V_\beta\times U_{jk}\mid x=hy,\, gh\in V_\gamma\}\to S^1$ satisfies the following cocycle relation which expresses that the product is associative:

$$\begin{aligned}&c_{ijk;\alpha_1\alpha_2,\alpha_{12}}(g_1,x,g_2,y)c_{ikl;\alpha_{12}\alpha_3,\alpha_{123}}(g_1g_2,y,g_3,z)\\ &\quad=c_{jkl;\alpha_2\alpha_3;\alpha_{23}}(g_2,y,g_3,z)c_{ijl;\alpha_1\alpha_{23},\alpha_{123}}(g_1,x,g_2g_3,z).\end{aligned}$$

Conversely, given a cocycle as above, then one can associate to it an S^1-central extension

$$S^1\to R\to\coprod_{i,j}G\times U_{ij}\rightrightarrows\coprod_i U_i.$$

The proof is elementary but tedious. We omit it here.

Remark 2.27. – There is a canonical map $H^3_G(M,\mathbb{Z})\to H^3(M,\mathbb{Z})$ induced by the inclusion of M to the unit space of $G\times M\rightrightarrows M$. This implies that an equivariant gerbe should induce a gerbe over M. From the picture of S^1-central extensions, such a gerbe over M is simply the restriction of the S^1-central extension $R'\to\Gamma'\rightrightarrows M'$ to the unit space, where $R'\to\Gamma'\rightrightarrows M'$ is an S^1-central extension representing this equivariant gerbe. In some cases, we have an isomorphism $H^3_G(M,\mathbb{Z})\cong H^3(M,\mathbb{Z})$. It is interesting to investigate how an S^1-gerbe over M can be made an equivariant one under this assumption.

2.4. Continuous case

The purpose of this subsection is to clarify the relation with [43]. To relate their constructions to ours, let $\mathcal{S}^1_{cont}$ be the sheaf of continuous S^1-valued functions. We need to determine whether the natural map $H^2(\Gamma^\bullet,\mathcal{S}^1)\to H^2(\Gamma^\bullet,\mathcal{S}^1_{cont})$ is an isomorphism. Unfortunately, we do not know the answer in general, but we can prove that it is an isomorphism in our main case of interest:

PROPOSITION 2.28. – *Let Γ be a proper Lie groupoid. Then the natural map*

$$H^2(\Gamma^\bullet,\mathcal{S}^1)\to H^2(\Gamma^\bullet,\mathcal{S}^1_{cont}) \tag{10}$$

is an isomorphism.

Proof. – Recall from Proposition 2.22 that $H^2(\Gamma^\bullet,\mathcal{S}^1)$ is isomorphic to $H^3(\Gamma^\bullet,\mathbb{Z})$. We claim that $H^2(\Gamma^\bullet,\mathcal{S}^1_{cont})$ is also isomorphic to $H^3(\Gamma^\bullet,\mathbb{Z})$. Indeed, Crainic's proof that smooth groupoid cohomology vanishes [19] also works for continuous cohomology since Crainic only uses integration and cutoff functions, and never uses differentiation. □

Exactly the same constructions can be performed in the category of locally compact groupoids: let us denote by $Tw^{lt}(\Gamma)$, $\mathcal{E}^{lt}(\Gamma)$ and $\mathrm{Ext}^{lt}(\Gamma,S^1)$ the groups thus obtained. The superscript "lt" stands for "locally trivial", since central extensions are required to be locally trivial S^1-principal bundles (in the continuous sense), and Morita equivalences between groupoids are required to be locally trivial principal bundles. An immediate consequence of Proposition 2.28 is the following

COROLLARY 2.29. – *Let Γ be a proper Lie groupoid. Then the natural map*

$$\mathrm{Ext}^{sm}(\Gamma,S^1)\to\mathrm{Ext}^{lt}(\Gamma,S^1)$$

is an isomorphism.

However, in [43], S^1-central extensions $S^1 \to R \to \Gamma$ are not required to be locally trivial: the homomorphism $R \to \Gamma$ is only required to be open surjective. Moreover, the notion of Morita equivalence in [43] is weaker since in their definition of an equivalence bimodule $\Gamma_1^{(0)} \overset{\tau}{\leftarrow} Z \overset{\sigma}{\rightarrow} \Gamma_2^{(0)}$, the maps σ and τ are just open surjective, and the actions of Γ_1 and Γ_2 on Z are free and proper, but Z is not necessarily a locally trivial Γ_i-principal bundle. Let us denote by $Tw^{lc}(\Gamma)$, $\mathcal{E}^{lc}(\Gamma)$ and $\mathrm{Ext}^{lc}(\Gamma, S^1)$ the groups constructed in [43]. There are obvious natural morphisms

$$\begin{tikzcd}
Tw^{sm}(\Gamma) \arrow[r] \arrow[d] & Tw^{lt}(\Gamma) \arrow[r] \arrow[d] & Tw^{lc}(\Gamma) \arrow[d] \\
\mathcal{E}^{sm}(\Gamma) \arrow[r] \arrow[d] & \mathcal{E}^{lt}(\Gamma) \arrow[r] \arrow[d] & \mathcal{E}^{lc}(\Gamma) \arrow[d] \\
Ext^{sm}(\Gamma, S^1) \arrow[r] & Ext^{lt}(\Gamma, S^1) \arrow[r] & Ext^{lc}(\Gamma, S^1).
\end{tikzcd}$$

Since any S^1-central extension of Lie groupoids is the pull-back of the central extension

$$S^1 \to U(\mathbb{H}) \to PU(\mathbb{H})$$

which is locally trivial (Section 2.6), the map $Tw^{lt}(\Gamma) \to Tw^{lc}(\Gamma)$ is an isomorphism. Therefore $\mathcal{E}^{lt}(\Gamma) \to \mathcal{E}^{lc}(\Gamma)$ and $\mathrm{Ext}^{lt}(\Gamma, S^1) \to \mathrm{Ext}^{lc}(\Gamma, S^1)$ are surjective.

From Proposition 2.13 and its analogue for $\mathcal{E}^{lt}$ and $\mathcal{E}^{lc}$ instead of $\mathcal{E}^{sm}$, an extension is zero in $\mathcal{E}^{lt}(\Gamma)$ if and only if there exists an open cover (U_i) such that its class is zero in $Tw^{lt}(\Gamma[U_i])$, and similarly for $\mathcal{E}^{lc}$. Therefore,

PROPOSITION 2.30. – *Let Γ be a Lie groupoid. Then*
(a) *the natural maps $\mathcal{E}^{lt}(\Gamma) \to \mathcal{E}^{lc}(\Gamma)$ and $\mathrm{Ext}^{lt}(\Gamma, S^1) \to Ext^{lc}(\Gamma, S^1)$ are isomorphisms.*
(b) *If Γ is proper, then $\mathrm{Ext}^{sm}(\Gamma, S^1) \xrightarrow{\sim} \mathrm{Ext}^{lt}(\Gamma, S^1) \xrightarrow{\sim} \mathrm{Ext}^{lc}(\Gamma, S^1)$.*

In [43] is defined the Brauer group $Br(\Gamma)$ of Γ. It is the group of locally trivial bundles of C^*-algebras over M endowed with an action of Γ, with fibers isomorphic to $\mathcal{K}$, divided by Morita equivalence. Let $Br_0(\Gamma)$ be the subgroup of $Br(\Gamma)$ consisting of those bundles whose Dixmier–Douady class in $H^3(M, \mathbb{Z})$ is zero. Then $Br_0(\Gamma)$ is the group of bundles of the form $M \times \mathcal{K}$ with the diagonal action $\gamma \cdot (s(\gamma), T) = (t(\gamma), \pi(\gamma)(T))$, where

$$\pi : \Gamma \to \mathrm{Aut}(\mathcal{K}) \cong PU(\mathbb{H}) \tag{11}$$

is a "projective representation" of Γ. The group structure is given by tensor product:

$$[\pi][\pi'] = [\pi \otimes \pi'],$$

where $(\pi \otimes \pi')(\gamma) \in \mathrm{Aut}(\mathcal{K}(\mathbb{H} \otimes \mathbb{H})) \cong \mathrm{Aut}(\mathcal{K})$.

Recall [43] that

$$Br_0(\Gamma) \cong \mathcal{E}^{lc}(\Gamma). \tag{12}$$

Indeed, from the data $(M \times \mathcal{K} \to M, \pi)$, one obtains an S^1-central extension as follows:

$$S^1 \to \{(\gamma, U) \in \Gamma \times U(\mathbb{H}) \mid \pi(\gamma) = \mathrm{Ad}(U)\} \to \Gamma.$$

For the construction of a bundle of C^*-algebras obtained from a central extension, see [43] or Section 2.6.

If $\{U_i\}$ is a cover of M by contractible open subspaces and if Γ' denotes $\Gamma[U_i]$, then $\mathrm{Ext}^{lc}(\Gamma, S^1) \cong Br(\Gamma) \cong Br(\Gamma') \cong \mathcal{E}^{lc}(\Gamma')$. To summarize,

PROPOSITION 2.31. – *If Γ is a proper Lie groupoid, then we have*

$$Br(\Gamma) \cong \mathrm{Ext}^{sm}(\Gamma, S^1) \cong H^2(\Gamma^\bullet, \mathcal{S}^1) \cong H^3(\Gamma^\bullet, \mathbb{Z}).$$

2.5. S^1-gerbes via principal G-bundles over groupoids

The purpose of this subsection is to present another construction of S^1-gerbes using principal G-bundles over groupoids together with an S^1-central extension of G. In fact, we show, in the next subsection, that every S^1-gerbe arises in this way when G is taken the projective unitary group $PU(\mathbb{H})$ of a separable Hilbert space $\mathbb{H}$. Let us recall the definition of principal G-bundles.

DEFINITION 2.32. – Let $\Gamma \rightrightarrows M$ be a Lie groupoid. A Γ-space consists of a smooth manifold P together with a smooth map $J: P \to M$ such that

(i) there is a map $\sigma: Q \to P$, where Q is the fibered product $Q = \Gamma \times_{s,M,J} P$. We write $\sigma(\gamma, x) = \gamma \cdot x$.

This map is subject to the constraints

(ii) for all $x \in P$ we have

$$J(x) \cdot x = x;$$

(iii) for all $x \in P$ and all $\gamma, \delta \in \Gamma$ such that $J(x) = s(\gamma)$ and $t(\gamma) = s(\delta)$ we have

$$(\delta \cdot \gamma) \cdot x = \delta \cdot (\gamma \cdot x).$$

Note that, as a consequence of the above definition, for any $r \in \Gamma$, the map

$$l_r : J^{-1}(u) \to J^{-1}(v), \quad x \to r \cdot x \tag{13}$$

must be a diffeomorphism, where $u = s(r)$ and $v = t(r)$.

Associated to any Γ-space $J: P \to M$, there is a natural groupoid $Q \rightrightarrows P$, called the *transformation groupoid*, which is defined as follows $Q = \Gamma \times_{s,M,J} P$, the source and target maps are, respectively, $s(\gamma, x) = x$, $t(\gamma, x) = \gamma \cdot x$, and the multiplication

$$(\gamma, y) \cdot (\delta, x) = (\gamma \cdot \delta, x), \quad \text{where } y = \delta \cdot x. \tag{14}$$

It is simple to check that the first projection defines a (strict) homomorphism of groupoids from $Q \rightrightarrows P$ to $\Gamma \rightrightarrows M$.

DEFINITION 2.33. – A principal G-bundle over $\Gamma \rightrightarrows M$ is a principal right G-bundle $P \xrightarrow{J} M$, which, at the same time, is also a Γ-space such that the following compatibility condition is satisfied: for all $x \in P$ and $\gamma \in \Gamma$, $s(\gamma) = J(x)$

$$(\gamma \cdot x) \cdot g = \gamma \cdot (x \cdot g). \tag{15}$$

In this case $Q \to \Gamma$ also becomes a principal (right) G-bundle.

Example 2.34. – Let Γ be the transformation groupoid $H \times M \rightrightarrows M$. Then a principal G-bundle over Γ corresponds exactly to an H-equivariant principal (right) G-bundle over M.

A principal G-bundle over a groupoid $\Gamma \rightrightarrows M$ can also be equivalently considered as a generalized homomorphism from $\Gamma \rightrightarrows M$ to $G \rightrightarrows \cdot$. As a consequence of Proposition 2.2, we see that principal bundles behave well under the "generalized homomorphisms" in the following sense.

PROPOSITION 2.35. – *Let f be a generalized homomorphism from $\Gamma_1 \rightrightarrows M_1$ to $\Gamma_2 \rightrightarrows M_2$ given by*

$$M_1 \xleftarrow{\tau} X \xrightarrow{\sigma} M_2.$$

Then for any principal G-bundle $P \to M_2$ over $\Gamma_2 \rightrightarrows M_2$,

$$f^* P \stackrel{def}{=} X \times_{M_2} P \to M_1$$

is a principal G-bundle over $\Gamma_1 \rightrightarrows M_1$. As a consequence, if $\Gamma_1 \rightrightarrows M_1$ and $\Gamma_2 \rightrightarrows M_2$ are Morita equivalent groupoids, then there is a bijection between their principal G-bundles.

Given a principal G-bundle $J : P \to M$ over $\Gamma \rightrightarrows M$, let $\frac{P\times P}{G} \rightrightarrows M$ be the gauge groupoid. We denote by (p_1, p_2) an element of $P \times P$ and by $\overline{(p_1, p_2)}$ the class of this element in $\frac{P\times P}{G}$. A map from Γ to $\frac{P\times P}{G}$ is defined by $\gamma \mapsto \overline{(\gamma p, p)}$ where p is any element that satisfies $J(p) = s(\gamma)$. Thus we obtain the following groupoid homomorphism:

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \frac{P\times P}{G} \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r] & M \end{tikzcd} \tag{16}$$

Since any transitive groupoid is Morita equivalent to its isotropy group, $\frac{P\times P}{G} \rightrightarrows M$ is Morita equivalent to $G \rightrightarrows \cdot$. It is not hard to check that the homomorphism (16) and the G-principal bundle P define the isomorphic generalized homomorphisms from Γ to G.

From Proposition 2.5, it follows that a generalized homomorphism f from $\Gamma_1 \rightrightarrows M_1$ to $\Gamma_2 \rightrightarrows M_2$ induces a natural homomorphism, called the *pull-back map*:

$$f^* : H^2(\Gamma_2^\bullet, \mathcal{S}^1) \to H^2(\Gamma_1^\bullet, \mathcal{S}^1).$$

In what follows, we describe a construction of S^1-gerbes over a stack which is similar to the construction in [14,16]. Assume that $S^1 \to \widetilde{G} \to G$ is an S^1-central extension of Lie groups, P is a G-bundle over $\Gamma \rightrightarrows M$. Then P defines a generalized homomorphism from $\Gamma \rightrightarrows M$ to $G \rightrightarrows \cdot$, and therefore induces a pull-back map $H^2(G^\bullet, \mathcal{S}^1) \to H^2(\Gamma^\bullet, \mathcal{S}^1)$. By pulling back the class of $S^1 \to \widetilde{G} \to G$ in $H^2(G^\bullet, \mathcal{S}^1)$ via this map, one obtains an element in $H^2(\Gamma^\bullet, \mathcal{S}^1)$, i.e., an S^1-gerbe over the stack $\mathfrak{X}_\Gamma$ associated to Γ.

Since the equivalence classes of G-bundles $P \xrightarrow{J} M$ over $\Gamma \rightrightarrows M$ are classified by $H^1(\Gamma^\bullet, G)$, we have a map

$$\Phi : H^1(\Gamma^\bullet, G) \times H^2(G^\bullet, \mathcal{S}^1) \to H^2(\Gamma^\bullet, \mathcal{S}^1). \tag{17}$$

Below we describe an explicit construction of the map Φ in a special case more relevant to us.

Besides the above assumption, we furthermore assume that, as a G-principal bundle, $P \to M$ can be lifted to a $\widetilde{G}$-principal bundle $\widetilde{P} \to M$. Note that if $\widetilde{P} \to M$ is a principal $\widetilde{G}$-bundle, there is a natural S^1-action on $\widetilde{P}$ defined as follows: $\forall \lambda \in S^1, \widetilde{p} \in \widetilde{P},\ \lambda \cdot \widetilde{p} = (\lambda \cdot 1_{\widetilde{G}})\widetilde{p}$, where $\lambda \cdot 1_{\widetilde{G}}$ is considered as an element in $\widetilde{G}$. Then $\widetilde{P}/S^1$ is a principal G-bundle over M, which is isomorphic to the reduced principal bundle induced by the group homomorphism $\widetilde{G} \to G$. We require that as a principal G bundle $P \cong \widetilde{P}/S^1$. In this case, it is simple to see that

$$\frac{\widetilde{P} \times \widetilde{P}}{\widetilde{G}} \to \frac{P \times P}{G} \rightrightarrows M$$

is an S^1-central extension, which is Morita equivalent to

$$\widetilde{G} \to G \rightrightarrows \cdot$$

Here the S^1-equivalent Morita equivalence bimodule $M \xleftarrow{\widetilde{\tau}} \widetilde{P} \xrightarrow{\widetilde{\sigma}} \cdot$ is given by the composition of the projection $\widetilde{P} \to P$ with $M \xleftarrow{\tau} P \xrightarrow{\sigma} \cdot$, the left action of $\frac{\widetilde{P} \times \widetilde{P}}{\widetilde{G}}$ on $\widetilde{P}$ is $[(\widetilde{p}_1, \widetilde{p}_2)] \cdot \widetilde{p}_3 = \widetilde{p}_1 \widetilde{g}$ where $\widetilde{g}$ is the unique element in $\widetilde{G}$ such that $\widetilde{p}_3 = \widetilde{p}_2 \widetilde{g}$, and the right $\widetilde{G}$-action is the usual one.

Let $R \to \Gamma$ denote the pull-back S^1-bundle of $\frac{\widetilde{P} \times \widetilde{P}}{\widetilde{G}} \to \frac{P \times P}{G}$ via the map $\Gamma \to \frac{P \times P}{G}$ as in Eq. (16).

PROPOSITION 2.36. – *Under the same hypothesis as above, $R \to \Gamma$ is a groupoid S^1-central extension, whose corresponding class in $H^2(\Gamma^\bullet, \mathcal{S}^1)$ is equal to $\Phi(\alpha, \beta)$. Here $\alpha \in H^1(\Gamma^\bullet, G)$ is the class defined by $P \to M$ and $\beta \in H^2(G^\bullet, \mathcal{S}^1)$ is the class corresponding to the central extension $S^1 \to \widetilde{G} \to G$.*

2.6. Hilbert bundle and Dixmier–Douady class

The purpose of this subsection is to show that every S^1-gerbe over a differential stack always arises from a principal $PU(\mathbb{H})$-bundle over the stack as in the case of manifolds [4]. However, unlike the manifold case, such projective bundles may not be unique. Nevertheless, we show that there always exists a canonical one. In the following, we describe an explicit construction of such a projective bundle.

We now fix a separable Hilbert space $\mathbb{H}$ and consider the canonical S^1-central extension:

$$S^1 \to U(\mathbb{H}) \to PU(\mathbb{H}), \tag{18}$$

which is a generator of $H^2(PU(\mathbb{H})^\bullet, \mathcal{S}^1)$ [14]. Thus Eq. (17) induces a group homomorphism

$$\Phi' : H^1(\Gamma^\bullet, PU(\mathbb{H})) \to H^2(\Gamma^\bullet, \mathcal{S}^1). \tag{19}$$

Note that $H^1(\Gamma^\bullet, PU(\mathbb{H}))$ can be endowed with the following abelian group structure: $[\pi][\pi'] = [\pi \otimes \pi']$ if π, $\pi' : \Gamma[U_i] \to PU(\mathbb{H})$ are groupoid homomorphisms and (U_i) is an open cover of Γ. See [14] for the case when Γ is a manifold $M \rightrightarrows M$.

In other words, any principal $PU(\mathbb{H})$-bundle over $\Gamma \rightrightarrows M$ defines an element in $H^2(\Gamma^\bullet, \mathcal{S}^1)$, or an S^1-gerbe over the associated stack $\mathfrak{X}_\Gamma$.

When Γ is a manifold $M \rightrightarrows M$, Φ' is indeed an isomorphism [4,14]. However, in general, Φ' may not be injective. [3] We will see below that Φ' admits a canonical left inverse. Therefore it is always surjective.

[3] For instance, let G be a compact Lie group and π any unitary representation of G such that $\pi(g)$ is not a scalar multiple of the identity for some $g \in G$. Then the associated element $[\pi] \in H^1(G^\bullet, PU(\mathbb{H}))$ is nonzero, but $\Phi'([\pi]) = 0$ since the composition $H^1(G^\bullet, U(\mathbb{H})) \to H^1(G^\bullet, PU(\mathbb{H})) \to H^2(G^\bullet, \mathcal{S}^1)$ is zero.

First of all, let us assume that $\alpha \in H^2(\Gamma^\bullet, \mathcal{S}^1)$ is the class defined by a groupoid S^1-central extension $R \to \Gamma \rightrightarrows M$.

DEFINITION 2.37. – A complex-valued function f on R is said to be equivariant if $f(\lambda\widetilde{\gamma}) = \lambda^{-1} f(\widetilde{\gamma})$ for any $\lambda \in S^1$ and any $\widetilde{\gamma} \in R$.

Let $\lambda = (\lambda^x)_{x\in M}$ be a Haar system on R, i.e. λ^x is a measure on R^x such that for any $\tilde{\gamma} \in R$ the map $L_{\tilde{\gamma}} : R^{s(\tilde{\gamma})} \to R^{t(\tilde{\gamma})}$ defined by $\tilde{\gamma}' \mapsto \tilde{\gamma}\tilde{\gamma}'$ preserves the measure.

By $\mathcal{L}^2_x$, we denote the space $L^2(R^x)^{S^1}$ consisting of S^1-equivariant functions defined on R^x which are L^2 with respect to the Haar measure. Let

$$\mathcal{H}_x = \mathcal{L}^2_x \otimes \mathbb{H}, \quad \text{and} \quad \widetilde{\mathcal{H}} = \coprod_x \mathcal{H}_x. \tag{20}$$

Then $\widetilde{\mathcal{H}} \to M$ is a countably generated continuous field of infinite dimensional Hilbert spaces over the finite dimensional space M, and therefore is a locally trivial Hilbert bundle (indeed globally trivial) according to Dixmier–Douady theorem [22].

For $x \in M$, let $\mathcal{B}_x$ be the set of orthonormal basis of $\mathcal{H}_x$ and $\mathcal{B} = \coprod_{x\in M} \mathcal{B}_x$. We endow $\mathcal{B}$ with the following topology: identify $\mathcal{B}_x$ with the space $U(\mathcal{H}_x, \mathbb{H})$ of unitary maps from $\mathcal{H}_x$ to $\mathbb{H}$. Then a section $x \mapsto u_x$ is continuous if and only if for every $\xi \in \mathbb{H}$, $x \mapsto u_x^{-1}\xi$ is a continuous section of the field $\widetilde{\mathcal{H}} \to M$. The fiber bundle $\mathcal{B} \to M$ is a principal $U(\mathbb{H})$-bundle. Now S^1 naturally acts on $\mathcal{B}$ by scalar multiplication. Let $P\mathcal{B} = \mathcal{B}/S^1$ be its quotient. Then $P\mathcal{B}$ is a principal $PU(\mathbb{H})$-bundle over M.

Let $U(x,y)$ be the set of unitary linear maps from $\mathcal{H}_x$ to $\mathcal{H}_y$, and

$$U(\mathcal{H}, \mathcal{H}) = \bigl\{U(x,y) \mid (x,y) \in M \times M\bigr\}.$$

Then $U(\mathcal{H}, \mathcal{H})$ is naturally a groupoid over M.

Let $PU(x,y)$ be the set of unitary projective maps from $\mathcal{H}_x$ to $\mathcal{H}_y$, and let

$$PU(\mathcal{H}, \mathcal{H}) = \{PU(x,y) \mid (x,y) \in M \times M\}.$$

Then $PU(\mathcal{H}, \mathcal{H})$ is a groupoid over M.

The groupoid R acts naturally on $\mathcal{H}$: for any element $\widetilde{\gamma} \in R$ with $x = s(\widetilde{\gamma})$ and $y = t(\widetilde{\gamma})$, and any equivariant function $f \in \mathcal{H}_x$, the action is given by:

$$f \mapsto \widetilde{\gamma} \cdot f, \quad \text{where } (\widetilde{\gamma} \cdot f)(r) = f(\widetilde{\gamma}^{-1} r).$$

Since this action preserves the measure λ, it induces a homomorphism of groupoids

$$i : R \to U(\mathcal{H}, \mathcal{H}).$$

Since i is equivariant under the S^1-actions, it induces a homomorphism of groupoids j:

$$j : \Gamma \to PU(\mathcal{H}, \mathcal{H}).$$

In short, we have the following diagram of groupoid homomorphisms:

$$
\begin{tikzcd}
R \arrow[r, "i"] \arrow[d] & U(\mathcal{H},\mathcal{H}) \arrow[d] \\
\Gamma \arrow[r, "j"] & PU(\mathcal{H},\mathcal{H})
\end{tikzcd}
\tag{21}
$$

It is obvious that $P\mathcal{B} \to M$ is a principal $PU(\mathbb{H})$-bundle over the groupoid $PU(\mathcal{H},\mathcal{H}) \rightrightarrows M$. By pushing forward the action using the above groupoid homomorphism, $P\mathcal{B} \to M$ is naturally a principal $PU(\mathbb{H})$-bundle over the groupoid $\Gamma \rightrightarrows M$.

PROPOSITION 2.38. – *If $\alpha \in H^1(\Gamma^\bullet, PU(\mathbb{H}))$ denotes the class defined by $P\mathcal{B} \to M$, then $\Phi'(\alpha)$ is equal to the class in $H^2(\Gamma^\bullet, \mathcal{S}^1)$ corresponding to the S^1-central extension $R \to \Gamma$.*

Proof. – Note that as groupoids,

$$U(\mathcal{H},\mathcal{H}) \rightrightarrows M$$

is isomorphic to $\frac{\mathcal{B}\times\mathcal{B}}{U(\mathbb{H})} \rightrightarrows M$, and

$$PU(\mathcal{H},\mathcal{H}) \rightrightarrows M$$

is isomorphic to $\frac{P\mathcal{B}\times P\mathcal{B}}{PU(\mathbb{H})} \rightrightarrows M$. Thus, the conclusion follows from Proposition 2.36 and diagram (21). □

Now let us return to the general case. Consider a Lie groupoid $\Gamma \rightrightarrows M$ and an element $\alpha \in H^2(\Gamma^\bullet, \mathcal{S}^1)$. There exists a Lie groupoid $\Gamma' \rightrightarrows M'$ Morita equivalent to Γ such that α is the class of an S^1-central extension

$$S^1 \to R \to \Gamma' \rightrightarrows M'.$$

Let $P\mathcal{B} \to M'$ be the corresponding principal $PU(\mathbb{H})$-bundle over $\Gamma' \rightrightarrows M'$ constructed above as in Proposition 2.38. Since $\Gamma' \rightrightarrows M'$ and $\Gamma \rightrightarrows M$ are Morita equivalent, there is an associated principal $PU(\mathbb{H})$-bundle $P_\alpha \to M$ over the groupoid $\Gamma \rightrightarrows M$. In fact, $P_\alpha = (Z \times_{M'} P\mathcal{B})/\Gamma'$, where $M \leftarrow Z \to M'$ is an equivalence bimodule between $\Gamma \rightrightarrows M$ and $\Gamma' \rightrightarrows M'$. By construction, P_α and $P\mathcal{B}$ represent the same generalized homomorphism, thus define the same element in $H^1(\Gamma^\bullet, PU(\mathbb{H}))$. Moreover, P_α does not depend on a particular choice of the S^1-central extension $S^1 \to R \to \Gamma' \rightrightarrows M'$ realizing the class α. This follows from the following

LEMMA 2.39. – *Assume that $p: Y \to M$ is a surjective submersion. Let $f: \Gamma[Y] \to \Gamma$ be the projection map. Assume that $S^1 \to R \to \Gamma \rightrightarrows M$ is an S^1-central extension, and denote by P (resp. P') the associated $PU(\mathbb{H})$-bundle over Γ (resp. $\Gamma[Y]$). Then P' is isomorphic to $P \circ f$ as generalized morphisms $\Gamma[Y] \to PU(\mathbb{H})$.*

Proof. – Let us first treat the case $Y = \amalg_{i\in I} U_i$. Let $I_x = \{i \in I \mid x \in U_i\}$ and $\mathcal{H}_I = \amalg_x \ell^2(I_x)$. Then $\mathcal{H}_I \to M$ is endowed with a structure of continuous field of Hilbert spaces over M (associated to the $C_0(M)$-Hilbert module $\bigoplus_{i\in I} C_0(U_i)$, see Proposition A.4).

It is easy to see that $P' = PU(f^*(\widetilde{\mathcal{H}} \otimes \mathcal{H}_I), \mathbb{H})$ (where "$\otimes$" denotes the tensor product of continuous fields over M). Now, $\widetilde{\mathcal{H}} \otimes \mathcal{H}_I \cong (\widetilde{\mathcal{H}} \otimes \mathbb{H}) \otimes \mathcal{H}_I \cong \widetilde{\mathcal{H}} \otimes (\mathcal{H}_I \otimes \mathbb{H}) \cong \widetilde{\mathcal{H}} \otimes \mathbb{H} \cong \widetilde{\mathcal{H}}$ since $\mathcal{H}_I \otimes \mathbb{H}$ is the trivial continuous field $M \times \mathbb{H} \to M$ (see the argument below (20)). It follows that $P' = PU(f^*\widetilde{\mathcal{H}}, \mathbb{H}) = P \circ f$.

In the general case, i.e. for a general Y, consider a continuous p-system $\mu = (\mu_x)_{x\in M}$, i.e. μ_x is a measure with support $p^{-1}(x)$ such that

$$\forall \varphi \in C_c(Y), \quad \left[x \mapsto \int \varphi(y)\, d\mu_x(y)\right] \in C_c(M).$$

A Haar system on $\Gamma[Y]$ is given by

$$\int_{\Gamma[Y]^y} \psi = \int_{\gamma\in\Gamma^{p(y)}} d\lambda^{p(y)}(\gamma) \int_{z\in p^{-1}(s(\gamma))} d\mu_{s(\gamma)}(y')\psi(y,\gamma,y').$$

Then, $\amalg_{x\in M} L^2(\mu_x) \to M$ is a continuous field of Hilbert spaces (associated to the $C_0(M)$-module obtained by the completion of $C_c(Y)$ with respect to the scalar product $\langle\varphi,\varphi\rangle(x) = \int |\varphi|^2\, d\mu_x$), such that $\amalg_{x\in M} L^2(\mu_x) \otimes \mathbb{H}$ is the trivial field $M \times \mathbb{H} \to M$. The proof is almost the same as above, except that notations are more complicated. We omit details. □

Therefore we have proved the following

PROPOSITION 2.40. – *Let $\Gamma \rightrightarrows M$ be a Lie groupoid. Associated to any element $\alpha \in H^2(\Gamma^\bullet, \mathcal{S}^1)$, there is a canonical $PU(\mathbb{H})$-bundle over $\Gamma \rightrightarrows M$, denoted by $P_\alpha \to M$, whose corresponding class in $H^1(\Gamma^\bullet, PU(\mathbb{H}))$ goes to α under the map Φ' in Eq.* (19).

Clearly, if α can be realized as an S^1-central extension over the groupoid $\Gamma \rightrightarrows M$ without the need of passing to Morita equivariance, then $P_\alpha = P\mathcal{B}$. As a consequence, when Γ is a transformation groupoid, we obtain the following:

COROLLARY 2.41. – *If G is a Lie group acting on M properly, then there is a group homomorphism*:

$$\left\{\textit{Isomorphism classes of } G\textit{-equivariant } PU(\mathbb{H})\textit{-bundles}\right\} \to H^3_G(M,\mathbb{Z}), \tag{22}$$

which admits a canonical inverse. Namely, to any element in $H^3_G(M,\mathbb{Z})$, there associates a canonical G-equivariant $PU(\mathbb{H})$-bundle $P_\alpha \to M$.

3. Twisted K-theory and Fredholm bundles

In this section, we introduce twisted K-theory groups of a Lie groupoid (or more precisely, a differential stack). In the case of proper Lie groupoids, we describe these K-groups in terms of homotopy classes of certain Γ-invariant sections of Fredholm operators associated to the projective Hilbert bundle as constructed in Section 2.6 (Theorem 3.14).

3.1. The reduced C^*-algebra of an S^1-central extension

Given an S^1-central extension of Lie groupoids $S^1 \to R \to \Gamma \rightrightarrows M$, let $L = R \times_{S^1} \mathbb{C}$ be its associated complex line bundle. Then $L \to M$ can be considered as a Fell bundle of C^*-algebras over the groupoid $\Gamma \rightrightarrows M$. Therefore one can construct a C^*-algebra out of it (see Appendix A.3).

DEFINITION 3.1. – Let Γ be a Lie groupoid and $S^1 \to R \to \Gamma \rightrightarrows M$ an S^1-central extension. Then the reduced C^*-algebra of the central extension $C^*_r(\Gamma; R)$ is defined to be $C^*_r(\Gamma; L)$, where $L = R \times_{S^1} \mathbb{C}$ is the associated complex line bundle considered as a Fell bundle of C^*-algebras over $\Gamma \rightrightarrows M$.

There is another picture for this C^*-algebra. Consider

$$C_c(R)^{S^1} = \{\xi \in C_c(R) \mid \xi(\lambda r) = \lambda^{-1}\xi(r),\ \forall\lambda \in S^1,\ r \in R\}.$$

One easily checks that $C_c(R)^{S^1}$ is stable under both the convolution and the adjoint, and that the map

$$C_c(R)^{S^1} \to C_c(\Gamma; L), \tag{23}$$

$\xi \mapsto \eta$, where $\eta(g) = [(r, \xi(r))] \in L_g = R_g \times_{S^1} \mathbb{C}$, is well-defined and is indeed an isomorphism of convolution algebras. Let us define

$$C_r^*(R)^{S^1} := \overline{C_c(R)^{S^1}} \subset C_r^*(R), \tag{24}$$

i.e. $C_r^*(R)^{S^1}$ is the norm-closure of $C_c(R)^{S^1}$ in $C_r^*(R)$ (see Ref. [61] for details on the construction of the reduced C^*-algebra $C_r^*(\Gamma)$ of a groupoid Γ).

The algebra $C^*(S^1) = C_r^*(S^1)$ acts on $C_r^*(R)$ by convolution operators. More precisely, there is a *-homomorphism

$$\Lambda : C^*(S^1) \to M\big(C_r^*(R)\big)$$

such that for every $f \in C(S^1)$ and every $\xi \in C_c(R)$,

$$(\Lambda(f)\xi)(r) = \int_{S^1} f(\lambda)\xi(\lambda^{-1}r)\, d\lambda$$

where $d\lambda$ is the normalized Haar measure $\frac{d\theta}{2\pi}$ on S^1. Indeed, one only needs to check that

$$(U_\lambda\xi)(r) = \xi(\lambda^{-1}r)$$

defines a unitary representation of S^1 into the unitary group of $M(C_r^*(R))$.

The map Λ is non-degenerate, for if f_n is a sequence in $C(S^1)$ converging to the delta function at 1, then $\Lambda(f_n)a$ converges to a for all $a \in C_r^*(R)$. That is, $\Lambda(f_n)$ converges strictly to the identity. Therefore, Λ extends to a unital strictly continuous $*$-homomorphism

$$M\big(C^*(S^1)\big) \to M\big(C_r^*(R)\big)$$

[58, Paragraphs 3.12.10 and 3.12.12].

Let $P_n \in C_r^*(S^1)$ be the convolution by z^n, i.e. P_n corresponds to the characteristic function of $\{n\}$ via the Fourier transformation $C_r^*(S^1) \cong C_0(\mathbb{Z})$. Let $Q_n = \Lambda(P_n)$. Then the Q_n's are pairwise orthogonal projections. Since Λ is non-degenerate, the sum $\sum Q_n$ is strictly convergent to 1. Moreover, since U_λ is in the center of $M(C_r^*(R))$, the projections Q_n also belong to the center of $M(C_r^*(R))$.

Using the formula $Q_n(\xi)(r) = \int_{S^1} \lambda^n \xi(\lambda^{-n} r)\, d\lambda$, one easily checks that the image of Q_n is the closure of the set of elements $\xi \in C_c(R)$ such that $\xi(\lambda r) = \lambda^n \xi(r)$ for all $(\lambda, r) \in S^1 \times R$. In particular, $C_r^*(R)^{S^1}$, the closure of $C_c(R)^{S^1}$ in $C_r^*(R)$, is $Q_{-1}(C_r^*(R))$.

Similarly as in Eq. (23), there is an isometric isomorphism of Hilbert $C_0(M)$-modules:

$$L^2(R)^{S^1} \to L^2(\Gamma; L). \tag{25}$$

If $\xi \in C_c(R)^{S^1}$ and $\eta(g) = [(r, \xi(r))]$, then the norm of ξ, as a convolution operator acting on $L^2(R)$, is equal to the norm of η, as a convolution operator acting on $L^2(\Gamma; L)$.

Noting that $L^2(R)^{S^1}$ is the image of the projection Q_{-1}, we have

$$\begin{aligned}\|\xi\|_{C_r^*(R)} = \|Q_{-1}\xi\|_{C_r^*(R)} = \|\xi Q_{-1}\|_{C_r^*(R)} &= \sup_{\|\varphi\|_{L^2(R)}=1} \|\xi * Q_{-1}\varphi\| \\ &= \sup_{\varphi \in L^2(R)^{S^1},\, \|\varphi\|_{L^2(R)}=1} \|\xi * \varphi\| = \sup_{\|\psi\|_{L^2(\Gamma;L)}=1} \|\eta * \psi\|_{L^2(\Gamma;L)} = \|\eta\|_{C_r^*(\Gamma;L)}.\end{aligned}$$

It follows that

$$C_r^*(\Gamma; R) \cong C_r^*(R)^{S^1}. \tag{26}$$

We summarize the above discussion in the following:

PROPOSITION 3.2. – *Let $S^1 \to R \to \Gamma$ be an S^1-central extension of Lie groupoids. Then there is a canonical isomorphism*

$$C_r^*(R) \cong \bigoplus_{n \in \mathbb{Z}} C_r^*(\Gamma; R^n),$$

where $C_r^*(\Gamma; R^n)$ is the C^*-algebra of the central extension

$$S^1 \to R^n = R \otimes \cdots \otimes R \to \Gamma$$

for all $n \neq 0$, and $C_r^*(\Gamma; R^0) = C_r^*(\Gamma)$ by convention.

The image of $C_r^*(\Gamma; R)$ in $C_r^*(R)$ consists of the closure of $C_c(\Gamma, R)^{S^1}$ defined in Eq. (24).

For $f \in C_c(R) \subset C_r^*(R)$, the image f_n of f in $C_r^*(\Gamma; R^n)$ is given by

$$f_n(r) = \int_{S^1} \lambda^{-n} f(\lambda^n r)\, d\lambda,$$

where $d\lambda$ is the normalized Haar measure on S^1.

For the S^1-central extension $\coprod_{ij} R_{ij} \to \coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i$ in Example 2.26(1), we refer to [60,63,42] for a detailed discussion on the C^*-algebra $C_r^*(\Gamma, R)$.

3.2. Definition of twisted K-theory and first properties

PROPOSITION 3.3. – *Let $R_i \to \Gamma_i \rightrightarrows M_i$ $(i = 1, 2)$ be Morita equivalent S^1-central extensions. Then $C_r^*(R_1)^{S^1}$ and $C_r^*(R_2)^{S^1}$ are Morita equivalent C^*-algebras.*

Proof. – This follows from [62] or from [55, Theorem 11]. □

We are now ready to define twisted K-theory.

DEFINITION 3.4. – Let Γ be a Lie groupoid and $\alpha \in H^2(\Gamma^\bullet, \mathcal{S}^1)$. We define the twisted K-theory as

$$K_\alpha^i(\Gamma^\bullet) = K_{-i}(C_r^*(R)^{S^1}),$$

where $S^1 \to R \to \Gamma' \rightrightarrows M'$ is any central extension realizing the class α and Γ' is Morita equivalent to Γ.

From Proposition 3.3, it follows that if two S^1-central extensions are Morita equivalent, their twisted K-theory groups are isomorphic and therefore only depend on the corresponding stack and the S^1-gerbe over the stack. Consequently, twisted K-theory is well-defined.

Example 3.5 (1). – When Γ is a manifold $M \rightrightarrows M$ and $\alpha \in H^3(M,\mathbb{Z}) \cong H^2(\Gamma^\bullet,\mathcal{S}^1)$, the above definition reduces to the one introduced by Rosenberg [63].

(2) Assume that a Lie group G acts on a smooth manifold M properly. According to Corollary 2.23, the equivariant cohomology $H^3_G(M,\mathbb{Z})$ is isomorphic to $H^2(\Gamma^\bullet,\mathcal{S}^1)$, where Γ denotes the transformation groupoid $G\times M \rightrightarrows M$. Let $\alpha \in H^3_G(M,\mathbb{Z})$. We define the twisted equivariant K-theory

$$K^i_{G,\alpha}(M) := K_{-i}\big(C^*_r(R)^{S^1}\big),$$

where $S^1 \to R \to \Gamma' \rightrightarrows M'$ is any S^1-central extension realizing the class α and Γ' is Morita equivalent to Γ. According to the observation following Definition 3.4, we have the following

PROPOSITION 3.6. – *If G acts on a smooth manifold M properly and freely so that M/G is a manifold, then*

$$K^i_{G,\alpha}(M) \cong K^i_{\alpha'}(M/G),$$

where α' is the image of α under the isomorphism $H^3_G(M,\mathbb{Z}) \xrightarrow{\sim} H^3(M/G,\mathbb{Z})$. More generally, if H is a normal subgroup of G which acts on M properly and freely, then

$$K^i_{G,\alpha}(M) \cong K^i_{G/H,\alpha'}(M/H),$$

where α' is the image of α under the isomorphism $H^3_G(M,\mathbb{Z}) \xrightarrow{\sim} H^3_{G/H}(M/H,\mathbb{Z})$.

Note that the proposition above is a non-trivial theorem even in the non-twisted case, i.e., $\alpha = 0$, in the ordinary equivariant K-theory of Segal [65]. The advantage of our approach is that these facts are encoded as a part of the definition since they are obvious consequences of the Morita equivalence between the transformation groupoids $G\times M \rightrightarrows M$ and $G/H \times M/H \rightrightarrows M/H$. The hard part is to prove that this definition coincides with the topological one which is more often used by geometers.

(3) Given an orbifold $\mathbf{X}$, let $\Gamma \rightrightarrows M$ be an étale groupoid representing this orbifold. Now given $\alpha \in H^3(\mathbf{X},\mathbb{Z}) \cong H^2(\Gamma^\bullet,\mathcal{S}^1)$, we define the twisted orbifold K-theory

$$K^i_\alpha(\mathbf{X}) := K_{-i}\big(C^*_r(R)^{S^1}\big),$$

where $S^1 \to R \to \Gamma' \rightrightarrows M'$ is any S^1-central extension realizing the class α and Γ' is Morita equivalent to Γ. It would be interesting to investigate the relation between our definition with the one given by Lupercio and Uribe [45].

Next let us deduce some properties that are immediate from the definition.

PROPOSITION 3.7 (Bott periodicity). – *Let $S^1 \to R \to \Gamma \rightrightarrows M$ be an S^1-central extension of Lie groupoids. Then for all i,*

$$K^i_\alpha(\Gamma^\bullet) \cong K^{i+2}_\alpha(\Gamma^\bullet),$$
$$K^{i+n}_\alpha(\Gamma^\bullet) \cong K^i_{\alpha_n}\big((\Gamma\times\mathbb{R}^n)^\bullet\big),$$

where α_n is the class of the extension $R\times\mathbb{R}^n \to \Gamma\times\mathbb{R}^n \rightrightarrows M\times\mathbb{R}^n$.

Note also that $K^i_{\alpha_n}((\Gamma\times\mathbb{R}^n)^\bullet)$ is the kernel of the morphism $K^i_{\alpha'_n}((\Gamma\times S^n)^\bullet)\to K^i_\alpha(\Gamma^\bullet)$ induced by the inclusion $\Gamma\times\{pt\}\subset\Gamma\times S^n$, where α'_n is the class of the extension

$$R\times S^n\to\Gamma\times S^n\rightrightarrows M\times S^n.$$

We say that a subgroupoid $\Gamma_1\rightrightarrows M_1$ of $\Gamma\rightrightarrows M$ is saturated if M_1 is an invariant subset of M (i.e. $\Gamma^{M_1}_{M_1}=\Gamma_{M_1}$) such that $\Gamma_1=\Gamma^{M_1}_{M_1}$.

PROPOSITION 3.8. – *Let $S^1\to R\to\Gamma\rightrightarrows M$ be an S^1-central extension of Lie groupoids and denote by α its class in $H^2(\Gamma^\bullet,\mathcal{S}^1)$. Suppose that Γ_1 is an open saturated subgroupoid of Γ and let α_1 be the class of the corresponding S^1-central extension of Γ_1. Then the inclusion $i:\Gamma_1\to\Gamma$ induces a canonical map*

$$i_*:K^n_{\alpha_1}(\Gamma_1^\bullet)\to K^n_\alpha(\Gamma^\bullet).$$

Proof. – Using the obvious notation, $C^*_r(R_1)^{S^1}$ is an ideal of the C^*-algebra $C^*_r(R)^{S^1}$. Indeed, it is not hard to check that $C_c(R_1)^{S^1}\subset C_c(R)^{S^1}$ is stable under the convolution and the adjoint. Since R_1 is a saturated subgroupoid of R, we have

$$\begin{aligned}\|f\|_{C^*_r(R_1)}&=\sup_{x\in R_1^{(0)}}\ \sup_{\xi\in C_c(R_x)}\|f*\xi\|_{L^2(R_x)}\\&=\sup_{x\in R^{(0)}}\ \sup_{\xi\in C_c(R_x)}\|f*\xi\|_{L^2(R_x)}\\&=\|f\|_{C^*_r(R)},\end{aligned}$$

and thus $C^*_r(R_1)^{S^1}$ is a sub-C^*-algebra of $C^*_r(R)^{S^1}$. Moreover, for all $f\in C_c(R_1)^{S^1}$ and $f'\in C_c(R)^{S^1}$ we have $f*f'\in C_c(R_1)^{S^1}$. Therefore $C^*_r(R_1)^{S^1}$ is an ideal in $C^*_r(R)^{S^1}$.

Recall (see for instance [37, Section 3]) that if I_1 and I_2 are two closed ideals in a C^*-algebra A such that $A=I_1+I_2$, then there is a six-term exact sequence

$$\begin{tikzcd} K_0(I_1\cap I_2) \arrow[r, "(j_1)_*\oplus(j_2)_*"] & K_0(I_1)\oplus K_0(I_2) \arrow[r, "(i_1)_*-(i_2)_*"] & K_0(A) \arrow[d, "\partial"] \\ K_1(A) \arrow[u, "\partial"] & K_1(I_1)\oplus K_1(I_2) \arrow[l, "(i_1)_*-(i_2)_*"'] & K_1(I_1\cap I_2) \arrow[l, "(j_1)_*\oplus(j_2)_*"'] \end{tikzcd}$$

where $j_k:I_1\cap I_2\to I_k$ and $i_k:I_k\to A$ are the inclusions ($k=1,2$). Therefore, we get

PROPOSITION 3.9 (Mayer–Vietoris sequence 1). – *Let $S^1\to R\to\Gamma\rightrightarrows M$ be an S^1-central extension of Lie groupoids and denote by α its class in $H^2(\Gamma^\bullet,\mathcal{S}^1)$. Suppose that Γ is the union of two open saturated subgroupoids Γ_1 and Γ_2. Let $\Gamma_{12}=\Gamma_1\cap\Gamma_2$, and let α_1, α_2 and α_{12} be the classes of the induced S^1-central extensions and denote by $\Gamma_{12}\xrightarrow{j_k}\Gamma_k\xrightarrow{i_k}\Gamma$ $(k=1,2)$ the inclusions. Then we have an hexagonal exact sequence*

$$\begin{tikzcd} K^0_{\alpha_{12}}(\Gamma_{12}^\bullet) \arrow[r, "(j_1)_*\oplus(j_2)_*"] & K^0_{\alpha_1}(\Gamma_1^\bullet)\oplus K^0_{\alpha_2}(\Gamma_2^\bullet) \arrow[r, "(i_1)_*-(i_2)_*"] & K^0_\alpha(\Gamma^\bullet) \arrow[d, "\partial"] \\ K^1_\alpha(\Gamma^\bullet) \arrow[u, "\partial"] & K^1_{\alpha_1}(\Gamma_1^\bullet)\oplus K^1_{\alpha_2}(\Gamma_2^\bullet) \arrow[l, "(i_1)_*-(i_2)_*"'] & K^1_{\alpha_{12}}(\Gamma_{12}^\bullet) \arrow[l, "(j_1)_*\oplus(j_2)_*"'] \end{tikzcd}$$

Proof. – It is clear that both $I_1 = C_r^*(R_1)^{S^1}$ and $I_2 = C_r^*(R_2)^{S^1}$ are ideals of $A = C_r^*(R)^{S^1}$.

To check that $I_1 \cap I_2 = C_r^*(R_{12})^{S^1}$, note that $I_1 \cap I_2 = I_1 I_2$ (this is a standard result in C^*-algebras) and that $f_1 * f_2 \in C_c(R_{12})^{S^1}$ if $f_1 \in C_c(R_1)^{S^1}$ and $f_2 \in C_c(R_2)^{S^1}$.

To check that $I_1 + I_2 = A$, take a partition of unity (φ_1, φ_2) associated to the cover $(\Gamma_i^{(0)}/\Gamma)_{i=1,2}$ of M/Γ. Let $f \in C_c(R)^{S^1}$. Then, considering φ_i as R-invariant functions on M, we have $f = (\varphi_1 f) + (\varphi_2 f) \in C_c(R_1)^{S^1} + C_c(R_2)^{S^1}$. □

PROPOSITION 3.10. – *Let $S^1 \to R \to \Gamma \rightrightarrows M$ be an S^1-central extension of Lie groupoids and denote by α its class in $H^2(\Gamma^\bullet, \mathcal{S}^1)$. Assume that Γ_1 is a closed saturated subgroupoid of Γ. Let $\alpha_1 \in H^2(\Gamma_1^\bullet, \mathcal{S}^1)$ be the class of the corresponding S^1-central extension. Then the inclusion $i : \Gamma_1 \to \Gamma$ induces a canonical map*

$$i^* : K_\alpha^n(\Gamma^\bullet) \to K_{\alpha_1}^n(\Gamma_1^\bullet).$$

Proof. – Using the obvious notation, $C_r^*(R_1)^{S^1}$ is a quotient of the C^*-algebra $C_r^*(R)^{S^1}$. Indeed, the restriction map $C_c(R)^{S^1} \to C_c(R_1)^{S^1}$ is a surjective $*$-homomorphism of convolution algebras and is norm-decreasing, and therefore induces a surjective $*$-homomorphism $C_r^*(R)^{S^1} \to C_r^*(R_1)^{S^1}$. □

Suppose that $A_1 = A/I_1$ and $A_2 = A/I_2$ are two quotients of a C^*-algebra such that $I_1 \cap I_2 = \{0\}$, and let $A_{12} = A/(I_1 + I_2)$. Denote by $p_k : A \to A_k$ and by $q_k : A_k \to A_{12}$ the quotient maps. Then there is a six-term exact sequence

$$\begin{tikzcd} K_0(A_{12}) \arrow[d, "\partial"] & K_0(A_1) \oplus K_0(A_2) \arrow[l, "(q_1)_* \oplus (q_2)_*"'] & K_0(A) \arrow[l, "(p_1)_* - (p_2)_*"'] \\ K_1(A) \arrow[r, "(p_1)_* - (p_2)_*"] & K_1(A_1) \oplus K_1(A_2) \arrow[r, "(q_1)_* \oplus (q_2)_*"] & K_1(A_{12}) \arrow[u, "\partial"'] \end{tikzcd}$$

Since we cannot locate this standard fact in the literature, here is a sketch of the proof. For every locally compact space X, we will denote by $A(X)$ the C^*-algebra $C_0(X, A)$. Consider the C^*-algebra

$$D = \big\{ (f_-, a, f_+) \in A_1(-1,0] \oplus A \oplus A_2[0,1) \mid f_-(0) = p_1(a) \text{ and } p_2(a) = f_+(0) \big\}.$$

There is an obvious exact sequence

$$0 \to I_2(-1,0] \oplus I_1[0,1) \to D \to A_{12}(-1,1) \to 0.$$

Since $I_2(-1,0] \oplus I_1[0,1)$ is contractible, the six-term exact sequence in K-theory yields an isomorphism $K_i(D) \cong K_i(A_{12}(-1,1))$, hence

$$K_i(D) \cong K_{i+1}(A_{12}). \tag{27}$$

Now, the obvious exact sequence

$$0 \to A_1(-1,0) \oplus A_2(0,1) \to D \to A \to 0$$

gives a six-term exact sequence in K-theory, which yields the result via Bott periodicity and Eq. (27).

PROPOSITION 3.11 (Mayer–Vietoris sequence 2). – *Let $S^1 \to R \to \Gamma \rightrightarrows M$ be an S^1-central extension of* proper *Lie groupoids and denote by α its class in $H^2(\Gamma^\bullet, \mathcal{S}^1)$. Suppose that Γ is the union of two closed saturated groupoids Γ_1 and Γ_2. Let $\Gamma_{12} = \Gamma_1 \cap \Gamma_2$. Let α_1, α_2 and α_{12} be the classes of their induced S^1-central extensions and denote by $\Gamma_{12} \xrightarrow{j_k} \Gamma_k \xrightarrow{i_k} \Gamma$ $(k = 1, 2)$ the inclusions. Then we have an hexagonal exact sequence*

$$\begin{tikzcd}
K^0_{\alpha_{12}}(\Gamma_{12}) \arrow[d, "\partial"] & K^0_{\alpha_1}(\Gamma_1) \oplus K^0_{\alpha_2}(\Gamma_2) \arrow[l, "(j_1)^* \oplus (j_2)^*"'] & K^0_\alpha(\Gamma) \arrow[l, "(i_1)^* - (i_2)^*"'] \\
K^1_\alpha(\Gamma) \arrow[r, "(i_1)^* - (i_2)^*"] & K^1_{\alpha_1}(\Gamma_1) \oplus K^1_{\alpha_2}(\Gamma_2) \arrow[r, "(j_1)^* \oplus (j_2)^*"] & K^1_{\alpha_{12}}(\Gamma_{12}) \arrow[u, "\partial"]
\end{tikzcd}$$

Proof. – Let Γ'_k ($k = 1, 2$ or 12) be the complementary of Γ_k. Since Γ_k is closed and saturated, it follows that Γ'_k is an open saturated subgroupoid of Γ. With the obvious notations, write $I_k = C^*_r(R'_k)^{S^1}$, $A = C^*_r(R)^{S^1}$ and $A_k = A/I_k$, where R'_k denotes the complementary of R_k. Since $\Gamma = \Gamma_1 \cup \Gamma_2$, we have $I_1 \cap I_2 = \{0\}$. It is also clear that $I_1 + I_2 = I_{12}$.

To obtain the Mayer–Vietoris sequence, it suffices to show that $C^*_r(R_k)^{S^1} = A/I_k$. This is not always true for every groupoid. I.e. the sequence

$$0 \to C^*_r(R'_k)^{S^1} \to C^*_r(R)^{S^1} \to C^*_r(R_k)^{S^1} \to 0$$

is not necessarily exact. However, the analogous sequence with C^* instead of C^*_r is always exact by the universal property of the full C^*-algebra of a groupoid, and we have $C^*_r = C^*$ for proper groupoids (or even for amenable groupoids). □

Example 3.12. – Assume that Γ is a transformation groupoid $G \times M \rightrightarrows M$, where G is a Lie group acting on M properly. Assume that U_1 and U_2 are G-invariant open submanifolds of M such that $M = U_1 \cup U_2$. Then Proposition 3.9 yields that

$$\begin{tikzcd}
K^0_{G,\alpha_{12}}(U_1 \cap U_2) \arrow[r, "(j_1)_* \oplus (j_2)_*"] & K^0_{G,\alpha_1}(U_1) \oplus K^0_{G,\alpha_2}(U_2) \arrow[r, "(i_1)_* - (i_2)_*"] & K^0_{G,\alpha}(M) \arrow[d, "\partial"] \\
K^1_{G,\alpha}(M) \arrow[u, "\partial"] & K^1_{G,\alpha_1}(U_1) \oplus K^1_{G,\alpha_2}(U_2) \arrow[l, "(i_1)_* - (i_2)_*"'] & K^1_{G,\alpha_{12}}(U_1 \cap U_2) \arrow[l, "(j_1)_* \oplus (j_2)_*"']
\end{tikzcd}$$

Similarly, if F_1 and F_2 are G-invariant closed submanifolds of M such that $F_1 \cup F_2 = M$, then Proposition 3.11 yields that

$$\begin{tikzcd}
K^0_{G,\alpha_{12}}(F_1 \cap F_2) \arrow[d, "\partial"] & K^0_{G,\alpha_1}(F_1) \oplus K^0_{G,\alpha_2}(F_2) \arrow[l, "(j_1)^* \oplus (j_2)^*"'] & K^0_{G,\alpha}(M) \arrow[l, "(i_1)^* - (i_2)^*"'] \\
K^1_{G,\alpha}(M) \arrow[r, "(i_1)^* - (i_2)^*"] & K^1_{G,\alpha_1}(F_1) \oplus K^1_{G,\alpha_2}(F_2) \arrow[r, "(j_1)^* \oplus (j_2)^*"] & K^1_{G,\alpha_{12}}(F_1 \cap F_2) \arrow[u, "\partial"]
\end{tikzcd}$$

3.3. The main theorem

Let $S^1 \to R \to \Gamma \rightrightarrows M$ be an S^1-central extension of groupoids, and $P\mathcal{B} \to M$ its associated principal $PU(\mathbb{H})$-bundle over the groupoid $\Gamma \rightrightarrows M$ as constructed in Proposition 2.38. Let

$$\mathcal{L}(\widetilde{\mathcal{H}}) = P\mathcal{B} \times_{PU(\mathbb{H})} \mathcal{L}(\mathbb{H}) \to M$$

and

$$\mathcal{K}(\widetilde{\mathcal{H}}) = P\mathcal{B} \times_{PU(\mathbb{H})} \mathcal{K}(\mathbb{H}) \to M$$

be its associated bundles of C^*-algebras, where $\mathcal{L}(\mathbb{H})$ denotes the algebra of bounded operators on $\mathbb{H}$ endowed with the $*$-strong topology, and $\mathcal{K}(\mathbb{H})$ denotes the C^*-algebra of compact operators on $\mathbb{H}$ endowed with the norm-topology. The group $PU(\mathbb{H})$ acts on $\mathcal{L}(\mathbb{H})$ and $\mathcal{K}(\mathbb{H})$ by conjugation. To justify the notation, we show that these bundles are isomorphic to those of bounded and compact operators associated to the Hilbert bundle $\bigcup_{x\in M} L^2(R^x)^{S^1} \otimes \mathbb{H}$ (see Eq. (20)) as in Appendix A (Propositions A.5 and A.6).

Indeed it is simple to see that the fiber of $\mathcal{L}(\widetilde{\mathcal{H}}) \to M$ and $\mathcal{K}(\widetilde{\mathcal{H}}) \to M$ at each $x \in M$ are, respectively, $\mathcal{L}(\mathcal{H}_x)$ and $\mathcal{K}(\mathcal{H}_x)$. The map

$$P\mathcal{B} \times_{PU(\mathbb{H})} \mathcal{L}(\mathbb{H}) \to \coprod_{x\in M} \mathcal{L}(\mathcal{H}_x),$$
$$(u, T) \mapsto uTu^{-1}$$

is clearly a bijection. To identify the topology of $P\mathcal{B} \times_{PU(\mathbb{H})} \mathcal{L}(\mathbb{H})$, we can assume that $\mathcal{H}$ is a trivial bundle (since it is locally trivial). Then

$$P\mathcal{B} \times_{PU(\mathbb{H})} \mathcal{L}(\mathbb{H}) \cong M \times PU(\mathbb{H}) \times_{PU(\mathbb{H})} \mathcal{L}(\mathbb{H}) \cong M \times \mathcal{L}(\mathbb{H})$$

is obviously the bundle of bounded operators associated to a continuous field of Hilbert spaces (see Proposition A.5). The proof for $\mathcal{K}(\widetilde{\mathcal{H}})$ is similar (see Proposition A.6).

The groupoid Γ-action on $P\mathcal{B} \to M$ induces an action on the C^*-algebra bundle $\mathcal{L}(\widetilde{\mathcal{H}}) \to M$ (and $\mathcal{K}(\widetilde{\mathcal{H}}) \to M$ respectively). On the other hand, the associated line bundle $L = R \times_{S^1} \mathbb{C} \to M$ can be considered as a Fell bundle over the groupoid $\Gamma \rightrightarrows M$. Therefore the general construction of Yamagami (see Appendix A.3) gives rise to a continuous action of the groupoid $\Gamma \rightrightarrows M$ on the C^*-bundle $\mathcal{L}(\widetilde{L}^2(\Gamma; L)) \to M$ (and $\mathcal{K}(\widetilde{L}^2(\Gamma; L)) \to M$ as well), which extends to an action on the C^*-bundle $\mathcal{L}(\widetilde{L}^2(\Gamma; L) \otimes \mathbb{H}) \to M$ (and $\mathcal{K}(\widetilde{L}^2(\Gamma; L) \otimes \mathbb{H}) \to M$). From Eq. (25), it follows that the Hilbert bundles $\widetilde{L}^2(\Gamma; L) \otimes \mathbb{H} \to M$ and $\widetilde{\mathcal{H}} \to M$ are canonically isomorphic. In fact we have the following:

LEMMA 3.13. – *The C^*-algebra bundles $\mathcal{L}(\widetilde{\mathcal{H}}) \to M$ ($\mathcal{K}(\widetilde{\mathcal{H}}) \to M$, respectively) and $\mathcal{L}(\widetilde{L}^2(\Gamma; L) \otimes \mathbb{H}) \to M$ (and $\mathcal{K}(\widetilde{L}^2(\Gamma; L) \otimes \mathbb{H}) \to M$ respectively) are canonically isomorphic, and the isomorphism respects the Γ-action.*

Recall that a section $x \mapsto T_x \in \mathcal{L}(\widetilde{\mathcal{H}}_x),\ x \in M$, is strongly continuous if $x \mapsto T_x\xi$ is norm-continuous for all continuous sections $\xi \in C(M, \widetilde{\mathcal{H}})$, and that $x \mapsto T_x$ is $*$-strongly continuous if $x \mapsto T_x$ and $x \mapsto T_x^*$ are strongly continuous.

By $C_b(M, \mathcal{L}(\widetilde{\mathcal{H}}))$ we denote the space of norm-bounded, $*$-*strongly*-continuous sections of bounded operators on $\mathcal{H}$, and by $C_b(M, \mathcal{L}(\widetilde{\mathcal{H}}))^\Gamma$ we denote the subalgebra of Γ-invariant sections. Similarly, by $C_0(M, \mathcal{K}(\widetilde{\mathcal{H}}))$ we denote the space of *norm*-continuous sections of compact operators vanishing at infinity.

Let $\mathcal{K}_\Gamma(\mathcal{H})$ be the space of norm-continuous Γ-invariant sections $\{T_x \mid x \in M\}$ of the C^*-algebra bundle $\mathcal{K}(\widetilde{\mathcal{H}}) \to M$ satisfying the boundary condition $\|T_x\| \to 0$ when $x \to \infty$ in M/Γ. Note that $\|T_x\|$ can be considered as a function on the orbit space M/Γ due to the invariance assumption.

Denote by $\mathcal{F}^0_\alpha$ the space of $T \in C_b(M, \mathcal{L}(\widetilde{\mathcal{H}}))^\Gamma$ such that there exists $S \in C_b(M, \mathcal{L}(\widetilde{\mathcal{H}}))^\Gamma$ satisfying $1 - TS, 1 - ST \in \mathcal{K}_\Gamma(\mathcal{H})$. In other words,

(i) T_x and S_x are Fredholm for all x, and the sections $x \mapsto T_x$ and $x \mapsto S_x$ are $*$-strongly continuous and Γ-invariant;

(ii) $1 - T_xS_x$, $1 - S_xT_x$ are compact operators for all x and the sections $x \mapsto 1 - T_xS_x$, $x \mapsto 1 - S_xT_x$ are norm-continuous and vanish at ∞ in M/Γ.

Denote by $\mathcal{F}^1_\alpha$ the space of self-adjoint elements in $\mathcal{F}^0_\alpha$. Our main theorem is

THEOREM 3.14. – *Let $\Gamma \rightrightarrows M$ be a proper Lie groupoid, $S^1 \to R \to \Gamma$ an S^1-central extension and denote by α its class in $H^2(\Gamma^\bullet, \mathcal{S}^1) (\cong H^3(\Gamma^\bullet, \mathbb{Z}))$. Then*

$$K^i_\alpha(\Gamma^\bullet) = \{[T] \mid T \in \mathcal{F}^i_\alpha\},$$

where $[T]$ denotes the homotopy class of T.

The proof of (a generalization of) Theorem 3.14 is the content of the next section.

Another way to formulate Theorem 3.14 is as follows.

Let $\Gamma \rightrightarrows M$ be a proper Lie groupoid and $\alpha \in H^3(\Gamma^\bullet, \mathbb{Z})$. Let $P_\alpha \to M$ be its corresponding canonical $PU(\mathbb{H})$-bundle over the groupoid $\Gamma \rightrightarrows M$ as in Proposition 2.40. Consider its associated bundles:

$$\mathit{Fred}^i_\alpha(\mathbb{H}) := P_\alpha \times_{PU(\mathbb{H})} \mathit{Fred}^i(\mathbb{H}) \to M, \tag{28}$$

$$\mathcal{K}_\alpha(\mathbb{H}) := P_\alpha \times_{PU(\mathbb{H})} \mathcal{K}(\mathbb{H}) \to M, \tag{29}$$

where $\mathit{Fred}^i(\mathbb{H})$ is endowed with the $*$-strong topology while $\mathcal{K}(\mathbb{H})$ is endowed with the norm-topology. By $\mathcal{F}^i_\alpha$, we denote the space of norm-bounded, Γ-invariant, continuous sections $x \mapsto T_x$ of the bundle $\mathit{Fred}^i_\alpha(\mathbb{H}) \to M$ such that there exists a norm-bounded, Γ-invariant, continuous section $x \mapsto S_x$ of $\mathit{Fred}^i_\alpha(\mathbb{H}) \to M$ with the property that $1 - T_xS_x$ and $1 - S_xT_x$ are continuous sections of $\mathcal{K}_\alpha(\mathbb{H})$ vanishing at infinity.[4] Then we have the following

THEOREM 3.15. – *Let $\Gamma \rightrightarrows M$ be a proper Lie groupoid, and $\alpha \in H^3(\Gamma^\bullet, \mathbb{Z})$. Then*

$$K^i_\alpha(\Gamma^\bullet) = \{[T] \mid T \in \mathcal{F}^i_\alpha\},$$

where $[T]$ denotes the homotopy class of T.

Remark 3.16. – Note that there may exist different $PU(\mathbb{H})$-principal bundles over $\Gamma \rightrightarrows M$ other than P_α, which also map to $\alpha \in H^2(\Gamma^\bullet, \mathcal{S}^1)$ under the map Φ'. However, only the construction using this particular principal bundle P_α gives the right answer for the Fredholm picture of twisted K-theory groups.

Example 3.17. – (1) When Γ is a compact manifold M, the principal $PU(\mathbb{H})$-bundle $P_\alpha \to M$ over M is represented by a 1-cocycle $g_{kl} : U_{kl} \to PU(\mathbb{H})$. A class $[T]$ in $K^i_\alpha(M)$ corresponds to a section of the bundle $P_\alpha \times_{PU(\mathbb{H})} \mathit{Fred}^i(\mathbb{H})$, thus to a family of $*$-strongly continuous maps $T_k : U_k \to \mathit{Fred}^i(\mathbb{H})$ satisfying $T_l = g_{kl}^{-1} T_k g_{kl}$ on U_{kl} [4,27].

[4] This definition of $\mathcal{F}^i_\alpha$ obviously agrees with the previous one.

(2) When Γ is a transformation groupoid $G \times M \rightrightarrows M$, for any $\alpha \in H^3_G(M,\mathbb{Z})$, the associated $PU(\mathbb{H})$-bundle $P_\alpha \to M$ over Γ is a G-equivariant $PU(\mathbb{H})$-bundle over M. Therefore, its associated bundles $\mathit{Fred}^i_\alpha(\mathbb{H}) \to M$ and $\mathcal{K}_\alpha(\mathbb{H}) \to M$ are G-equivariant. Thus $K^i_{G,\alpha}(M)$ can be represented as the group of homotopy classes of G-invariant $*$-strongly continuous sections of the Fredholm bundle $\mathit{Fred}^i_\alpha(\mathbb{H}) \to M$.

In terms of local charts, the principal $PU(\mathbb{H})$-bundle P_α is represented by a 1-cocycle $\varphi_{kl} : (G \times M)^{U_l}_{U_k} \to PU(\mathbb{H})$ (see Eqs. (11) and (12)). Then a class $[T]$ in $K^i_{G,\alpha}(M)$ corresponds to a family of $*$-strongly continuous maps $T_k : U_k \to \mathit{Fred}^i(\mathbb{H})$ satisfying

$$T_l(y) = \varphi_{lk}(g,x) T_k(x) \varphi_{kl}(g^{-1}, y)$$

if $x \in U_k$, $y \in U_l$ and $y = gx$.

4. C^*-algebras of Fell bundles over proper groupoids

The goal of this section is to prove a general result in C^*-algebras, which includes Theorem 3.14 as a special case. More precisely, we prove that if $\Gamma \rightrightarrows M$ is a proper groupoid and $E = \coprod_{g\in\Gamma} E_g$ is a u.s.c. Fell bundle over Γ (see Appendix A), then its associated C^*-algebra $C^*_r(\Gamma; E)$ is isomorphic to the space of Γ-equivariant continuous sections of the C^*-algebra bundle $\coprod_{x\in M} \mathcal{K}(L^2(\Gamma_x; E)) \to M$ that vanish at infinity in M/Γ (Proposition 4.3), and to deduce that the K-theory groups of $C^*_r(\Gamma; E)$ are isomorphic to homotopy classes of certain generalized Fredholm operators (Theorem 4.6).

Let us recall

DEFINITION 4.1. – Let $\Gamma \rightrightarrows M$ be a proper groupoid with Haar system $\{\lambda^x \mid x \in M\}$. A continuous function $c : M \to \mathbb{R}_+$ is called a cutoff function if

(i) for all $x \in M$, $\int_{\gamma\in\Gamma^x} c(s(\gamma))\lambda^x(d\gamma) = 1$; and
(ii) for all $K \subset M$ compact, the support of $(c \circ s)_{|\Gamma^K}$ is compact.

The condition (ii) means that if $F \subset M$ is the saturate of a compact set, then $F \cap \operatorname{supp}(c)$ is compact. It is known that a cutoff function exists if and only if Γ is proper [68, Proposition 6.7].

Cutoff functions allow us to make “averages”. Namely, let

$$T^\Gamma_x = \int_{\gamma\in\Gamma^x} \alpha_\gamma(T_{s(\gamma)}) c\big(s(\gamma)\big) \lambda^x(d\gamma) \in \mathcal{L}\big(L^2(\Gamma_x; E)\big). \tag{30}$$

Then $T \mapsto T^\Gamma$ is a linear projection of norm one from $\mathcal{L}(L^2(\Gamma; E))$ onto $\mathcal{L}(L^2(\Gamma; E))^\Gamma$. (If $\Gamma = G$ is a compact group then T^G is the average $\int_G \alpha_g(T)\, dg$.)

More generally, let $\Gamma \rightrightarrows M$ be a proper groupoid with Haar system acting continuously on a u.s.c. field of C^*-algebras $\mathcal{A} \to M$. By $A = C_0(M, \mathcal{A})$ we denote its corresponding C^*-algebra of continuous sections vanishing at infinity. As in Proposition A.5, there exists a (not u.s.c.) field of C^*-algebras $M(\mathcal{A}) \to M$ with the fiber at $x \in M$ being $M(\mathcal{A}_x)$, such that

(a) a section $x \mapsto T_x \in M(\mathcal{A}_x)$ is a continuous section of $M(\mathcal{A}) \to M$ if and only if for every continuous section $x \mapsto a_x$ of $\mathcal{A} \to M$, $x \mapsto T_x a_x$ and $x \mapsto T^*_x a_x$ are continuous sections of $\mathcal{A} \to M$.
(b) The algebra $C_b(M, M(\mathcal{A}))$ of continuous, norm-bounded sections is isomorphic to the multiplier algebra $M(A)$.

It is clear that the Γ-action on $\mathcal{A} \to M$ induces a natural Γ-action on $M(\mathcal{A}) \to M$. By $M(A)^\Gamma$, we denote the C^*-subalgebra of $M(A)$ consisting of Γ-invariant sections. For any $T \in A$, let

$T^\Gamma \in M(A)$ be the element such that

$$(T^\Gamma)_x = \int\limits_{\gamma\in\Gamma^x} \alpha_\gamma(T_{s(\gamma)})c\big(s(\gamma)\big)\lambda^x(d\gamma) \in M(\mathcal{A}_x).$$

Of course, the averaging map $T \mapsto T^\Gamma$ depends on the choice of the cutoff function.

Let us introduce some notations. If $\mathcal{E}$ is a $C_0(M)$-Hilbert module, let

$$\mathcal{C}(\mathcal{E}) = \big\{T \in \mathcal{L}(\mathcal{E}) \mid \varphi T \in \mathcal{K}(\mathcal{E})\ \forall\varphi \in C_0(M)\big\}. \tag{31}$$

If, moreover, $\mathcal{E}$ is a (Γ, E)-equivariant Hilbert module, let

$$\mathcal{K}_\Gamma(\mathcal{E}) = \big\{T \in \mathcal{C}(\mathcal{E})^\Gamma \mid \|T_x\| \to 0 \text{ when } x \to \infty \text{ in } M/\Gamma\big\}. \tag{32}$$

More generally, if Γ acts on a $C_0(M)$-algebra A, let

$$\begin{aligned} A_\Gamma = \{T \in M(A)^\Gamma \mid \forall\varphi \in C_0(M), \varphi T \in A \\ \text{and } \|T_x\| \to 0 \text{ when } x \to \infty \text{ in } M/\Gamma\big\}. \end{aligned} \tag{33}$$

For example, if $A = C_0(\mathbb{R})$ and $\Gamma = \mathbb{Z}$ acts on A by translations, then A_Γ is the space of 1-periodic continuous functions on $\mathbb{R}$, while the algebra A^Γ, consisting of Γ-invariant elements in A, is $\{0\}$.

LEMMA 4.2. – *With the above assumptions, A_Γ is a C^*-subalgebra of $M(A)$ and is equal to* $\{T^\Gamma \mid T \in A\}$.

Proof. – The first assertion is easy.

To show $\{T^\Gamma \mid T \in A\} \subset A_\Gamma$, let $T \in A$. By a density argument, we may assume that T is supported on a compact subset of M. That is, $T_x = 0$ for x outside a compact set. Let $\varphi \in C_c(M)$. Then $(\varphi \cdot T^\Gamma)_x = \int_{\gamma\in\Gamma^x} \varphi(x)\alpha_\gamma(T_{s(\gamma)})c(s(\gamma))\, d\gamma$ is the integral on Γ^x of a compactly supported element of t^*A. Thus $\varphi \cdot T^\Gamma$ belongs to A. Moreover, it is clear that T^Γ is zero outside a compact subset of M/Γ.

To show that $\{T^\Gamma \mid T \in A\} \supset A_\Gamma$, let $T \in A_\Gamma$. Assume first that $T_x = 0$ outside a compact set K of M/Γ. Let $L = (\operatorname{supp}(c)) \cap \pi^{-1}(K)$ where $\pi : M \to M/\Gamma$ is the projection. Then, for all $\varphi \in C_0(M)$ such that $\varphi = 1$ on L, one has

$$\begin{aligned} T_x &= T_x \int\limits_{g\in\Gamma^x} c\big(s(g)\big)\varphi\big(s(g)\big)\,\lambda^x(dg) \\ &= \int\limits_{g\in\Gamma^x} T_x c\big(s(g)\big)\varphi\big(s(g)\big)\,\lambda^x(dg) \\ &= \int\limits_{g\in\Gamma^x} \alpha_g(T_{s(g)})c\big(s(g)\big)\varphi\big(s(g)\big)\lambda^x(dg). \end{aligned}$$

Thus $T = (\varphi T)^\Gamma$.

In the general case, one has $T = \sum_n T_n$ where T_n is zero outside a compact subset of M/Γ and $\|T_n\| \leqslant 2^{-n}$ for n large enough. From the previous paragraph, we have $T_n = (\varphi_n T_n)^\Gamma$, and therefore $T = (\sum_n \varphi_n T_n)^\Gamma$. □

PROPOSITION 4.3. – *If* $\Gamma \rightrightarrows M$ *is a proper groupoid with Haar system and* E *is a u.s.c. Fell bundle over* Γ, *then*

$$C_r^*(\Gamma;E) = \{T^\Gamma \mid T \in \mathcal{K}\big(L^2(\Gamma;E)\big)\} = \mathcal{K}_\Gamma\big(L^2(\Gamma;E)\big).$$

Proof. – Let us explain the idea of the proof in the case that Γ is a compact group G. In this case $C_r^*(G)$ is the closure of the space of convolution operators on $L^2(G)$. Since these operators have a G-invariant kernel $K \in C(G\times G)$, they are compact G-invariant. Conversely, any compact invariant operator T is the limit of operators with G-invariant kernel, and such kernels are of the form $a(gh^{-1})$ where $a \in C(G)$. It follows that T is the limit of convolution operators.

Now we consider the general case of a proper groupoid. Let us first show that $C_r^*(\Gamma;E) \supset \{T^\Gamma \mid T\in\mathcal{K}(L^2(\Gamma;E))\}$. Let $T\in\mathcal{K}(L^2(\Gamma;E))$. We need to show that T^Γ lies in the image of $C_r^*(\Gamma;E)$. We may assume that $T = T_{b,b'} := (\xi \mapsto b\langle b',\xi\rangle)$, where b, $b' \in C_c(\Gamma;E)$, i.e.

$$(T\xi)(g) = b(g)\int_{h\in\Gamma_{s(g)}} b'(h)^*\xi(h)\,\lambda_{s(g)}(dh).$$

Then,

$$(T^\Gamma\xi)(g) = \int_{h\in\Gamma_{s(g)}}\int_{\gamma\in\Gamma^{s(h)}} b(g\gamma)b'(h\gamma)^*\xi(h)c\big(s(\gamma)\big)\,\lambda^{s(h)}(d\gamma)\,\lambda_{s(g)}(dh).$$

Set $f(g) = \int_{\gamma\in\Gamma^{s(g)}} b(g\gamma)b'(\gamma)^*c(s(\gamma))\,\lambda^{s(g)}(d\gamma)$. Let us check that $f\in C_c(\Gamma;E)$. By Proposition A.10, $(g,h)\mapsto b(g)b'(h^{-1})^*$ can be approximated by sums of the form $\sum_i f_i(g,h)\zeta_i(gh)$, where $f_i\in C_c(\Gamma)$ and $\zeta_i\in C_c(\Gamma;E)$. Therefore $(g,\gamma)\mapsto b(g\gamma)d(\gamma)^*$ is approximated by $\sum_i f_i(g\gamma,\gamma^{-1})\zeta_i(g)$. Then, approximating $f_i(g\gamma,\gamma^{-1})$ by $\sum_j f_{1,i,j}(g)f_{2,i,j}(\gamma)$, we see that $f(g)$ is approximated by $\sum_i f_i'(g)\zeta_i(g)$ where $f_i'\in C_c(\Gamma)$. Hence f is a continuous section. Moreover, since f is obviously compactly supported, we have $f\in C_c(\Gamma;E)$. Now,

$$f(gh^{-1}) = \int_{\gamma\in\Gamma^{s(g)}} b(g\gamma)b'(h\gamma)^*c\big(s(\gamma)\big)\,\lambda^{s(g)}(d\gamma),$$

and $\pi_l(f)\xi(g) = \int_{h\in\Gamma_{s(g)}} f(gh^{-1})\xi(h)\,\lambda_{s(g)}(dh)$, where $\pi_l : C_c(\Gamma;E)\to\mathcal{L}(L^2(\Gamma;E))$ denotes the left regular representation. Therefore, $T_{b,b'}^\Gamma = \pi_l(f)$.

Next we show that $C_r^*(\Gamma;E)\subset\overline{\{T^\Gamma\mid T\in\mathcal{K}(L^2(\Gamma;E))\}}$. Assume that $a\in C_c(\Gamma;E)$. Let K be a compact subset of M such that Γ_K^K contains the support of a, and K_1 a compact subset such that the interior of K_1 contains K. By the definition of c, the set $L = (\operatorname{supp}(c))\cap\pi^{-1}(\pi(K_1))$, where $\pi : M\to M/\Gamma$ is the projection, is compact. By Proposition A.10, one may approximate $a(gh^{-1})$ on the compact set $\Gamma_L^{K_1}\times\Gamma_L^{K_1}$ uniformly by elements of the form

$$\sum_i b_i(g)b_i'(h)^*,$$

where b_i, $b_i'\in E$. Therefore $(g,h)\mapsto a(gh^{-1})$ is approximated uniformly on

$$\{(g,h)\in\Gamma^{K_1}\times\Gamma^{K_1}\mid s(g)=s(h)\}$$

by elements of the form $\sum_i \int_{\gamma\in\Gamma^{s(g)}} b_i(g\gamma)b_i'(h\gamma)^*c(s(\gamma))\,d\gamma$. Replacing $b_i(g)$ by $\varphi(t(g))b_i(g)$ and $b_i'(g)$ by $\varphi(t(g))b_i'(g)$, where $\varphi\in C_c(M)$, $0\leqslant\varphi\leqslant 1$, $\varphi=1$ on K and $\varphi=0$ on $M-K_1$, we define an operator

$$T=\sum T_{b_i,b_i'}.$$

Then $\pi_l(a)\in\mathcal{L}(L^2(\Gamma;E))$ is approximated by the operator T^Γ.

The other inclusions are proved using Lemma 4.2 with $A=\mathcal{K}(L^2(\Gamma;E))$. □

To continue, let us introduce the following convention. For any C^*-algebras A and B such that $B\subset M(A)$, we say that B contains an approximate unit for A if there exist $u_i\in B$ such that $0\leqslant u_i\leqslant 1$ and $u_ia\to a$ for all $a\in A$. This terminology is slightly abusive since u_i may not belong to A.

LEMMA 4.4. – *Let $\Gamma\rightrightarrows M$ be a proper groupoid with Haar system acting on a u.s.c. field of C^*-algebras $\mathcal{A}\to M$, and $A=C_0(M,\mathcal{A})$. Then*

(a) *A_Γ contains an approximate unit for A;*

(b) $\overline{A_\Gamma A}=A$;

(c) $M(A_\Gamma)=M(A)^\Gamma$.

Proof. – (a) Let $(u_i)_{i\in I}$ be an approximate unit in A (it is standard that this always exists, see [21]). Let $\tilde u_i=(u_i)^\Gamma$. Then $\tilde u_i\in A_\Gamma$ by Lemma 4.2. It suffices to show that for all $a\in A_U$, where U is a relatively compact open subset of M, we have $\tilde u_ia\to a$.

Let $b(g)=a_{t(g)}c(s(g))$. Then $b=(t^*a)(c\circ s)\in t^*A$. Set $v_i(g)=\alpha_g((u_i)_{s(g)})$. Since $\alpha: s^*A\xrightarrow{\sim}t^*A$ is an isomorphism and $v_i=\alpha(s^*u_i)$, it follows that v_i is an approximate unit for t^*A. Thus $v_ib\to b$, i.e.,

$$\sup_{g\in\Gamma}\bigl\|a_{t(g)}c\bigl(s(g)\bigr)-\alpha_g(u_i)_{s(g)}a_{t(g)}c\bigl(s(g)\bigr)\bigr\|\to 0.$$

By integration on Γ^x ($x\in M$), it follows easily that $\|a-\tilde u_ia\|\to 0$.

(b) clearly follows from (a).

(c) The map $M(A)^\Gamma\to M(A_\Gamma),\ a\mapsto\mu(a)$, where $\mu(a)b=ab$, $\forall a\in M(A)^\Gamma,\ b\in A_\Gamma$, is well-defined and $*$-linear. To obtain its inverse, by identifying $M(A)$ with $\mathcal{L}(A)$ (the space of A-linear adjointable operators on the A-Hilbert module A), the map

$$\nu: T\in M(A_\Gamma)=\mathcal{L}(A_\Gamma)\mapsto T\otimes 1\in\mathcal{L}(A_\Gamma\otimes_A A)=\mathcal{L}(\overline{A_\Gamma A})=\mathcal{L}(A)=M(A)$$

takes its value in $M(A)^\Gamma$. It is clear that $\nu: M(A_\Gamma)\to M(A)^\Gamma$ and μ are inverse of each other. □

COROLLARY 4.5. – *If Γ is a proper groupoid with Haar system and E is a u.s.c. Fell bundle over Γ, then*

$$M\bigl(\mathcal{K}_\Gamma\bigl(L^2(\Gamma;E)\otimes\mathbb{H}\bigr)\bigr)=\mathcal{L}\bigl(L^2(\Gamma;E)\otimes\mathbb{H}\bigr)^\Gamma.$$

Proof. – For a C^*-algebra A and an A-Hilbert module $\mathcal{E}$, we have $\mathcal{L}(\mathcal{E})=M(\mathcal{K}(\mathcal{E}))$ [11, Theorem 13.4.1]. Hence the result follows from Lemma 4.4(c). □

Let $\mathcal{F}^0(\Gamma,E)$ be the set consisting of all $T\in\mathcal{L}(L^2(\Gamma;E)\otimes\mathbb{H})^\Gamma$ which are invertible modulo $\mathcal{K}_\Gamma(L^2(\Gamma;E)\otimes\mathbb{H})$, and $\mathcal{F}^1(\Gamma,E)$ the subset of $\mathcal{F}^0(\Gamma,E)$ consisting of self-adjoint elements. We denote by $[T]$ the homotopy class of T.

THEOREM 4.6. – *Let* Γ *be a proper groupoid with a Haar system. Suppose that* $E = (E_g)_{g \in \Gamma}$ *is a u.s.c. Fell bundle over* Γ. *Then*

$$K_i\big(C_r^*(\Gamma; E)\big) = \big\{[T] \mid T \in \mathcal{F}^i(\Gamma, E)\big\}.$$

Proof. – Recall that if B is a C^*-algebra then $K_0(B)$ is the set of homotopy classes of elements $T \in M(B \otimes \mathcal{K}(\mathbb{H}))$ which are invertible modulo $B \otimes \mathcal{K}(\mathbb{H})$, and $K_1(B)$ is the set of homotopy classes of elements $T \in M(B \otimes \mathcal{K}(\mathbb{H}))$ which are self-adjoint and invertible modulo $B \otimes \mathcal{K}(\mathbb{H})$ ([11, Corollary 12.2.3], [72, Theorem 17.3.11]).

The theorem is thus a consequence of Proposition 4.3 and Corollary 4.5, by taking $B = \mathcal{K}_\Gamma(L^2(\Gamma, E))$. □

5. Twisted vector bundles

In many situations, it is desirable to describe the K-theory groups in terms of geometrical objects such as vector bundles. For the twisted K-theory group $K^0_\alpha(\Gamma^\bullet)$, a natural candidate will be twisted vector bundles. However, these vector bundles do not always exist. In fact, a necessary condition is that the twisted class $\alpha \in H^2(\Gamma^\bullet, \mathcal{S}^1)$ must be a torsion class. The main purpose of this section is to explore the conditions under which $K^0_\alpha(\Gamma^\bullet)$ can be expressed by twisted vector bundles. More precisely, we prove that given an S^1-central extension $S^1 \to R \to \Gamma \rightrightarrows M$ of a proper Lie groupoid Γ such that M/Γ is compact, the K-theory group $K^0_\alpha(\Gamma^\bullet)$ twisted by the class α of the above central extension is the Grothendieck group of twisted vector bundles $K^{vb}_\alpha(\Gamma^\bullet)$, provided some conditions are fulfilled (see Theorem 5.28).

The proof is divided into five steps outlined as follows. Let $L = R \times_{S^1} \mathbb{C}$ be the associated line bundle over Γ.

Step 1: From the previous section, it is known that $K^0_\alpha(\Gamma^\bullet)$ is isomorphic to

$$K_0\big(\mathcal{K}_\Gamma\big(L^2(\Gamma; L) \otimes \mathbb{H}\big)\big).$$

Therefore, if $\mathcal{K}_\Gamma(L^2(\Gamma; L) \otimes \mathbb{H})$ has an approximate unit consisting of projections, then $K^0_\alpha(\Gamma^\bullet)$ is the Grothendieck group of projections in $\mathcal{K}_\Gamma(L^2(\Gamma; L) \otimes \mathbb{H})$ [11, Proposition 5.5.5].

Step 2: $\mathcal{K}_\Gamma(L^2(\Gamma; L) \otimes \mathbb{H})$ has an approximate unit consisting of projections if and only if the (Γ, L)-equivariant Hilbert module $L^2(\Gamma; L) \otimes \mathbb{H}$ satisfies a certain property that we denote by AFGP.

Step 3: If $L^2(\Gamma) \otimes \mathbb{H}$ is AFGP and if there exists a twisted vector bundle, then $L^2(\Gamma; L) \otimes \mathbb{H}$ is AFGP.

Step 4: Projections in $\mathcal{K}_\Gamma(L^2(\Gamma; L) \otimes \mathbb{H})$ correspond to (Γ, L)-equivariant Hilbert modules $\mathcal{E}$ such that $\mathrm{Id}_\mathcal{E} \in \mathcal{K}_\Gamma(\mathcal{E})$ (see notation (32)).

Step 5: (Γ, L)-equivariant Hilbert modules $\mathcal{E}$ such that $\mathrm{Id}_\mathcal{E} \in \mathcal{K}_\Gamma(\mathcal{E})$ correspond to twisted vector bundles, which can be considered as a generalization of Swan's theorem.

5.1. Definition of twisted vector bundles

In this subsection, we give the definition of a twisted vector bundle and show that if such a vector bundle exists, then the S^1-central extension must be a torsion.

Let us first recall the definition of a Γ-vector bundle.

DEFINITION 5.1. – Let $\Gamma \rightrightarrows M$ be a groupoid. A Γ-vector bundle is a vector bundle $J: E \to M$ such that E is a Γ-space in the sense of Definition 2.32, and the map (13) is a linear map.

Note that in this case for any $r \in \Gamma$, the map

$$l_r : J^{-1}(u) \to J^{-1}(v), \quad x \to r \cdot x, \tag{34}$$

where $u = s(r)$ and $v = t(r)$, must be a linear isomorphism.

For example, given a G-bundle P over $\Gamma \rightrightarrows M$ and a representation $G \to \operatorname{End} V$, the associated vector bundle $E := (P \times V)/G \to M$ naturally becomes a Γ-vector bundle.

DEFINITION 5.2. – Let $S^1 \to R \to \Gamma \rightrightarrows M$ be an S^1-central extension of Lie groupoids. By a (Γ, R)-twisted vector bundle, we mean an R-vector bundle satisfying the compatibility condition:

$$(\lambda \cdot r) \cdot x = \lambda(r \cdot x), \quad \lambda \in S^1, r \in R \text{ and } x \in E \text{ such that } s(r) = J(x).$$

Here S^1 is considered as the unit circle in $\mathbb{C}$.

The following gives an equivalent definition of twisted vector bundles.

LEMMA 5.3. – *Let $S^1 \to R \xrightarrow{\pi} \Gamma \rightrightarrows M$ be an S^1-central extension of Lie groupoids. An R-vector bundle $E \to M$ is a (Γ, R)-twisted vector bundle if and only if $\ker \pi \cong M \times S^1$ acts on E by scalar multiplication, where S^1 is identified with the unit circle of $\mathbb{C}$.*

When M is a point, the definition above reduces to the usual projective representations of a group.

Example 5.4. – (1) Consider the S^1-central extension as in Example 2.26(1). A twisted vector bundle $E \to \coprod_i U_i$ of rank n corresponds to vector bundles $E_i \cong U_i \times \mathbb{C}^n$, where the transition functions $a_{ij} : U_{ij} \to GL(n, \mathbb{C})$ satisfy the twisted cocycle condition

$$a_{ij} a_{jk} a_{ki} = c_{ijk}.$$

Note that when the central extension is trivial, i.e., $c_{ijk} = 1$, the transition functions (a_{ij}) define an ordinary vector bundle over M. In other words, a vector bundle over the groupoid $\coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i$ corresponds exactly to a vector bundle over M in the usual sense.

(2) Consider the S^1-central extension as in Example 2.26(2). Let $E \to \coprod_i U_i$ be a twisted vector bundle of rank n. Then $E|_{U_i} \cong U_i \times \mathbb{C}^n$. For all $x \in U_i$ and $\xi \in \mathbb{C}^n$, denote by $[(i, x, \xi)]$ the corresponding element of $E|_{U_i}$. Write

$$(\alpha, g, x, \lambda) \cdot \left[(j, x, \xi)\right] = \left[\left(i, gx, \lambda a_{ij;\alpha}(g, x)\xi\right)\right],$$

where $a_{ij;\alpha} : G \times U_{ij} \to GL_n(\mathbb{C})$. Then we have the cocycle relation

$$a_{ij;\alpha}(g, x) a_{jk;\beta}(h, y) = c_{ijk;\alpha\beta,\gamma}(g, x, h, y) a_{ik;\gamma}(gh, y).$$

(3) Consider the case that R is topologically trivial and the S^1-central extension is given by a groupoid S^1-valued 2-cocycle $c(x, y)$ as in Eq. (2). Let $E \to M$ be a (non-equivariant) trivial vector bundle over M, i.e., $E \cong M \times \mathbb{C}^n$. Then $E \to M$ defines a twisted vector bundle of R if and only if there is a smooth map $\phi : \Gamma \to GL(n, \mathbb{C})$ satisfying the condition:

$$\phi(x)\phi(y) = c(x, y)\phi(xy), \quad \forall (x, y) \in \Gamma^{(2)}.$$

In the proposition below, we show that twisted vector bundles exist only when the S^1-central extension defines a torsion class in $H^2(\Gamma^\bullet, \mathcal{S}^1)$.

PROPOSITION 5.5. – *Let $S^1 \to R \to \Gamma \rightrightarrows M$ be an S^1-central extension of Lie groupoids. Consider the following properties*:

(i) *there exists a rank n twisted vector bundle*;

(ii) *there exists an S^1-equivariant generalized homomorphism $R \to GL_n(\mathbb{C})$, where $GL_n(\mathbb{C})$ is naturally considered as an S^1-central extension of $PGL_n(\mathbb{C})$*:

$$S^1 \to GL_n(\mathbb{C}) \to PGL_n(\mathbb{C});$$

(ii)$'$ *there exists a generalized homomorphism $\Gamma \to PGL_n(\mathbb{C})$ such that R is the pull-back of $GL_n(\mathbb{C}) \to PGL_n(\mathbb{C})$*;

(iii) *there exists an open cover (U_i) and $\psi : R[U_i] \to \mathbb{C}^*$ such that $\psi(\lambda r) = \lambda^n \psi(r)$ for all $\lambda \in S^1$ and $r \in R$*;

(iv) *R^n is a trivial extension*;

(v) *there exists an open cover (U_i) and a $\mathbb{Z}_n$-central extension $\mathbb{Z}_n \to R'[U_i] \to \Gamma[U_i] \rightrightarrows \coprod U_i$ such that $R[U_i] = R' \times_{\mathbb{Z}_n} S^1$, where $\mathbb{Z}_n$ is identified with the group of n-th roots of unity in $S^1 \subset \mathbb{C}$.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v).

Proof. – (i) $\Rightarrow$ (ii): let E be a rank n twisted vector bundle. Since E is locally trivial, replacing R by $R[U_i]$ one may assume that $E \cong M \times \mathbb{C}^n$ as a (non-equivariant) vector bundle. Hence the action of R on $M \times \mathbb{C}^n$ defines an S^1-equivariant homomorphism.

(ii) $\Rightarrow$ (i): let (U_i) be an open cover of M such that there exists an S^1-equivariant strict homomorphism $R[U_i] \to GL_n(\mathbb{C})$ (see Proposition 2.3). Let $Z = \coprod R_{U_i} = \{(r,i) \mid s(r) \in U_i\}$. Then Z is naturally endowed with a right $R[U_i]$-action. Let $E = Z \times_{R[U_i]} \mathbb{C}^n$. Then E is a twisted vector bundle of rank n (where the map $E \to M$ is $(r,i,\xi) \mapsto t(r)$).

(ii) $\Rightarrow$ (iii): compose with the determinant $GL_n(\mathbb{C}) \to \mathbb{C}^*$.

(iii) $\Rightarrow$ (iv): replacing ψ by $\psi/|\psi|$, we may assume that the image of ψ lies in S^1. Define $\varphi(\lambda[r,\ldots,r]) = \lambda\psi(r)$. Then φ is a well-defined S^1-equivariant homomorphism from $R^n[U_i]$ to S^1. Hence R^n is a trivial extension (see Proposition 2.13).

(iv) $\Rightarrow$ (iii): If $\varphi : R^n[U_i] \to S^1$ is S^1-equivariant, then $\psi(r) := \varphi[(r,\ldots,r)] : R[U_i] \to S^1$ is the function satisfying the desired property.

(iii) $\Rightarrow$ (v): take $R' = \psi^{-1}(1) \subset R[U_i]$. Then $(r,\lambda) \in R' \times_{\mathbb{Z}_n} S^1 \mapsto \lambda r \in R[U_i]$ is an isomorphism.

(v) $\Rightarrow$ (iii): the map $[(r,\lambda)] \in R' \times_{\mathbb{Z}_n} S^1 \mapsto \lambda^n \in S^1$ is well-defined and satisfies (iii). □

Remark 5.6. – It is worth noting that (v) means that the class in $H^2(\Gamma^\bullet, \mathcal{S}^1)$ defined by the S^1-central extension $R \to \Gamma$ lies in the image of the homomorphism $H^2(\Gamma^\bullet, \mathbb{Z}_n) \to H^2(\Gamma^\bullet, \mathcal{S}^1)$.

By $\widetilde{K}^0(\Gamma, R)$, we denote the Grothendieck group of twisted vector bundles. As an immediate consequence of (i) $\Rightarrow$ (ii), we have

COROLLARY 5.7. – *Assume that $S^1 \to R_i \to \Gamma_i \rightrightarrows M_i$, $i = 1,2$, are Morita equivalent S^1-central extensions of groupoids. Then*

$$\widetilde{K}^0(\Gamma_1, R_1) \cong \widetilde{K}^0(\Gamma_2, R_2).$$

This allows us to introduce the following

DEFINITION 5.8. – Let $\Gamma \rightrightarrows M$ be a Lie groupoid. For any $\alpha \in H^2(\Gamma^\bullet, \mathcal{S}^1)$, denote by $K_\alpha^{vb}(\Gamma^\bullet)$ the Grothendieck group of (Γ', R')-twisted vector bundles, where $S^1 \to R' \to \Gamma' \rightrightarrows M'$ is any S^1-central extension realizing the class α.

This definition coincides with the definition of twisted orbifold K-theory in the special case considered by Adem and Ruan [1].

Similarly, one can also work in the category of locally compact groupoids and introduce the K-theory group $K_\alpha^{vb,cont}(\Gamma^\bullet)$. We will later find conditions which guarantee that the canonical morphism $K_\alpha^{vb}(\Gamma^\bullet) \to K_\alpha^{vb,cont}(\Gamma^\bullet)$ is an isomorphism (see Theorem 5.33).

Remark 5.9. – (1) In general, (iv) does not imply (i). Even more, there does not exist a function $f:\mathbb{N} \to \mathbb{N}$ such that every S^1-central extension satisfying (iv) has a twisted vector bundle of rank $\leqslant f(n)$.

Let us assume the contrary. Choose any integer $N > f(n)$ such that N and n are not mutually prime, for instance $N = nf(n)$. We show that there exists an S^1-central extension of Lie groups $S^1 \to R \to \Gamma \rightrightarrows \cdot$ such that any twisted vector bundle has rank $\geqslant N$ and R^n is trivial.

Let $R' = U(N)$, $\Gamma = R'/\mathbb{Z}_n$ where $\mathbb{Z}_n$ is identified with the group of nth roots of unity. We consider the central extension (of order dividing n)

$$S^1 \to R' \times_{\mathbb{Z}_n} S^1 \to \Gamma \rightrightarrows \cdot.$$

Suppose that there exists a rank n' twisted vector bundle with $n' \leqslant f(n)$. Then by (ii) there exists a $\mathbb{Z}_n$-equivariant group morphism $\pi : R' \to GL_{n'}(\mathbb{C})$. Since $R' = U(N)$ is compact, we may assume that π is an irreducible unitary representation, and since $\dim \pi < N$, we have $\pi = (\det)^p$ for some p.

Since π is $\mathbb{Z}_n$-equivariant, for $\omega = e^{2i\pi/n}$ we get $\pi(\omega r) = \omega\pi(r)$ and thus $\omega^{Np} = \omega$. This is impossible since N and n have a common factor.

(2) Consider the Lie group $SL_2(\mathbb{R})$. Its fundamental group is $\mathbb{Z}$. Let H be its connected double covering. Then H is a $\mathbb{Z}_2$-central extension over $SL_2(\mathbb{R})$. Let $R = S^1 \times_{\mathbb{Z}_2} H$ be its associated S^1-central extension over $SL_2(\mathbb{R})$. Then clearly R defines a torsion class of degree 2.

Let us show that $R \to SL_2(\mathbb{R})$ does not admit any finite dimensional twisted vector bundle, i.e., a projective representation. It is known (see [59, p. 13]) that any group homomorphism ϕ from the universal extension $\hat{G}$ of $SL_2(\mathbb{R})$ to $GL_n(\mathbb{C})$ satisfies the following property:

$$\phi(z \cdot g) = \phi(g), \quad \forall z \in \mathbb{Z}, g \in \hat{G}. \tag{35}$$

Assume that $\psi : R \to GL_n(\mathbb{C})$ is an S^1-equivariant group homomorphism. Let

$$\psi' : H \to GL_n(\mathbb{C})$$

be its restriction to H. Then ψ' is a $\mathbb{Z}_2$-equivariant map. Let $\pi : \hat{G} \to H$ and $p : \mathbb{Z} \to \mathbb{Z}_2$ be the canonical projections. Since $\psi' \circ \pi : \hat{G} \to GL_n(\mathbb{C})$ is a group homomorphism, according to Eq. (35), we have, for any $z \in \mathbb{Z}$ and $g \in \hat{G}$,

$$\psi'\big(\pi(z \cdot g)\big) = \psi'\big(p(z) \cdot \pi(g)\big) = \psi'\big(\pi(g)\big).$$

Since both p and π are onto, it follows that for any $a \in \mathbb{Z}_2$ and any $g \in H$

$$\psi'(a \cdot g) = \psi'(g). \tag{36}$$

This contradicts to the assumption that ψ' is $\mathbb{Z}_2$-equivariant.

5.2. Proof of step 2

Recall that a positive element a in a C^*-algebra A is said to be strictly positive if $A = \overline{aA}$, which is also equivalent to $A = \overline{aAa}$, and that every separable C^*-algebra has a strictly positive element, i.e., A is σ-unital [58, 3.10.6].

LEMMA 5.10. – *Let Γ be a proper groupoid with Haar system acting on a u.s.c. field of C^*-algebras $\mathcal{A} \to M$, and $A = C_0(M, \mathcal{A})$. Let $(u_i) \in A_\Gamma$ such that $0 \leqslant u_i \leqslant 1$ (see notation* (33)). *The following are equivalent*:

(i) (u_i) *is an approximate unit for* A_Γ;

(ii) (u_i) *is an approximate unit for* A.

(Recall that in (ii) above, we mean that $u_i a \to a$ for all $a \in A$, but u_i does not necessarily belong to A.)

Proof. – (i) $\Rightarrow$ (ii): it is clear since $\overline{A_\Gamma A} = A$ according to Lemma 4.4.

(ii) $\Rightarrow$ (i): by assumption, $u_i a \to a$ for all $a \in A$. Since $a \mapsto a^\Gamma$ is linear and norm-decreasing, we have $(u_i a)^\Gamma \to a^\Gamma$. On the other hand, it is simple to see that $(u_i a)^\Gamma = u_i a^\Gamma$. Thus, from Lemma 4.2, $u_i b \to b$ for all $b \in A_\Gamma$. □

PROPOSITION 5.11. – *Let Γ be a proper groupoid with a Haar system acting on a u.s.c. field of C^*-algebras $\mathcal{A} \to M$, and $A = C_0(M, \mathcal{A})$. Then* (i)–(iii) *are equivalent, and* (i) $\Rightarrow$ (iv) *if A is σ-unital.*

(i) $\exists P_i \in A_\Gamma$ *approximate unit of A_Γ consisting of projections*;

(ii) $\exists P_i \in A_\Gamma$ *approximate unit of A consisting of projections*;

(iii) *for all $x \in A_+$ and $\varepsilon > 0$, there exists $a_{\varepsilon,x} \in (A_\Gamma)_+$ such that $(a_{\varepsilon,x})$ does not contain any interval $[0,\delta]$ ($\delta > 0$) and $x \leqslant \varepsilon + a_{\varepsilon,x}$*;

(iv) *there exists $x \in A_+$ strictly positive with the property that for all $\varepsilon > 0$, there exists $a_{\varepsilon,x} \in (A_\Gamma)_+$ such that $sp(a_{\varepsilon,x})$ does not contain any interval $[0,\delta]$ ($\delta > 0$) and $x \leqslant \varepsilon + a_{\varepsilon,x}$.*

Proof. – (i) $\Rightarrow$ (ii) follows from Lemma 5.10.

(ii) $\Rightarrow$ (iii): $\forall \varepsilon > 0$, by (ii), there exists i such that $\|x - P_i x P_i\| < \varepsilon$. Since

$$\begin{aligned} x = (x - P_i x P_i) + P_i x P_i &\leqslant \|x - P_i x P_i\| + P_i \|x\| P_i \\ &= \|x - P_i x P_i\| + \|x\| P_i, \end{aligned}$$

we see that $a_{\varepsilon,x} = \|x\| P_i$ satisfies (iii).

(iii) $\Rightarrow$ (iv): obvious.

(iii) $\Rightarrow$ (ii): let $x_1, \ldots, x_n \in A$ and $\varepsilon > 0$. We want to find a projection $P \in A_\Gamma$ such that $\|(1-P)x_i\| \leqslant \varepsilon$ for all $i = 1, \ldots, n$.

Let $x = \sum x_i x_i^*$. Choose a real number η such that $0 < \eta < \varepsilon^2/2$ and η does not belong to the spectrum of $a_{\varepsilon^2/2,x}$. Then the spectral projection $P = 1_{[\eta,\infty)}(a_{\varepsilon^2/2,x})$ of $a_{\varepsilon^2/2,x}$ on $[\eta, \infty)$ is an element of A_Γ. Since $1 - P$ is the spectral projection of $a_{\varepsilon^2/2,x}$ on $[0, \eta]$, we have $(1-P)a_{\varepsilon^2/2,x}(1-P) \leqslant \eta(1-P)$. Now, for any i, we have

$$\begin{aligned} (1-P)x_i x_i^*(1-P) &\leqslant (1-P)x(1-P) \\ &\leqslant (1-P)(\varepsilon^2/2 + a_{\varepsilon^2/2,x})(1-P) \\ &= (1-P)(\varepsilon^2/2 + \eta)(1-P) \\ &\leqslant \varepsilon^2(1-P) \leqslant \varepsilon^2, \end{aligned}$$

so $\|(1-P)x_i\| = \|(1-P)x_i x_i^*(1-P)\|^{1/2} \leqslant \varepsilon$ for all $i = 1, \ldots, n$.

(iv) ⇒ (ii): the same proof shows that if x satisfies (iv), there exist projections $P_i \in A_\Gamma$ such that $(1-P_i)x^{1/2} \to 0$, and therefore $P_i y \to y$ for all $y \in \overline{x^{1/2}A} = A$. □

We note that the approximate unit is not necessarily increasing. In fact, we have the following:

PROPOSITION 5.12. – *Let Γ be a proper groupoid with a Haar system acting on a countably generated u.s.c. field of C^*-algebras $\mathcal{A} \to M$. Let $A = C_0(M, \mathcal{A})$. The following are equivalent*:

(i) *there exist projections $P_1 \leqslant P_2 \leqslant \cdots \leqslant P_n$ in A_Γ such that $P_i a \to a$ for all $a \in A_\Gamma$*;
(ii) *there exist projections $P_1 \leqslant P_2 \leqslant \cdots \leqslant P_n$ in A_Γ such that $P_i a \to a$ for all $a \in A$*;
(iii) *there exists $a \in A_\Gamma$ strictly positive with countable spectrum.*

Proof. – (i) ⇒ (ii): follows from Lemma 5.10.

(ii) ⇒ (iii): take $a = \sum_{n=0}^{\infty} 2^{-n}(P_{n+1} - P_n)$ (with $P_0 = 0$ by convention). Then $1_{[2^{-n},1]}(a) = P_{n+1}$, and hence $P_{n+1}A \subset aA$. It follows that $A = \overline{\bigcup_n P_n A} \subset \overline{aA}$, so a is strictly positive with spectrum in $\{0\} \cup \{2^{-n} \mid n \in \mathbb{N}\}$.

(iii) ⇒ (i): take $P_n = 1_{[\alpha_n,\infty)}(a)$, where α_n is a sequence decreasing to 0 and $\alpha_n \notin \mathrm{sp}(a)$. Then clearly $P_n a \to a$. Thus $P_n b \to b$ for all $b \in \overline{aA} = A$. □

DEFINITION 5.13. – Let E be a u.s.c. Fell bundle over a locally compact groupoid Γ, and let $A = C_0(M;E)$. A (Γ, E)-equivariant Hilbert module is an A-Hilbert module $\mathcal{E}$ with isomorphisms of $A_{s(g)}$-Hilbert modules

$$\mathcal{E}_{t(g)} \otimes_{A_{t(g)}} E_g \to \mathcal{E}_{s(g)}$$

such that $(\xi\eta)\zeta = \xi(\eta\zeta)$ whenever $(g,h) \in \Gamma^{(2)}$ and $(\xi,\eta,\zeta) \in \mathcal{E}_{t(g)} \times E_g \times E_h$. The product is required to be continuous in the following sense: for all $\xi \in \mathcal{E}$ and $\eta \in C_0(\Gamma;E)$, $g \mapsto \xi(t(g))\eta(g)$ belongs to $s^*\mathcal{E}$.

Note that $\mathcal{E}$ can be canonically identified with a field of Banach spaces over M such that for any $x \in M$ the fiber $\mathcal{E}_x$ at x is an A_x-Hilbert module (see Proposition A.4). $\mathcal{E}$ being (Γ, E)-equivariant means, roughly speaking, that this field of Banach spaces is equipped with an E-action.

DEFINITION 5.14. – Let Γ be a locally compact groupoid with Haar system, E a u.s.c. Fell bundle over Γ, $A = C_0(M;E)$ and $\mathcal{E}$ a (Γ, E)-equivariant A-Hilbert module. Then $\mathcal{E}$ is said to be approximately finitely generated projective (AFGP) if there exist projections P_i in $\mathcal{K}_\Gamma(\mathcal{E})$ such that $P_i\xi \to \xi$ for all $\xi \in \mathcal{E}$.

For the notation $\mathcal{K}_\Gamma(\mathcal{E})$, see Eq. (32).

LEMMA 5.15. – *If A is a C^*-algebra and $\mathcal{E}$ an A-Hilbert module, then $\mathrm{Id}_{\mathcal{E}} \in \mathcal{K}(\mathcal{E})$ implies that $\mathcal{E}$ is finitely generated projective, and the converse holds if A is unital. This explains the terminology.*

Proof. – This is proved in the unital case in [72, Theorem 15.4.2, Remark 15.4.3]. Below we outline a proof for the direction " ⇒ ".

If $\mathrm{Id}_{\mathcal{E}}$ is compact, then $\mathrm{Id}_{\mathcal{E}}$ can be approximated by finite rank operators, i.e. there exist ξ_i, η_i such that $S = \sum_{i=1}^n T_{\xi_i,\eta_i}$ satisfies $\|\mathrm{Id} - S\| < 1$. In particular, S is invertible, and therefore

$$\mathrm{Id}_{\mathcal{E}} = S^{-1}S = \sum_{i=1}^n T_{S^{-1}\xi_i,\eta_i}.$$

Replacing ξ_i by $S^{-1}\xi_i$, we may assume that $\mathrm{Id}_{\mathcal{E}} = \sum_{i=1}^n T_{\xi_i,\eta_i}$. Now, define

$$\begin{aligned} U:\mathcal{E} &\to A^n, \\ \xi &\mapsto \big(\langle \eta_1,\xi\rangle,\dots,\langle \eta_n,\xi\rangle\big), \\ V: A^n &\to \mathcal{E}, \\ (a_1,\dots,a_n) &\mapsto \xi_1 a_1+\cdots+\xi_n a_n. \end{aligned}$$

Then $VU=\mathrm{Id}_{\mathcal{E}}$. Hence $P=UV$ is an idempotent in $\mathcal{L}(A^n)=M_n(A)$ and $\mathcal{E}\cong PA^n$ as right A-Hilbert modules. □

PROPOSITION 5.16. – *Let Γ be a proper groupoid with a Haar system, E a u.s.c. Fell bundle over Γ, $A=C_0(M;E)$ and $\mathcal{E}$ a (Γ,E)-equivariant A-Hilbert module. Then $\mathcal{E}$ is AFGP if and only if $\mathcal{K}_\Gamma(\mathcal{E})$ has an approximate unit consisting of projections.*

Proof. –

$$\begin{aligned} \forall\xi\in\mathcal{E}, P_n\xi\to\xi &\iff \forall\xi\in\mathcal{E}, (1-P_n)T_{\xi,\xi}(1-P_n)=T_{(1-P_n)\xi,(1-P_n)\xi}\to 0 \\ &\iff \forall\xi,\eta\in\mathcal{E}, (1-P_n)T_{\xi,\eta}(1-P_n)\to 0 \\ &\qquad \text{since } T_{\xi,\eta}=(1/4)\sum_{\omega^4=1}\omega T_{\xi+\omega\eta,\xi+\omega\eta} \\ &\iff \forall T\in\mathcal{K}(\mathcal{E}),\ (1-P_n)T(1-P_n)\to 0 \\ &\iff (P_n) \text{ is an approximate unit for } \mathcal{K}(\mathcal{E}) \\ &\iff (P_n) \text{ is an approximate unit for } \mathcal{K}_\Gamma(\mathcal{E}) \\ &\qquad \text{by Lemma 5.10.} \end{aligned}$$

In the second from the last equivalence, we used the fact that

$$\|T(1-P_n)\|=\|(T(1-P_n))^*T(1-P_n)\|^{1/2}=\|(1-P_n)T^*T(1-P_n)\|^{1/2}. \quad \square$$

An immediate consequence is the following:

COROLLARY 5.17. – *Let $S^1\to R\to\Gamma\rightrightarrows M$ be an S^1-central extension of proper Lie groupoids, and let $L=R\times_{S^1}\mathbb{C}$ be the associated complex line bundle. Then $L^2(\Gamma;L)\otimes\mathbb{H}$ is AFGP if and only if $C^*_r(\Gamma;R)\otimes\mathcal{K}(\mathbb{H})$ has an approximate unit consisting of projections.*

Proof. – Apply Proposition 5.16 to $\mathcal{E}=L^2(\Gamma;L)\otimes\mathbb{H}$ and use Proposition 4.3. □

COROLLARY 5.18. – *Let Γ be a proper Lie groupoid, then $L^2(\Gamma)\otimes\mathbb{H}$ is AFGP if and only if $C^*_r(\Gamma)\otimes\mathcal{K}(\mathbb{H})$ has an approximate unit consisting of projections.*

Proof. – Apply Corollary 5.17 to the trivial S^1-central extension. □

We end this subsection by listing some examples of AFGP modules.

PROPOSITION 5.19. – *If G is a compact group and π a unitary representation of G on a separable Hilbert space $\mathbb{H}$ (considered as a G-equivariant $\mathbb{C}$-Hilbert module), then $\mathbb{H}$ is AFGP.*

Proof. – Choose a strictly positive element $a\in\mathcal{K}(\mathbb{H})^G$. Since a is a compact operator on the Hilbert space $\mathbb{H}$, its spectrum is countable. By Propositions 5.16 and 5.12, $\mathbb{H}$ is AFGP. □

The following well known result (see [21, Corollary 15.1.4] and [41]) is a direct consequence of the above Proposition 5.19.

COROLLARY 5.20. – *If G is a compact group, then every irreducible unitary representation of G is finite dimensional.*

Proof. – Assume that π is an irreducible representation on H_π. Let P_n be a sequence of compact, G-invariant projections in H_π such that $P_n\xi \to \xi$ for all ξ. Since the representation is irreducible, we have either $P_n = 0$ or $P_n = \mathrm{Id}$. Therefore $P_n = \mathrm{Id}$ for n large enough. Since P_n is a compact projection on a Hilbert space, its range H_π is finite dimensional. □

COROLLARY 5.21. – *If Γ is a transformation groupoid $G \times M \rightrightarrows M$, where M is a compact space and G is a compact group, then $L^2(\Gamma) \otimes \mathbb{H}$ is AFGP.*

Proof. – Since

$$L^2(\Gamma) \otimes \mathbb{H} \cong C(M) \otimes L^2(G) \otimes \mathbb{H},$$

the C^*-algebra $\mathcal{K}(L^2(\Gamma, \mathbb{H})) \cong C(M) \otimes \mathcal{K}(L^2(G)) \otimes \mathcal{K}(\mathbb{H})$ is the tensor product of three C^*-algebras having approximate units consisting of invariant projections. □

5.3. Proof of step 3

We need a sequence of lemmas.

LEMMA 5.22. – *Let M be a locally compact space, F a Hermitian vector bundle and $\mathcal{F} = C_0(M, F)$ its space of continuous sections vanishing at infinity considered as a $C_0(M)$-Hilbert module. Then $\mathrm{Id}_{\mathcal{F}} \in \mathcal{C}(\mathcal{F})$ (see notation* (31)*).*

Proof. – For every compact subspace K of M, the restriction of $\mathcal{F}$ to K, i.e. the $C(K)$-Hilbert module $\mathcal{F}_K = \mathcal{F} \otimes_{C_0(M)} C(K)$, is the space of sections of $F_{|K}$, thus by Swan theorem, is a projective finitely generated $C(K)$-module. Therefore, from Remark 5.15, the identity map on $\mathcal{F}_K$ is compact.

Let us show that this implies $\mathrm{Id}_{\mathcal{F}} \in \mathcal{C}(\mathcal{F})$. Let φ, $\psi \in C_c(M)$. Choose an open set U and a compact set K such that $U \subset K \subset M$ and U contains the supports of both φ and ψ. Since $\mathrm{Id}_{\mathcal{F}_K}$ is compact, there exist ξ_i, $\eta_i \in \mathcal{F}_K$ such that $\mathrm{Id}_{\mathcal{F}_K} = \sum_i T_{\xi_i,\eta_i}$. Let $\xi'_i = \overline{\varphi}\xi_i$ and $\eta'_i = \psi\eta_i$. Then $\varphi\psi = \sum_i T_{\xi'_i,\eta'_i} \in \mathcal{L}(\mathcal{F}_K)$, where $\varphi\psi$ denotes the multiplication operator acting on $\mathcal{F}$. However since φ, ψ and all ξ'_i, η'_i are all supported in U, it is not hard to check that the equality $\varphi\psi = \sum_i T_{\xi'_i,\eta'_i}$ also holds in $\mathcal{L}(\mathcal{F})$. Therefore, $\varphi\psi$ is compact for all φ, $\psi \in C_c(M)$. By a density argument, φ is compact for all $\varphi \in C_0(M)$, i.e. $\mathrm{Id}_{\mathcal{F}} \in \mathcal{C}(\mathcal{F})$. □

LEMMA 5.23. – *Let $S^1 \to R_i \to \Gamma \rightrightarrows M$, $i = 1, 2$, be S^1-central extensions of the Lie groupoid Γ, and $\mathcal{E}_i$, $i = 1, 2$, (Γ, R_i)-equivariant $C_0(M)$-Hilbert modules. Suppose that $\mathcal{E}_1$ is AFGP and $\mathrm{Id}_{\mathcal{E}_2} \in \mathcal{C}(\mathcal{E}_2)$. Then $\mathcal{E}' := \mathcal{E}_1 \otimes_{C_0(M)} \mathcal{E}_2$ is AFGP as a $(\Gamma, R_1 \otimes R_2)$-equivariant $C_0(M)$-Hilbert module.*

Proof. – By assumption, there exists an approximate unit $P_n \in \mathcal{K}_\Gamma(\mathcal{E}_1)$ consisting of projections. Let $P'_n = P_n \otimes_{C_0(M)} \mathrm{Id}_{\mathcal{E}_2}$. It is clear that P'_n is an invariant projection, and that $\|P'_n(x)\| \to 0$ when $x \to 0$ in M/Γ. Let us show that $P'_n \in \mathcal{C}(\mathcal{E}')$. For all φ, $\psi \in C_0(M)$, $(\varphi\psi)P'_n = (\varphi P_n) \otimes_{C_0(M)} \psi \in \mathcal{K}(\mathcal{E}')$. It follows that $\phi P'_n \in \mathcal{K}(\mathcal{E}')$ for all $\phi \in C_0(M)$, i.e. $P'_n \in \mathcal{C}(\mathcal{E}')$. Therefore $P'_n \in \mathcal{K}_\Gamma(\mathcal{E}')$, and it is clear that P'_n is an approximate unit consisting of projections. □

Before we proceed, we need to introduce some notation.

Let E be a u.s.c. Fell bundle over the groupoid Γ, $A = C_0(M; E)$, and let $\mathcal{E}$ be a (possibly non-equivariant) A-Hilbert module. Consider the field of Banach spaces over Γ with fiber $\mathcal{E}_{t(g)} \otimes_{A_{t(g)}} E_g$, determined by sections of the form $\eta(t(g)) \otimes \zeta(g)$ where $\eta \in \mathcal{E}$ and $\zeta \in C_0(\Gamma; E)$ (see Proposition A.2). Denote by $C_c(\Gamma; E, \mathcal{E})$ the space of compactly supported sections.

Endow $C_c(\Gamma; E, \mathcal{E})$ with an A-valued scalar product

$$\langle \xi, \eta\rangle(x) = \int\limits_{g\in\Gamma_x} \big\langle \xi(g), \eta(g)\big\rangle\, \lambda_x(dg), \quad \forall x \in M,$$

and denote by $L^2(\Gamma; E, \mathcal{E})$ its completion. Since $L^2(\Gamma; E, \mathcal{E})$ is an A-Hilbert module, it can be considered as a field of Banach spaces over M; denote by $L^2(\Gamma_x; E, \mathcal{E})$ its fiber at x, which is an A_x-Hilbert module.

The usual action of (Γ, E) on $L^2(\Gamma_x; E)$ (see Appendix A) extends naturally to an action on $L^2(\Gamma; E, \mathcal{E})$, which is defined as follows:

$$L^2(\Gamma_x; E, \mathcal{E}) \otimes_{A_x} E_{\gamma^{-1}} \cong L^2(\Gamma_y; E, \mathcal{E}),$$

$$\xi\otimes\eta \mapsto \big(g \mapsto \xi(g\gamma)\otimes\eta\big) \in \mathcal{E}_{t(g)} \otimes E_{g\gamma}\otimes E_{\gamma^{-1}} \cong \mathcal{E}_{t(g)}\otimes E_g.$$

If moreover $\mathcal{E}$ is a (Γ, E)-equivariant module, then since $\mathcal{E}_{t(g)} \otimes_{A_{t(g)}} E_g \cong \mathcal{E}_{s(g)}$, we get

$$L^2(\Gamma; E, \mathcal{E}) \cong L^2(\Gamma) \otimes_{C_0(M)} \mathcal{E}, \tag{37}$$

where on the left-hand side we *forget* the equivariant structure on $\mathcal{E}$, while the right-hand side is endowed with a "diagonal" action. The isomorphism above is well-known in the case of a Hilbert space $\mathbb{H}$ endowed with a unitary representation $g \mapsto U_g$ of a locally compact group G. In this case, the isomorphism $L^2(G, \mathbb{H}) \to L^2(G) \otimes \mathbb{H}$ (where $\mathbb{H}$ is endowed with the *trivial* action of G on the left-hand side but G acts *diagonally* on the right-hand side) is given by

$$\xi \mapsto \eta(g) = U_g\big(\xi(g)\big).$$

PROPOSITION 5.24. – *Assume that $\Gamma \rightrightarrows M$ is a proper Lie groupoid such that M/Γ is compact and $S^1 \to R \to \Gamma$ is an S^1-central extension. Let $L = R \times_{S^1} \mathbb{C}$. If $L^2(\Gamma) \otimes \mathbb{H}$ is AFGP, and if there exists a topological (i.e. without differentiable structure) (Γ, R)-twisted vector bundle (of finite rank), then $L^2(\Gamma; L) \otimes \mathbb{H}$ is AFGP.*

Proof. – Let F be a (Γ, R)-twisted vector bundle. Since Γ is proper, F can be endowed with an invariant Hermitian metric, and therefore can be considered as a (Γ, R)-equivariant Hilbert module. As a (non-equivariant) continuous field of Hilbert spaces over M, $F \times \mathbb{H} \to M$ is locally trivial with infinite dimensional fibers. According to the triviality theorem of Dixmier and Douady [22], $F \times \mathbb{H} \to M$ is isomorphic to $M \times \mathbb{H} \to M$. Moreover, the space of continuous sections $\mathcal{F} = C_0(M, F)$ of $F \to M$ can be considered as a (Γ, R)-equivariant $C_0(M)$-module such that $\mathrm{Id}_{\mathcal{F}} \in \mathcal{K}_\Gamma(\mathcal{F})$ (see Lemma 5.22).

Since $L^2(\Gamma) \otimes \mathbb{H}$ is AFGP as a Γ-equivariant Hilbert module, according to Lemma 5.23 and Lemma 5.22, we see that $L^2(\Gamma) \otimes \mathbb{H} \otimes_{C_0(M)} \mathcal{F}$ is AFGP as a (Γ, R)-equivariant Hilbert module. Using the isomorphism (37), we deduce that $L^2(\Gamma; L, \mathbb{H} \otimes \mathcal{F})$ is AFGP. By the triviality of the Hilbert bundle $F \times \mathbb{H}$, we get $L^2(\Gamma; L, \mathbb{H} \otimes \mathcal{F}) \cong L^2(\Gamma; L, C_0(M) \otimes \mathbb{H}) \cong L^2(\Gamma; L) \otimes \mathbb{H}$. Therefore $L^2(\Gamma; L) \otimes \mathbb{H}$ is AFGP. $\square$

5.4. Proof of step 4

PROPOSITION 5.25 (Stabilization theorem). – *Let $\Gamma \rightrightarrows M$ be a proper groupoid with a Haar system, and E a u.s.c. Fell bundle over Γ. Let $A = C_0(M; E)$. Assume that $\mathcal{E}$ is a (Γ, E)-equivariant countably generated A-Hilbert module. Then we have the following isomorphism* *of* (Γ, E)*-equivariant Hilbert* C^**-modules*:

$$\mathcal{E} \oplus L^2(\Gamma; E) \otimes \mathbb{H} \cong L^2(\Gamma; E) \otimes \mathbb{H}.$$

Proof. – Since Γ is proper, $C_0(M)$ is (as a Γ-equivariant $C_0(M)$-Hilbert module) a direct factor of $L^2(\Gamma)$ [68]. Hence $\mathcal{E}$ is a direct factor of $L^2(\Gamma) \otimes_{C_0(M)} \mathcal{E} \cong L^2(\Gamma; E, \mathcal{E})$. By Kasparov's stabilization theorem for non equivariant modules ([72, Theorem 15.4.6]), $\mathcal{E}$ is a direct factor of $A \otimes \mathbb{H}$, and thus $\mathcal{E}$ is a direct factor of $L^2(\Gamma; E, A \otimes \mathbb{H}) \cong L^2(\Gamma; E) \otimes \mathbb{H}$. That is, there exists $\mathcal{E}'$ such that $\mathcal{E} \oplus \mathcal{E}' \cong L^2(\Gamma; E) \otimes \mathbb{H}$. Therefore, we have

$$\begin{aligned} L^2(\Gamma; E) \otimes \mathbb{H} &\cong L^2(\Gamma; E) \otimes (\mathbb{H} \oplus \mathbb{H} \oplus \cdots) \cong \mathcal{E} \oplus \mathcal{E}' \oplus \mathcal{E} \oplus \mathcal{E}' \oplus \cdots \\ &\cong \mathcal{E} \oplus L^2(\Gamma; E) \otimes \mathbb{H}. \quad \square \end{aligned}$$

COROLLARY 5.26. – *Let* $\Gamma \rightrightarrows M$ *be a proper Lie groupoid,* $S^1 \to R \to \Gamma \rightrightarrows M$ *an* S^1*-central extension, and* $L = R \times_{S^1} \mathbb{C}$*. Then there is an equivalence of categories between the category of* (Γ, R)*-equivariant* $C_0(M)$*-Hilbert modules* $\mathcal{E}$ *such that* $\mathrm{Id}_{\mathcal{E}} \in \mathcal{K}_\Gamma(\mathcal{E})$ *and the category of projections in* $C_r^*(\Gamma; R) \otimes \mathcal{K}(\mathbb{H})$.

Proof. – As usual, let

$$L = R \times_{S^1} \mathbb{C}.$$

Recall from Proposition 4.3 that $C_r^*(\Gamma; R) \otimes \mathcal{K}(\mathbb{H})$ is isomorphic to $\mathcal{K}_\Gamma(L^2(\Gamma; L) \otimes \mathbb{H})$. Given a projection $P \in \mathcal{K}_\Gamma(L^2(\Gamma; L) \otimes \mathbb{H})$, $\mathcal{E} = P(L^2(\Gamma; L) \otimes \mathbb{H})$ is a (Γ, R)-equivariant Hilbert module. It is clear that $\mathrm{Id}_{\mathcal{E}} \in \mathcal{K}_\Gamma(\mathcal{E})$.

Conversely, if $\mathcal{E}$ is a countably generated (Γ, R)-equivariant Hilbert module such that $\mathrm{Id}_{\mathcal{E}} \in \mathcal{K}_\Gamma(\mathcal{E})$, we know, from the stabilization Theorem 5.25, that there is an invariant projection P such that $\mathcal{E} = P(L^2(\Gamma; L) \otimes \mathbb{H})$. Since $\mathrm{Id}_{\mathcal{E}} \in \mathcal{K}_\Gamma(\mathcal{E})$, we have $P \in \mathcal{K}_\Gamma(L^2(\Gamma; L) \otimes \mathbb{H})$.

A standard argument shows that two projections P_1 and $P_2 \in \mathcal{K}_\Gamma(L^2(\Gamma; L) \otimes \mathbb{H})$ are Murray–Von Neumann equivalent if and only if the associated Hilbert modules are isomorphic. $\square$

5.5. Proof of step 5

The next proposition generalizes Serre–Swan theorem: if M is a compact space, there is an equivalence of categories between vector bundles on M and finitely generated projective $C(M)$-modules (and thus $K^0(M) = K_0(C(M))$).

PROPOSITION 5.27. – *Assume that* $\Gamma \rightrightarrows M$ *is a proper Lie groupoid such that* M/Γ *is compact, and* $S^1 \to R \to \Gamma \rightrightarrows M$ *is an* S^1*-central extension.*

(a) *The forgetful functor from the category of topological* (*i.e. without differentiable structure*) (Γ, R)*-twisted vector bundles endowed with an* R*-invariant Hermitian metric to the category of* (Γ, R)*-twisted vector bundles is an equivalence of categories.*

(b) *The functor from the category of topological* (Γ, R)*-twisted vector bundles endowed with an* R*-invariant metric to the category of* (Γ, R)*-equivariant* $C_0(M)$*-Hilbert modules* $\mathcal{E}$ *such that* $\mathrm{Id}_{\mathcal{E}} \in \mathcal{K}_\Gamma(\mathcal{E})$*, defined by*

$$\Phi : F \mapsto C_0(M, F),$$

is an equivalence of categories.

Proof. – To prove (a), note that by an averaging procedure using cutoff functions (see Definition 4.1), every twisted vector bundle can be endowed with an invariant Hermitian metric.

If two Hermitian R-equivariant vector bundles F_1 and F_2 are isomorphic as R-equivariant vector bundles, then by the polar decomposition, they must be isometrically isomorphic. Indeed, if $T_x : F_{1,x} \to F_{2,x}$ is an R-equivariant isomorphism, then $U_x := T_x(T_x^* T_x)^{-1/2}$, $\forall x \in M$, defines an R-equivariant isometric isomorphism.

Let us prove (b). From Lemma 5.22 and the fact that M/Γ is compact, it is easy to see that $\mathcal{E} := \Phi(F)$ satisfies $\mathrm{Id}_{\mathcal{E}} \in \mathcal{K}_\Gamma(\mathcal{E})$, and can be endowed with a (Γ, R)-action so that Φ is equivariant. Therefore Φ is well-defined and functorial.

Now, for every (Γ, R)-equivariant $C_0(M)$-Hilbert module $\mathcal{E}$ satisfying $\mathrm{Id}_{\mathcal{E}} \in \mathcal{K}_\Gamma(\mathcal{E})$, $\mathcal{E}$ is isomorphic to $C_0(M, F)$, where F is a continuous field of Hilbert spaces on M with fiber $F_x = \mathcal{E} \otimes_{ev_x} \mathbb{C}$ (see Appendix A). For every compact $K \subset M$, since $\mathcal{E}_K$ is a finitely generated projective module over $C(K)$, it follows from Swan theorem that $F_{|K}$ is a vector bundle over K (i.e., a locally trivial field of finite dimensional Hilbert spaces). Since this is true for every compact K, it follows that F is a vector bundle.

Define $\Psi(\mathcal{E}) = F$. It is clear that Φ and Ψ are inverse from each other. □

5.6. The main theorem: continuous case

THEOREM 5.28. – *Let $\Gamma \rightrightarrows M$ be a Lie groupoid, and $S^1 \to R \to \Gamma \rightrightarrows M$ an S^1-central extension. Denote by α its corresponding class in $H^2(\Gamma^\bullet, \mathcal{S}^1)$. Assume that*

(a) $\Gamma \rightrightarrows M$ *is proper*;

(b) M/Γ *is compact*;

(c) $L^2(\Gamma) \otimes \mathbb{H}$ *is AFGP. In other words, there exists a sequence* (P_n) *such that*

  (i) $P_n = (P_n(x))_{x \in M}$ *is a continuous section of the field of compact operators* $\mathcal{K}(\widetilde{L}^2(\Gamma) \otimes \mathbb{H}) \to M$;

  (ii) $x \mapsto P_n(x)$ *is* Γ*-equivariant*;

  (iii) $P_n(x)$ *is a finite rank projection for all* x;

  (iv) *for every compactly supported continuous section* ξ *of* $\widetilde{L}^2(\Gamma) \otimes \mathbb{H}$, $(P_n \xi)(x) \to \xi(x)$ *uniformly on* M *when* $n \to \infty$.

(d) *there exists a* (Γ, R)*-twisted vector bundle* (*of finite rank*).

Then $K^0_\alpha(\Gamma^\bullet)$ *is isomorphic to* $K^{vb,cont}_\alpha(\Gamma^\bullet)$.

Proof. – It is known that if B is a stably unital C^*-algebra, i.e., $B \otimes \mathcal{K}(\mathbb{H})$ has an approximate identity consisting of projections, then $K_0(B)$ is the Grothendieck group of projections in $B \otimes \mathcal{K}(\mathbb{H})$ [11, Proposition 5.5.5]. We want to apply this fact to $B = C^*_r(\Gamma; L)$.

Since $L^2(\Gamma; L) \otimes \mathbb{H}$ is AFGP according to Proposition 5.24, it follows from Proposition 4.3 and Proposition 5.16 that $C^*_r(\Gamma; R)$ is stably unital. Hence $K_0(C^*_r(\Gamma; R))$ is the Grothendieck group of projections in $C^*_r(\Gamma; R) \otimes \mathcal{K}(\mathbb{H})$. Therefore it is the Grothendieck group of (Γ, R)-twisted vector bundles according to Corollary 5.26 and Proposition 5.27. □

Remark 5.29. – Note that conditions (a), (b), (c) and (d) are invariant under Morita equivalence according to Lemma 2.21, Corollary 5.18 and Proposition 5.5.

We end this subsection by describing an explicit isomorphism $K^0_\alpha(\Gamma^\bullet) \to K^{vb,cont}_\alpha(\Gamma^\bullet)$. We will use the Fredholm picture for $K^0_\alpha(\Gamma^\bullet)$ (Theorem 4.6).

Let $T \in \mathcal{F}^0_\alpha$. By definition, there exists $S \in \mathcal{F}^0_\alpha$ such that $ST = 1 + K$ where $K \in \mathcal{K}_\Gamma(L^2(\Gamma; L) \otimes \mathbb{H})$. By Lemma 4.2, we have $K = K_0^\Gamma + K_1^\Gamma$ where $K_0 = \sum_{i=1}^n T_{\xi_i, \eta_i}$ and $\|K_1\| < 1$ is compact. Since $L^2(\Gamma; L) \otimes \mathbb{H}$ is AFGP, we may assume that $\eta_i \in P(L^2(\Gamma; L) \otimes \mathbb{H})$, where $P \in \mathcal{K}_\Gamma(L^2(\Gamma; L) \otimes \mathbb{H})$ is a projection. Then

$$
\begin{aligned}
(1 + K_1^\Gamma)^{-1} ST(1 - P) &= (1 + K_1^\Gamma)^{-1}(1 + K_1^\Gamma + K_0^\Gamma)(1 - P) \\
&= 1 - P + \left((1 + K_1^\Gamma)^{-1} K_0 (1 - P)\right)^\Gamma
\end{aligned}
$$

$$= 1 - P + \left(\sum_i T_{(1+K_1^\Gamma)^{-1}\xi_i,(1-P)\eta_i}\right)^\Gamma$$
$$= 1 - P.$$

Replacing T by $T(1-P)$ and S by $(1+K_1^\Gamma)^{-1}S$, we may assume that ST equals to the projection $1-P$. Thus TS is also a projection. Let $Q = 1 - TS$. Then the image of $[T] \in K_\alpha^0(\Gamma^\bullet)$ is $[P] - [Q] \in K_\alpha^{vb,cont}(\Gamma^\bullet)$.

Conversely, assume that P is a projection in $\mathcal{K}_\Gamma(L^2(\Gamma; L) \otimes \mathbb{H})$. Let $\mathcal{E} = P(L^2(\Gamma; L) \otimes \mathbb{H})$, and

$$T : L^2(\Gamma; L) \otimes \mathbb{H} \cong \mathcal{E} \oplus L^2(\Gamma; L) \otimes \mathbb{H} \to L^2(\Gamma; L) \otimes \mathbb{H}$$

be the projection. More explicitly,

$$T : L^2(\Gamma; L) \otimes \mathbb{H} \otimes \ell^2(\mathbb{N}) \to L^2(\Gamma; L) \otimes \mathbb{H} \otimes \ell^2(\mathbb{N}),$$

$$(\xi_n)_{n\geqslant 0} \mapsto \big(P\xi_{n+1} + (1-P)\xi_n\big)_{n\geqslant 0}.$$

Then the map $K_\alpha^{vb,cont}(\Gamma^\bullet) \to K_\alpha^0(\Gamma^\bullet)$ is given by $[P] \to [T]$.

5.7. Discussion on the conditions in Theorem 5.28

We would like to remark that conditions (a)–(d) are all necessary for Theorem 5.28 to hold. Let us go over these conditions one by one.

1) Condition (a) cannot be avoided even when Γ is a group G. Note that the K_0-group of $C_r^*(G)$ in general is not equal to the (finite dimensional) representation ring of G when G is not compact.

2) When Γ is the manifold $\mathbb{R}$, $K_0(C_r^*(\Gamma)) = \{0\}$ while vector bundles on $\mathbb{R}$ are obviously classified by their rank. Thus, condition (b) cannot be removed.

3) Condition (c) is not always true for every proper Lie groupoid. For instance, let $G = SU(2)$, and Γ be the transformation groupoid $G \times G \rightrightarrows G$, where G acts on itself by conjugation. It is known that $H^3(\Gamma^\bullet, \mathbb{Z}) = H_G^3(G, \mathbb{Z}) = \mathbb{Z}$ [47]. Let $S^1 \to R \to \Gamma \rightrightarrows M$ be an S^1-central extension corresponding to the generator of $H_G^3(G, \mathbb{Z})$. Then R is clearly a proper Lie groupoid, and $C_r^*(R) = \bigoplus_{n\in\mathbb{Z}} C_r^*(\Gamma; R^n)$ according to Proposition 3.2. Assume that $C_r^*(R)$ is stably unital. Then $C_r^*(\Gamma; R)$ is stably unital since a quotient of a stably unital C^*-algebra is obviously stably unital. Therefore it follows that there exists a projection in $C_r^*(\Gamma; R) \otimes \mathcal{K}(\mathbb{H})$, and hence a (Γ, L)-twisted vector bundle by Corollary 5.26. This contradicts Proposition 5.5. In fact the above argument shows that (c) fails for any non-torsion S^1-central extension of a proper Lie groupoid.

However, note that condition (c) is fulfilled when Γ is a transformation groupoid $G \times M \rightrightarrows M$, where G is a compact Lie group acting on a compact manifold M (Corollary 5.21), or when Γ is a compact étale groupoid (since in this case $C_r^*(\Gamma)$ is unital).

4) Condition (d) implies that the class α of the S^1-central extension in $H^2(\Gamma^\bullet, \mathcal{S}^1)$ must be a torsion. We conjecture that the converse holds:

CONJECTURE. – *Let $\Gamma \rightrightarrows M$ be a proper Lie groupoid such that M/Γ is compact. Assume that $S^1 \to R \to \Gamma \rightrightarrows M$ is an S^1-central extension of Lie groupoids which corresponds to a torsion class in $H^2(\Gamma^\bullet, \mathcal{S}^1)$. Then there exists a (Γ, R)-twisted vector bundle.*

It is known by Serre–Grothendieck theorem [23,31] that Conjecture 5.7 holds when Γ is Morita equivalent to a compact manifold. It also holds if Γ is a compact group. In this case,

R is also a compact group, so $C_r^*(R)$ is stably unital (see, for instance, Corollary 5.17 and Proposition 5.19). Therefore there always exists a twisted vector bundle, i.e., a finite dimensional projective representation. However, the conjecture remains open even for orbifold groupoids (i.e. étale proper groupoids).

One possibility to prove this conjecture is to generalize Grothendieck's proof [31, Theorem 1.6] to the simplicial CW-complex $\Gamma^\bullet$ corresponding to the groupoid Γ. This requires some sophisticated study of homotopy theory of simplicial manifolds. In particular, the following question arises naturally:

Question. – Let $PU(\infty)$ be the inductive limit of $PU(n)$, and $\Gamma \rightrightarrows M$ a proper Lie groupoid such that M/Γ is compact. Let γ be an element in $H^2(\Gamma^\bullet, \mathbb{Q}/\mathbb{Z})$. Does γ always induce a map from simplicial manifolds $\widetilde{\Gamma}^\bullet$ to $PU(\infty)^\bullet$, where $\widetilde{\Gamma}$ is some Lie groupoid Morita equivalent to Γ?

Finally, we list some consequences of Theorem 5.28 in various special cases.

COROLLARY 5.30. – *Let M be a compact manifold and α a torsion class in $H^3(M, \mathbb{Z})$. Then $K^0_\alpha(M)$ is isomorphic to $K^{vb,cont}_\alpha(M)$.*

Proof. – Use Remark 5.29 and the discussion following the conjecture above. □

COROLLARY 5.31. – *Let M be a compact manifold and G a compact Lie group. Assume that $\alpha \in H^3_G(M, \mathbb{Z})$ is a torsion class which admits at least one twisted vector bundle. Then $K^0_{G,\alpha}(M)$ is isomorphic to $K^{vb,cont}_\alpha(\Gamma^\bullet)$, where Γ is the transformation groupoid $G \times M \rightrightarrows M$.*

Note that in the above case when $\alpha = 0$, twisted vector bundles simply correspond to G-equivariant vector bundles over M, which always exist. Corollary 5.31 simply implies that the original definition of equivariant K-theory of Segal [65] is equivalent to the K-theory of the crossed product C^*-algebra $C_0(M) \rtimes G$.

COROLLARY 5.32. – *Let $\mathbf{X}$ be a compact orbifold. Assume that $\mathbf{X}$ is reduced, or that $\mathbf{X}$ can be represented by a compact étale groupoid. If $\alpha \in H^3(\mathbf{X}, \mathbb{Z})$ is a torsion which admits a twisted vector bundle, then $K^0_\alpha(\mathbf{X})$ is isomorphic to $K^{vb,cont}_\alpha(\mathbf{X})$.*

Proof. – Recall that if an orbifold is reduced, it can be represented by a crossed-product of a manifold by a compact group, and therefore the result follows from Corollary 5.31.

If Γ is a compact étale groupoid, then $C_r^*(\Gamma)$ is unital (the unit being represented by the characteristic function of $\Gamma^{(0)}$), and therefore condition (c) is fulfilled. □

5.8. The main theorem: smooth case

Our goal in this subsection is to prove the analogue of Theorem 5.28 for smooth vector bundles. The main result is the following

THEOREM 5.33. – *Under the same hypothesis as in Theorem* 5.28, *we have the following commutative diagram of isomorphisms*:

(38)
$$\begin{array}{ccc} K_0(C_c^\infty(\Gamma, R)) & \xrightarrow{V} & K^{vb}_\alpha(\Gamma^\bullet) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\ K_0(C_r^*(\Gamma; R)) & \xrightarrow{V'} & K^{vb,cont}_\alpha(\Gamma^\bullet), \end{array}$$

where i and i' are naturally defined; V and V' are defined as follows. For every projection $P \in C_r^*(\Gamma; R) \otimes \mathcal{K}(\mathbb{H})$,

$$V'(P) = P\big(\widetilde{L}^2(\Gamma; L) \otimes \mathbb{H}\big),$$

and for every projection $P \in C_c^\infty(\Gamma; R) \otimes \mathcal{K}(\mathbb{H}_n)$,

$$V(P) = P\big(\widetilde{L}^2(\Gamma; L) \otimes \mathbb{H}_n\big).$$

Here $\mathbb{H}_n$ denotes the n-dimensional Hilbert space $\mathbb{C}^n \subset \mathbb{H}$.

It follows from Theorem 5.28 that V' is well-defined and is an isomorphism. To prove that i and i' are isomorphisms, we will first show that $C_c^\infty(\Gamma, R)$ is stable under holomorphic functional calculus. Let us recall the definition below.

DEFINITION 5.34. – Assume that $\mathcal{B}$ is a subalgebra of a Banach algebra B. Let $\widetilde{\mathcal{B}}$ and $\widetilde{B}$ be the unitization of $\mathcal{B}$ and B respectively. $\mathcal{B}$ is said to be stable under holomorphic functional calculus if for any $b \in \widetilde{\mathcal{B}}$ and any f holomorphic on a neighborhood of $sp(b)$, we have $f(b) \in \widetilde{\mathcal{B}}$.

If furthermore $\mathcal{B}$ is endowed with a structure of Fréchet algebra such that the inclusion $\mathcal{B} \to B$ is continuous, then the following are equivalent (see [13, Appendix] or [64, Lemma 1.2, Theorem 2.1]):

(i) $\mathcal{B}$ is stable under holomorphic calculus;
(ii) for all n, $M_n(\mathcal{B})$ is stable under holomorphic calculus;
(iii) every element in $\widetilde{\mathcal{B}}$ which is invertible in $\widetilde{B}$ is actually invertible in $\widetilde{\mathcal{B}}$.

When any of the conditions above is satisfied, the inclusion $\mathcal{B} \to B$ induces an isomorphism of K-theory.

Assume now that Γ and M are manifolds and $s : \Gamma \to M$ is a submersion (Γ is not necessarily a groupoid). Let $E \to \Gamma$ be a Hermitian vector bundle. Assume that there exists a smooth s-system $\mu = (\mu_x)_{x \in M}$, i.e. μ_x is a measure on Γ whose support is $\Gamma_x = s^{-1}(x)$ such that for every $f \in C_c^\infty(\Gamma)$ the function $x \mapsto \int_{g \in \Gamma_x} f(g)\, \mu_x(dg)$ is smooth.

Remark 5.35. – We will be interested in the case that $S^1 \to R \to \Gamma \rightrightarrows M$ is an S^1-central extension of Lie groupoids, s is the source map, E is the associated line bundle and μ is a smooth Haar system. It is well-known that such a Haar system exists on any Lie groupoid.

Let $F \subset \Gamma$ be a closed subset such that the restriction $s_{|F} : F \to M$ is proper. Let

$$\mathcal{A}_F = \big\{ a \in C^\infty\big(\Gamma \times_s \Gamma, pr_1^*(E) \otimes pr_2^*(E^*)\big) \mid \operatorname{supp}(a) \subset F \times_s F \big\},$$

where $pr_1, pr_2 : \Gamma \times_s \Gamma \to \Gamma$ are the projections. We endow $\mathcal{A}_F$ with the convolution product

$$(a * b)(g,h) = \int_{\Gamma_{s(g)}} a(g,k) \cdot b(k,h)\, \mu_{s(g)}(dk),$$

where $\cdot$ denotes the obvious product $E_g \otimes E_k^* \otimes E_k \otimes E_h^* \to E_g \otimes E_h^*$, and the adjoint

$$(a^*)(g,h) = a(h,g)^*.$$

For any $\xi \in C_c^\infty(\Gamma_x; E)$, let

$$\big(\pi_x(a)(\xi)\big)(g) = (a * \xi)(g) = \int_{h \in \Gamma_x} a(g,h) \xi(h)\, \mu_x(dh).$$

Then $a \mapsto \pi_x(a)$ defines a $*$-representation of $\mathcal{A}_F$ in $\mathcal{L}(L^2(\Gamma_x;E))$. Assume now that we are given a directed system (ordered by inclusion) of closed subsets $F_i \subset \Gamma$ such that $s_{|F_i}$ is proper for all i. Let $\mathcal{A} = \lim_i \mathcal{A}_{F_i}$, and A be the completion of $\mathcal{A}$ under the norm

$$\|a\| = \sup_{x\in M}\big\|\pi_x(a)\big\|.$$

Denote by $\widetilde{A}$ and $\widetilde{\mathcal{A}}$ the unitization of A and $\mathcal{A}$, respectively.

LEMMA 5.36. – *$\mathcal{A}$ is a subalgebra of A, and is stable under holomorphic functional calculus.*

Proof. – Let $\widetilde{b} \in \widetilde{\mathcal{A}}$ be invertible in A. We need to show that $\widetilde{b}$ is invertible in $\widetilde{\mathcal{A}}$. Since $\widetilde{\mathcal{A}}$ is dense in $\widetilde{A}$, there exists $x \in \widetilde{\mathcal{A}}$ such that $\|1-\widetilde{b}x\| < 1/3$. Since $\widetilde{b}^{-1} = x(\widetilde{b}x)^{-1}$, we may assume that $\|1-\widetilde{b}\| < 1/3$. Let $\widetilde{a} = 1-\widetilde{b}$. We have $\widetilde{a} = \lambda + b$, where $\lambda \in \mathbb{C}$, $|\lambda| < 1/3$ and $b \in \mathcal{A}$. Thus $\|b\| = \|\widetilde{a}-\lambda\| < 2/3$. Let $a = (1-\lambda)^{-1}b$. Since $(1-\widetilde{a})^{-1} = (1-\lambda)^{-1}(1-a)^{-1}$, it suffices to prove that $(1-a)^{-1} \in \widetilde{\mathcal{A}}$ whenever $a \in \mathcal{A}$ and $\|a\| < 1$.

Let $a_n = a * a * \cdots * a$ (n times). We show that the sum $\sum_{n=1}^{\infty} a_n$, and as well as all its derivatives, converges uniformly on every compact set.

Since $a_n(g,h) = [a_{n-1} * a(\cdot,h)](g)$, we have

$$\big\|a_n(\cdot,h)\big\|_{L^2(\Gamma_{s(h)})} = \big\|a_{n-1} * a(\cdot,h)\big\| \leqslant \|a\|^{n-1}\big\|a(\cdot,h)\big\|_{L^2(\Gamma_{s(h)})}$$

and similarly,

$$\big\|a_m(g,\cdot)\big\|_{L^2(\Gamma_{s(g)})} \leqslant \|a\|^{m-1}\big\|a(g,\cdot)\big\|_{L^2(\Gamma_{s(g)})}.$$

From the Cauchy–Schwarz inequality,

$$\begin{aligned}\big|a_{m+n}(g,h)\big| &\leqslant \big\|a_m(g,\cdot)\big\|_{L^2(\Gamma_{s(g)})}\big\|a_n(\cdot,h)\big\|_{L^2(\Gamma_{s(h)})}\\ &\leqslant \|a\|^{m+n-2}\big\|a(g,\cdot)\big\|_{L^2(\Gamma_{s(g)})}\big\|a(\cdot,h)\big\|_{L^2(\Gamma_{s(h)})}.\end{aligned}$$

It follows that $\sum_n a_n$ converges uniformly on every compact subset of $\Gamma \times_s \Gamma$. Similarly, one shows that all derivatives converge uniformly on any compact subset. □

PROPOSITION 5.37. – *Let $S^1 \to R \to \Gamma \rightrightarrows M$ be an S^1-central extension of Lie groupoids. Assume that Γ is proper. Then the subalgebra $C_c^\infty(\Gamma;R)$ of $C_r^*(\Gamma;R)$ is stable under holomorphic functional calculus.*

Proof. – We use the construction above, where $s : \Gamma \to M$ is the source map and the fiber bundle E is $L = R \times_{S^1} \mathbb{C}$. Let $K \subset M$ be a compact subset, and $F_K = \Gamma^K$. By the properness of Γ, $s_{|F_K}$ is a proper map. As above, define $\mathcal{A} = \lim_K \mathcal{A}_{\Gamma^K}$. Denote by $\mathcal{A}^\Gamma$ the subspace of $\mathcal{A}$ consisting of Γ-invariant elements, i.e. elements satisfying $a(g\gamma,h\gamma) = a(g,h)$, where $L_g \otimes L_h^*$ and $L_{g\gamma} \otimes L_{h\gamma}^*$ are both identified with $L_{gh^{-1}}$. Consider the map

$$\begin{aligned}C_c(\Gamma;R) &\to \mathcal{A}^\Gamma\\ f &\mapsto a,\end{aligned} \tag{39}$$

given by $a(g,h) = f(gh^{-1}) \in L_{gh^{-1}} \cong L_g \otimes L_h^*$. This map is well-defined. Indeed, if f is compactly supported, then there exists a compact subset K of Γ such that $\mathrm{supp}(f) \subset \Gamma_K^K$. Therefore it follows that $a \in \mathcal{A}_{\Gamma^K} \cap \mathcal{A}^\Gamma$. Conversely, if $F = \Gamma^K$ and $a \in \mathcal{A}_F \cap \mathcal{A}^\Gamma$, then $f(g) = a(g,s(g))$ is supported on Γ_K^K, which is compact by the properness assumption, and $a(g,h) = f(gh^{-1})$ since a is Γ-invariant. Therefore, the map defined by Eq. (39) is bijective.

It is not hard to check that it is an isometric $*$-isomorphism which extends to an isomorphism $C_r^*(\Gamma;R) \xrightarrow{\sim} A$. The conclusion thus follows from Lemma 5.36. □

Remark 5.38. – Proposition 5.37 was proved in [6, Lemma 7.5] in the non-twisted case for the crossed-product of a discrete group acting properly on a manifold.

As an immediate consequence, we have the following

COROLLARY 5.39. – *The inclusion $i : C_c^\infty(\Gamma;R) \to C_r^*(\Gamma;R)$ induces an isomorphism of K-theory.*

We now return to the diagram (38), and show that V is well-defined. We first need two preliminary lemmas.

LEMMA 5.40. – *Let $P \in C_r^*(\Gamma;R) \otimes \mathcal{K}$ be a projection and $\varepsilon > 0$. Then there exists a projection $P' \in C_c^\infty(\Gamma;R) \otimes \mathcal{K}_0$, where $\mathcal{K}_0$ denotes the algebra of finite rank operators on $\mathbb{H}$ such that $\|P' - P\| < \varepsilon$.*

Proof. – Let $a \in C_c^\infty(\Gamma;R) \otimes \mathcal{K}_0$ such that $\|a - P\| < \varepsilon/2$. Then the spectrum of a is contained in the open set $U = B(0,\varepsilon/2) \cup B(1,\varepsilon/2) \subset \mathbb{C}$. Let $f : U \to \mathbb{C}$ be the function which is equal to 0 on $B(0,\varepsilon/2)$ and is equal to 1 on $B(1,\varepsilon/2)$. Then $P' := f(a)$ is a projection such that $\|P' - a\| < \varepsilon/2$, and $P' \in C_c^\infty(\Gamma;R) \otimes \mathcal{K}_0$ by Proposition 5.37. □

LEMMA 5.41. – *Suppose that M is a manifold and $\pi : E \to M$ a Hermitian vector bundle in the topological sense. Assume that we are given a subspace $\mathcal{S} \subset C(M,E)$ such that*
(a) *$\mathcal{S}$ is a $C^\infty(M)$-module;*
(b) *for all $\xi, \eta \in \mathcal{S}$, $x \mapsto \langle \xi(x), \eta(x) \rangle$ is a smooth function on M;*
(c) *$\{\xi(x) \mid \xi \in \mathcal{S}\}$ is dense in E_x for all x.*
Then there exists a unique smooth structure on the vector bundle E such that $\mathcal{S}$ consists of smooth sections.

Proof. – By the Gram–Schmidt orthonormalization process, there exists an open cover (U_i) of M and sections $\xi_{i,1}, \ldots, \xi_{i,n} \in \mathcal{S}$ such that for all $x \in U_i$, $(\xi_{i,1}(x), \ldots, \xi_{i,n}(x))$ is an orthonormal basis of E_x. Thus, we get local trivializations $\varphi_i : \pi^{-1}(U_i) \cong U_i \times \mathbb{C}^n$. Since $\langle \xi_{i,k}, \xi_{j,l} \rangle$ is smooth for all i,j,k,l, the change of coordinates $\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times \mathbb{C}^n \to (U_i \cap U_j) \times \mathbb{C}^n$ is smooth, thus we get a smooth structure on E. From (b), it is clear that all elements of $\mathcal{S}$ are smooth sections.

Conversely, it is clear that if E has a second smooth structure such that all elements of $\mathcal{S}$ are smooth sections, then φ_i must be smooth for all i, and therefore the two smooth structures coincide. □

Now we return to the proof of Theorem 5.33.

Proof of Theorem 5.33. – By assumption (see proof of Theorem 5.28), there exists an approximate unit (P_n) in $C_r^*(\Gamma;R) \otimes \mathcal{K}$ consisting of projections. According to Lemma 5.40, there is a projection $P_n' \in C_c^\infty(\Gamma;R) \otimes \mathcal{K}_0$ such that $\|P_n' - P_n\| < 1/n$. It is clear that (P_n') is an approximate unit of $C_c^\infty(\Gamma;R) \otimes \mathcal{K}_0$ consisting of projections. Hence according to [12, Proposition 5.5.5], $K_0(C_c^\infty(\Gamma;R) \otimes \mathcal{K}_0)$ is the Grothendieck group of projections in $C_c^\infty(\Gamma;R) \otimes \mathcal{K}_0$.

Assume now that $P \in C_c^\infty(\Gamma;R) \otimes \mathcal{K}_0 \subset \mathcal{L}(L^2(\Gamma;L) \otimes \mathbb{H})$ is a projection. Let $E = P(\widetilde{L}^2(\Gamma;R) \otimes \mathbb{H})$. Then E is a twisted vector bundle in the topological sense. We say that a section of E is smooth if it is of the form $x \mapsto P_x \xi_x$, where $\xi \in C_c^\infty(\Gamma;L) \otimes \mathbb{H}_n$ for some n. Since for any two smooth sections η and ζ, $x \mapsto \langle \eta(x), \zeta(x) \rangle$ is smooth, the space of smooth sections defines a smooth structure on E according to Lemma 5.41. It follows that the map V in (38) is well-defined. Also it is clear that the diagram (38) is commutative.

Finally, we prove that all maps in (38) are isomorphisms. For i and V', this follows from Theorem 5.28 and Corollary 5.39. It remains to show that i' is injective.

Assume that E and F are smooth twisted vector bundles such that $[E]-[F]\in\ker i'$. Then there exists a topological twisted vector bundle G such that $E\oplus G\cong F\oplus G$. From the proof of Theorem 5.28, we know that there exists a projection $P\in C_r^*(\Gamma;R)\otimes\mathcal{K}$ such that $G\cong P(\widetilde{L}^2(\Gamma;R)\otimes\mathbb{H})$. According to Lemma 5.40, there exists a projection $P'\in C_c^\infty(\Gamma;R)\otimes\mathcal{K}_0$ such that $\|P'-P\|<1$. This implies that $G\cong P'(\widetilde{L}^2(\Gamma;R)\otimes\mathbb{H})$. Therefore we can assume that G is a smooth vector bundle. Replacing E by $E\oplus G$ and F by $F\oplus G$, we see that E and F are isomorphic as topological twisted vector bundles. Let $T=(T_x)_{x\in M}$ be an isomorphism from E to F. As in the proof of Proposition 5.27, we can assume that T_x is isometric for all x. Let $T':E\to F$ be a (fiberwise linear, non equivariant) smooth morphism of vector bundles such that $\|T'_x-T_x\|\leqslant 1/2$ for all x. Choose a smooth cutoff function $c:M\to\mathbb{R}_+$ for the proper groupoid R. Let

$$T''_x=\int_{R^x}\alpha_r\big(T'_{s(r)}\big)c\big(s(r)\big)\,\lambda^x(dr).$$

Since T is R-invariant,

$$\|T''_x-T_x\|\leqslant\int_{R^x}\big\|\alpha_r(T'_{s(r)}-T_{s(r)})\big\|c\big(s(r)\big)\,\lambda^x(dr)\leqslant 1/2\int_{R^x}c\big(s(r)\big)\,\lambda^x(dr)=1/2.$$

Therefore T''_x is an isomorphism for all $x\in M$. Moreover, it is clear that $x\mapsto T''_x$ is equivariant. It follows that E and F are isomorphic as smooth twisted vector bundles, and thus $[E]-[F]=0$. This completes the proof of the theorem. □

6. The product $K^i_\alpha(\Gamma^\bullet)\otimes K^j_\beta(\Gamma^\bullet)\to K^{i+j}_{\alpha+\beta}(\Gamma^\bullet)$

6.1. The main idea

Let $S^1\to R\to\Gamma\rightrightarrows M$ and $S^1\to R'\to\Gamma\rightrightarrows M$ be S^1-central extensions, and let α and β be their corresponding classes in $H^2(\Gamma^\bullet,\mathcal{S}^1)$. It is simple to see that there is a bilinear product

$$K^{vb}_\alpha(\Gamma^\bullet)\otimes K^{vb}_\beta(\Gamma^\bullet)\to K^{vb}_{\alpha+\beta}(\Gamma^\bullet)$$

defined as follows: let E be a (Γ,R)-twisted vector bundle and E' a (Γ,R')-twisted vector bundle, then the product of $[E]$ and $[E']$ is $[E\otimes E']$.

The objective of this section is to prove the following

THEOREM 6.1. – *Let $\Gamma\rightrightarrows M$ be a proper Lie groupoid such that M/Γ is compact, and α, $\beta\in H^2(\Gamma^\bullet,\mathcal{S}^1)$. Then there exists a bilinear, associative product*

$$K^i_\alpha(\Gamma^\bullet)\otimes K^j_\alpha(\Gamma^\bullet)\to K^{i+j}_{\alpha+\beta}(\Gamma^\bullet)\tag{40}$$

$(i,\ j\in\{0,1\})$, which is compatible with the canonical map $K^{vb}_\gamma(\Gamma^\bullet)\to K^0_\gamma(\Gamma^\bullet)$ $(\gamma\in H^2(\Gamma^\bullet,\mathcal{S}^1))$.

In the theorem above, the canonical map $\iota : K^{vb}_\gamma(\Gamma^\bullet) \to K^0_\gamma(\Gamma^\bullet)$ is constructed as in Section 5.6. Note that the construction of ι only requires the groupoid Γ to be proper, while the construction of the inverse of ι as described in Section 5.6 requires all the hypotheses in Theorem 5.28.

Recall that in the Fredholm picture of twisted K-theory (see Theorem 3.14), the difficulty in constructing the product (40) is to obtain a Fredholm operator T out of two Fredholm operators T_1 and T_2. Exactly the same difficulty appears in the construction of the Kasparov product [11]. The existence of the product $KK(A,D) \times KK(D,B) \to KK(A,B)$ (with A, D and B separable C^*-algebras) is proved using non constructive methods (in particular the Hahn–Banach theorem), although explicit computations are possible in particular cases.

As a matter of fact, one can show, using Theorem 3.14, that for a proper groupoid with compact orbit space $\Gamma \rightrightarrows M$, the *untwisted* K-theory groups $K^i(\Gamma^\bullet)$ are isomorphic to the equivariant KK-groups $KK^i_\Gamma(C_0(M), C_0(M))$ defined by Le Gall [44]. The existence of the product thus follows from the product in KK_Γ-theory.

From the above discussion, it is quite natural to generalize the KK-bifunctor further, and then try to identify it with the twisted K-theory groups: this is the object of this section. We will assume that the reader has some basic knowledge about KK-theory [11, Chapter 8]. Since most of the theory is already done in [11] or [44], we will only give those definitions and proofs that need substantial modification.

6.2. The KK bifunctor

Let us first recall a definition:

DEFINITION 6.2. – Let A and B be C^*-algebras. A C^*-correspondence from A to B is a pair $(\mathcal{E}, f)$ where $\mathcal{E}$ is a B-Hilbert module and f is a non-degenerate $*$-homomorphism from A to $\mathcal{L}(\mathcal{E})$.

Recall that a $*$-homomorphism $\pi : A \to \mathcal{L}(\mathcal{E})$ is said to be non-degenerate if the closed linear span of $\pi(A)\mathcal{E}$ is equal to $\mathcal{E}$; by Cohen's theorem, this is equivalent to $\pi(A)\mathcal{E} = \mathcal{E}$. In particular, if $f : A \to B$ is a $*$-homomorphism, then f induces a C^*-correspondence (B, f).

Correspondences can be composed using the internal tensor product of Hilbert modules: if $(\mathcal{E}, f)$ is a C^*-correspondence from A to B and $(\mathcal{E}', g)$ is a C^*-correspondence from B to D, then $(\mathcal{E}', g) \circ (\mathcal{E}, f) = (\mathcal{E} \otimes_g \mathcal{E}', f \otimes 1)$ is a C^*-correspondence from A to D. Therefore, there is a category $\mathcal{C}$ whose objects are C^*-algebras and morphisms are C^*-correspondences. And also there is a functor from the usual category of C^*-algebras $\mathbf{C}^*$ to $\mathcal{C}$ (given by the map $f \mapsto (B, f)$ as above). Moreover, isomorphism in the category $\mathcal{C}$ is Morita equivalence.

Recall that given a locally compact group G, Kasparov constructed a bifunctor from the category of G-C^*-algebras to abelian groups $(A,B) \mapsto KK_G(A,B)$ which is covariant in B and contravariant in A, and which is endowed with an associative product

$$KK_G(A,B) \otimes KK_G(B,D) \to KK_G(A,D).$$

This construction was generalized by Le Gall to locally compact groupoids admitting Haar systems [44]. Our goal in this subsection is to generalize this construction further, by allowing the groupoid to act by Morita equivalences on the C^*-algebras instead of by $*$-automorphisms, i.e. to work in the category $\mathcal{C}$ instead of $\mathbf{C}^*$. This idea was communicated to us by Le Gall.

For convenience, let us introduce some terminology.

DEFINITION 6.3. – Let $\Gamma \rightrightarrows M$ be a locally compact groupoid. Let A be a C^*-algebra. A *generalized action* of Γ on A is given by

(i) a u.s.c. Fell bundle $\mathcal{A}$ over Γ;
(ii) an isomorphism $A \cong C_0(M;\mathcal{A})$.

For instance, if Γ acts on A in the usual sense, then there exists a u.s.c. field of C^*-algebras $\mathcal{A}'$ with fiber $\mathcal{A}'_x \cong A_x$ at $x \in M$ such that $A \cong C_0(M;\mathcal{A}')$, and the action of Γ on A induces $*$-isomorphisms $\alpha_g : \mathcal{A}'_{s(g)} \to \mathcal{A}'_{t(g)}$. Let $\mathcal{A} = s^*\mathcal{A}'$, with the product

$$\begin{aligned}\mathcal{A}_g \otimes \mathcal{A}_h \cong A_{s(g)} \otimes A_{s(h)} \to A_{s(h)} \cong \mathcal{A}_{gh},\\ (a,b) \mapsto \alpha_{h^{-1}}(a)b\end{aligned}$$

and the involution

$$\begin{aligned}\mathcal{A}_g \cong A_{s(g)} \to A_{t(g)} \cong \mathcal{A}_{g^{-1}},\\ a \mapsto \alpha_g(a^*).\end{aligned}$$

Then $\mathcal{A}$ is a u.s.c. Fell bundle over Γ, and thus defines a generalized action of Γ on A.

If A and B are C^*-algebras endowed with a Γ-action, there is a notion of equivariant $*$-homomorphism $f : A \to B$. More generally, we want to introduce the definition of an equivariant correspondence (Definition 6.4).

We first introduce some notation: let A be a C^*-algebra endowed with a generalized action $\mathcal{A}$ of a locally compact groupoid Γ. Denote by $\hat{\mathcal{A}}$ the space of norm-bounded continuous maps vanishing at infinity $g \mapsto a'_g \in \mathcal{A}_{g^{-1}}$. $\hat{\mathcal{A}}$ is naturally a t^*A-Hilbert module with the module structure

$$(a'a)_g = a'_g \cdot a_g \quad (a' \in \hat{\mathcal{A}},\ a \in t^*A)$$

and the scalar product

$$\langle a', a'' \rangle_g = {a'_g}^* a''_g \in A_{t(g)}.$$

DEFINITION 6.4. – Let A and B be C^*-algebras endowed with generalized actions $\mathcal{A}$ and $\mathcal{B}$ of a locally compact groupoid Γ. Let $\mathcal{E}$ be a C^*-correspondence from A to B. We say that the correspondence $\mathcal{E}$ is equivariant if there is an isomorphism of s^*A, t^*B correspondences

$$W : s^*\mathcal{E} \otimes_{s^*B} \hat{\mathcal{B}} \to \hat{\mathcal{A}} \otimes_{t^*A} t^*\mathcal{E}$$

such that for every $(g,h) \in \Gamma^{(2)}$,

$$\begin{aligned}&(\mathrm{Id}_{\mathcal{A}_{h^{-1}}} \otimes W_g) \circ (W_h \otimes \mathrm{Id}_{\mathcal{B}_{g^{-1}}})\\ &\quad \in \mathcal{L}(\mathcal{E}_{s(h)} \otimes_{B_{s(h)}} \mathcal{B}_{h^{-1}} \otimes_{B_{t(h)}} \mathcal{B}_{g^{-1}}, \mathcal{A}_{h^{-1}} \otimes_{A_{t(h)}} \mathcal{A}_{g^{-1}} \otimes_{A_{t(g)}} \mathcal{E}_{t(g)})\end{aligned}$$

is equal to

$$W_{gh} \in \mathcal{L}(\mathcal{E}_{s(h)} \otimes_{B_{s(h)}} \mathcal{B}_{h^{-1}g^{-1}}, \mathcal{A}_{h^{-1}g^{-1}} \otimes_{A_{t(g)}} \mathcal{E}_{t(gh)})$$

via the identifications $\mathcal{A}_{(gh)^{-1}} \cong \mathcal{A}_{h^{-1}} \otimes_{A_{t(h)}} \mathcal{A}_{g^{-1}}$ and $\mathcal{B}_{(gh)^{-1}} \cong \mathcal{B}_{h^{-1}} \otimes_{B_{t(h)}} \mathcal{B}_{g^{-1}}$.

When the action of Γ on $A \simeq C_0(M, \widetilde{\mathcal{A}})$ is an action in the usual sense, and $\mathcal{E}$ is an equivariant correspondence, then $\mathcal{E}$ is a $(\Gamma, \mathcal{B})$-equivariant B-Hilbert module (see Definition 5.13). Let $\mathcal{L}(\widetilde{\mathcal{E}}) = \coprod_{x \in M} \mathcal{L}(\mathcal{E}_x)$ be the bundle defined in Appendix A, preceding Proposition A.5. Then the map $A \to \mathcal{L}(\mathcal{E})$ induces a Γ-equivariant bundle map $\widetilde{\mathcal{A}} \to \mathcal{L}(\widetilde{\mathcal{E}})$.

Note that there is a category $\mathcal{C}_\Gamma$ whose objects consist of C^*-algebras endowed with generalized actions of Γ, and whose morphisms are equivariant correspondences.

To define the KK-groups, we first recall that if $F_2 \in \mathcal{L}(\mathcal{E}_2)$ then $\mathrm{Id} \otimes F_2$ does not make sense. Instead, one has to use the notion of connection [18, Appendix A, pp. 1174–1178]:

DEFINITION 6.5. – Let $\mathcal{E}_1$ be a D-Hilbert module and $\mathcal{E}_2$ a D, B-correspondence. Let $\mathcal{E} = \mathcal{E}_1 \otimes_D \mathcal{E}_2$, $F_2 \in \mathcal{L}(\mathcal{E}_2)$ and $F \in \mathcal{L}(\mathcal{E})$. We say that F is a F_2-connection for $\mathcal{E}_1$ if for every $\xi \in \mathcal{E}_1$,

$$T_\xi F_2 - (-1)^{\partial\xi\partial F_2} F T_\xi \in \mathcal{K}(\mathcal{E}_2, \mathcal{E})$$
$$F_2 T_\xi^* - (-1)^{\partial\xi\partial F_2} T_\xi^* F \in \mathcal{K}(\mathcal{E}, \mathcal{E}_2).$$

The above operator $T_\xi \in \mathcal{L}(\mathcal{E}_2, \mathcal{E})$ is defined by

$$T_\xi(\eta) = \xi \otimes \eta. \tag{41}$$

Note the slight ambiguity, since $\mathcal{E}_2$ does not appear in the notation T_ξ.

Recall also that F is a F_2 connection if and only if for all $\xi \in \mathcal{E}_1$, the graded commutator $\left[\theta_\xi, \left(\begin{smallmatrix} F_2 & 0 \\ 0 & F \end{smallmatrix}\right)\right]$ belongs to $\mathcal{K}(\mathcal{E}_2 \oplus \mathcal{E})$, where $\theta_\xi = \left(\begin{smallmatrix} 0 & T_\xi^* \\ T_\xi & 0 \end{smallmatrix}\right)$.

Let us now define the KK-groups. If $\mathcal{E}$ is an equivariant A, B-correspondence and $F \in \mathcal{L}(\mathcal{E})$. Denote by $t^*F \in \mathcal{L}(t^*\mathcal{E})$ and $s^*F \in \mathcal{L}(s^*\mathcal{E})$ the pull-backs of F by t and s respectively. Let

$$\sigma(F) = W(s^*F \otimes \mathrm{Id})W^* \in \mathcal{L}(\hat{\mathcal{A}} \otimes_{t^*A} t^*\mathcal{E}).$$

DEFINITION 6.6. – Let A and B be C^*-algebras endowed with generalized actions of Γ. An equivariant Kasparov A, B-bimodule [5] is a pair $(\mathcal{E}, F)$, where $\mathcal{E}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded, equivariant A, B-correspondence and $F \in \mathcal{L}(\mathcal{E})$ is a degree 1 operator such that for all $a \in A$,
- (i) $a(F - F^*) \in \mathcal{K}(\mathcal{E})$;
- (ii) $a(F^2 - 1) \in \mathcal{K}(\mathcal{E})$;
- (iii) $[a, F] \in \mathcal{K}(\mathcal{E})$;
- (iv) $\sigma(F)$ is a t^*F-connection for $\hat{\mathcal{A}}$.

If Γ is a discrete group, (iv) holds if and only if $Ad_g(F) - F$ is compact for all $g \in \Gamma$. Thus we will refer to condition (iv) as the condition of invariance modulo compacts.

As usual, unitarily equivalent Kasparov bimodules are identified. Let $\mathbf{E}_\Gamma(A, B)$ be the set of (unitary equivalence classes of) Kasparov A, B-bimodules. A *homotopy* in $E_\Gamma(A, B)$ is given by an element of $\mathbf{E}_\Gamma(A, B[0,1])$. The set of homotopy classes of elements of $\mathbf{E}_\Gamma(A, B)$ is denoted by $KK_\Gamma(A, B)$. Then $KK_\Gamma(A, B)$ is an abelian group, and $(A, B) \mapsto KK_\Gamma(A, B)$ is a bifunctor, covariant in B, contravariant in A (in the category $\mathcal{C}_\Gamma$).

6.3. The technical theorem

The main ingredient in the construction of the product

$$KK_\Gamma(A, D) \times KK_\Gamma(D, B) \to KK_\Gamma(A, B)$$

is the so-called technical theorem [34, pp. 108–109]. We first need a lemma:

LEMMA 6.7. – *Let J and J' be two C^*-algebras. Let $\pi : J \to J'$ be a $*$-homomorphism, $\varepsilon > 0$, $h_0 \in J_+$ such that $\|h_0\| < 1$, $h \in J$, $h' \in J'$, $\mathcal{K} \subset Der(J)$ compact, $\mathcal{K}' \subset Der(J')$ compact. Then there exists $u \in J$ such that*
1) $h_0 \leqslant u$, $\|u\| < 1$;
2) $\|uh - h\| \leqslant \varepsilon$;

[5] “Kasparov correspondence” might be a more appropriate terminology but is not the usual one.

2′) $\|\pi(u)h' - h'\| \leqslant \varepsilon$;

3) $\forall d \in \mathcal{K}$, $\|[d,u]\| \leqslant \varepsilon$;

3′) $\forall d' \in \mathcal{K}'$, $\|[d',\pi(u)]\| \leqslant \varepsilon$.

The proof is almost the same as in [34]. Let us now come to the technical theorem.

THEOREM 6.8. – *Let*

- A_1 *and* A_1' *be two* C^**-algebras such that* A_1 *is* σ*-unital*;
- J *and* J' *equivariant ideals in* A_1 *and* A_1' *respectively*;
- $\pi : A_1 \to A_1'$ *such that* $\pi(J) \subset J'$;
- $\mathcal{F}$ (*resp.* $\mathcal{F}'$) *a separable subspace of* $Der(A_1)$ (*resp. of* $Der(A_1')$);
- $a_2 \in M(J)_+$ *such that* $a_2 A_1 \subset J$;
- $a_2' \in M(J')_+$ *such that* $a_2' A_1' \subset J'$.

Then there exists an element $M \in M(A_1)$*, of degree* 0*, such that*

1) M *and* $1-M$ *are strictly positive*;

2) $(1-M)a_2 \in J$;

2′) $\pi(1-M)a_2' \in J'$;

3) $MA_1 \subset J$;

3′) $\pi(M)A_1' \subset J'$;

4) $[\mathcal{F}, M] \subset J$;

4′) $[\mathcal{F}', \pi(M)] \subset J'$.

Again, the proof is almost the same as in [34].

6.4. The Kasparov product

THEOREM 6.9. – *Let* A*,* D *and* B *be separable* C^**-algebras endowed with generalized actions of a groupoid* Γ*. Let* $(\mathcal{E}_1, F_1) \in \mathbf{E}_\Gamma(A,D)$ *and* $(\mathcal{E}_2, F_2) \in \mathbf{E}_\Gamma(D,B)$*. Denote by* $\mathcal{E}$ *the equivariant* A*,* B*-correspondence* $\mathcal{E} = \mathcal{E}_1 \hat{\otimes}_D \mathcal{E}_2$*. Then the set* $F_1 \hat{\#}_\Gamma F_2$ *of operators* $F \in \mathcal{L}(\mathcal{E})$ *such that*

- $(\mathcal{E}, F) \in \mathbf{E}_\Gamma(A,B)$;
- F *is a* F_2*-connection for* $\mathcal{E}_1$;
- $\forall a \in A$*,* $a[F_1 \hat{\otimes}_D 1, F]a^* \geqslant 0$ *modulo* $\mathcal{K}(\mathcal{E})$

is non-empty.

Proof. – Choose a F_2-connection T for $\mathcal{E}_1$, and define

$$\begin{aligned}
J &= \mathcal{K}(\mathcal{E}),\\
A_1 &= \mathcal{K}(\mathcal{E}_1)\hat{\otimes}_D \operatorname{Id}_{\mathcal{E}_2} + J \subset \mathcal{L}(\mathcal{E}),\\
J' &= \big\{ S \in \mathcal{L}(\hat{\mathcal{A}} \otimes_{t^*A} t^*\mathcal{E} \oplus t^*\mathcal{E}) \mid \forall \chi \in C_0(\Gamma),\ \chi S \in \mathcal{K} \big\}\\
A_1' &= \big\{ S \in \mathcal{L}(\hat{\mathcal{A}} \otimes_{t^*A} t^*\mathcal{E} \oplus t^*\mathcal{E}) \mid \\
&\qquad \forall \chi \in C_0(\Gamma),\ \chi S \in \mathcal{K}(\hat{\mathcal{A}} \otimes_{t^*A} t^*\mathcal{E}_1 \oplus t^*\mathcal{E}_1) \otimes \operatorname{Id}_{\mathcal{E}_2} \big\} + J',\\
\mathcal{F} &= \operatorname{Vect}\big(Ad(F_1 \hat{\otimes} \operatorname{Id}_{\mathcal{E}_2}), Ad(T), Ad(a)\,(a \in A)\big) \subset Der(A_1).
\end{aligned}$$

Let $\pi : \mathcal{L}(\mathcal{E}) \to \mathcal{L}(\hat{\mathcal{A}} \otimes_{t^*A} t^*\mathcal{E} \oplus t^*\mathcal{E})$ defined by

$$\pi(S) = \begin{pmatrix} \sigma(S) & 0 \\ 0 & t^*S \end{pmatrix}.$$

Then $\pi(A_1) \subset A_1'$ and $\pi(J) \subset J'$. Let a_2' be a strictly positive element of the C^*-algebra generated by $[\theta_{a'}, \pi(T)]$ ($a' \in \hat{\mathcal{A}}$), where

$$\theta_{a'} = \begin{pmatrix} 0 & T_{a'} \\ T_{a'}^* & 0 \end{pmatrix} \in \mathcal{L}(\hat{\mathcal{A}} \otimes_{t^*A} t^*\mathcal{E} \oplus t^*\mathcal{E}).$$

Let $\mathcal{F}' = \{Ad_{\theta_a} \mid a' \in \hat{\mathcal{A}}\}$. Let A_2 be the sub-C^*-algebra of $\mathcal{L}(\mathcal{E})$ generated by

$$\left\{T - T^*; 1 - T^2; [T, F_1 \hat{\otimes}_D \mathrm{Id}_{\mathcal{E}_2}]; [T, a], \forall a \in A\right\}$$

and let a_2 be a strictly positive element of A_2. We can apply the technical Theorem 6.8 and thus obtain an operator $M \in \mathcal{L}(\mathcal{E})$ which satisfies the properties in Theorem 6.8. Let

$$F = M^{1/2}(F_1 \hat{\otimes}_D \mathrm{Id}_{\mathcal{E}_2}) + (1 - M)^{1/2} T.$$

As in the non-equivariant case, we have

$$(\mathcal{E}, F) \in \mathbf{E}(A, B)$$

and thus it just remains to prove that F satisfies the condition of "invariance modulo compacts" (Definition 6.6(iv)).

Let $M_1 = M^{1/2}(F_1 \otimes \mathrm{Id})$ and $M_2 = (1 - M)^{1/2} T$. We show that both M_1 and M_2 satisfy the invariance condition. Since $F = M_1 + M_2$, it follows that F also satisfies the invariance condition.

We have $[\theta_{a'}, \pi(M_1)] = [\theta_{a'}, \pi(M)^{1/2}]\pi(F_1 \otimes 1) + \pi(M)^{1/2}[\theta_{a'}, \pi(F_1 \otimes 1)]$. From property (4′) of M, we have $[\theta_{a'}, \pi(M)^{1/2}] \in J'$. Since J' is an ideal, $[\theta_{a'}, \pi(M)^{1/2}]\pi(F_1 \otimes 1) \in J'$. From property (iv) in Definition 6.6 for F_1, we have

$$\left[\theta_{a'}, \pi(F_1 \otimes 1)\right] \in \mathcal{K}(\hat{\mathcal{A}} \otimes_{t^*A} t^*\mathcal{E}_1 \oplus t^*\mathcal{E}_1) \otimes \mathrm{Id}_{\mathcal{E}_2} \subset A_1'.$$

Since $\pi(M)A_1' \subset J'$, we have $\pi(M)^{1/2}[\theta_{a'}, \pi(F_1 \otimes 1)] \in J'$. Finally, $[\theta_{a'}, \pi(M_1)] \in J'$, which means that $\sigma(M_1)$ is a t^*M_1-connection.

Let us show that M_2 satisfies the invariance condition (Definition 6.6 (iv)). We have

$$\left[\theta_{a'}, \pi\big((1 - M)^{1/2} T\big)\right] = \left[\theta_{a'}, \pi(1 - M)^{1/2}\right]\pi(T) + \pi(1 - M)^{1/2}\left[\theta_{a'}, \pi(T)\right].$$

By the property (4′) of M,

$$\left[\theta_{a'}, \pi(1 - M)\right] = -\left[\theta_{a'}, \pi(M)\right] \in J',$$

and thus $[\theta_{a'}, \pi(1 - M)^{1/2}] \in J'$. Since J' is an ideal, we obtain $[\theta_{a'}, \pi(1 - M)^{1/2}]\pi(T) \in J'$.

Since $\pi(1 - M)a_2' \in J'$, we have $\pi(1 - M)^{1/2} a_2' \in J'$ and thus $\pi(1 - M)^{1/2}[\theta_{a'}, \pi(T)] \in J'$. Finally, we get $[\theta_{a'}, \pi(M_2)] \in J'$. This is equivalent to the fact that $\sigma(M_2)$ is a t^*M_2-connection.

It now remains to show that the condition $a_2' A_1' \subset J'$ is fulfilled. It suffices to show that for all $a' \in \hat{\mathcal{A}}$ and all $T' \in \mathcal{K}(\hat{\mathcal{A}}_{t^*A} \otimes t^*\mathcal{E}_1 \oplus t^*\mathcal{E}_1) \otimes \mathrm{Id}_{t^*\mathcal{E}_2}$, the operator $[\theta_{a'}, \pi(T)]T'$ is compact.

Since $\mathcal{K}(\hat{\mathcal{A}} \otimes_{t^*A} t^*\mathcal{E}_1 \oplus t^*\mathcal{E}_1)$ is the closed vector subspace generated by operators of the form $T_\zeta \circ T_{\zeta'}^*$ ($\zeta, \zeta' \in \hat{\mathcal{A}} \otimes t^*\mathcal{E}_1 \oplus t^*\mathcal{E}_1$), it suffices to show that $[\theta_{a'}, \pi(T)]T_\zeta \in \mathcal{K}$ for all ζ, i.e. that

(a) $(T_{a'} t^*T - (-1)^{\partial a'} \sigma(T) T_{a'}) T_{\xi_1'} \in \mathcal{K}$, $\forall \xi_1' \in t^*\mathcal{E}_1$;

(b) $(T^*_{a'}\sigma(T) - (-1)^{\partial a'}(t^*T)T^*_{a'})T_{a''\otimes\xi'_1} \in \mathcal{K}, \forall a''\otimes\xi'_1 \in \hat{\mathcal{A}} \otimes_{t^*A} t^*\mathcal{E}_1$.

Let us show (a). Since T is a F_2-connection, t^*T is a t^*F_2-connection. Hence

$$(t^*T)T_{\xi'_1} - (-1)^{\partial\xi'_1}T_{\xi'_1}t^*F_2 \in \mathcal{K},$$

which in turn implies that

$$T_{a'}(t^*T)T_{\xi'_1} - (-1)^{\partial\xi'_1}T_{a'\otimes\xi'_1}(t^*F_2) \in \mathcal{K}.$$

Therefore, it suffices to show that

$$T_{a'\otimes\xi'_1}(t^*F_2) - (-1)^{\partial(a'\otimes\xi'_1)}\sigma(T)T_{a'\otimes\xi'_1} \in \mathcal{K}. \tag{42}$$

Let $W_1 : s^*\mathcal{E}_1 \otimes_{s^*D} \hat{\mathcal{D}} \xrightarrow{\sim} \hat{\mathcal{A}} \otimes_{t^*A} t^*\mathcal{E}_1$ be the isomorphism induced from the generalized action of Γ on $\mathcal{E}_1$. Similarly, we introduce the obvious notations W_2 and W.

To show Eq. (42), it suffices to prove that for all $\xi''\otimes d' \in s^*\mathcal{E}_1 \otimes_{s^*D} \hat{\mathcal{D}}$,

$$T_{W_1(\xi''_1\otimes d')}(t^*F_2) - (-1)^{\partial(\xi''_1\otimes d')}\sigma(T)T_{W_1(\xi''_1\otimes d')} \in \mathcal{K}.$$

Now $T_{W_1(\xi''_1\otimes d')} = (W_1\otimes \mathrm{Id})(T_{\xi''_1\otimes d'})$. Moreover, from the invariance condition on F_2 (Definition 6.6(iv)), we get $T_{d'}\circ t^*F_2 - (-1)^{\partial d'}\sigma(F_2)\circ T_{d'} \in \mathcal{K}$. Thus it suffices to show that

$$\big((W_1\otimes \mathrm{Id})\circ T_{\xi''_1}\circ\sigma(F_2) - (-1)^{\partial\xi''_1}\sigma(T)\circ(W_1\otimes\mathrm{Id})\circ T_{\xi''_1}\big)\circ T_{d'} \in \mathcal{K}.$$

Now, recall that

$$\sigma(T) = W(s^*T\otimes \mathrm{Id}_{\hat{\mathcal{B}}})W^{-1},$$
$$\sigma(F_2) = W_2(s^*F_2\otimes \mathrm{Id}_{\hat{\mathcal{B}}})W_2^{-1}$$

with $W = (W_1\otimes \mathrm{Id}_{t^*\mathcal{E}_2})\circ(\mathrm{Id}_{s^*\mathcal{E}_1}\otimes W_2)$. Hence, we are reduced to

$$(W_1\otimes\mathrm{Id})T_{\xi''_1}W_2(s^*F_2\otimes\mathrm{Id}_{\hat{\mathcal{B}}})W_2^{-1} - (-1)^{\partial\xi''_1}W(s^*T\otimes\mathrm{Id}_{\hat{\mathcal{B}}})W^{-1}(W_1\otimes\mathrm{Id})T_{\xi''_1} \overset{?}{\in} \mathcal{K}$$

and then (multiplying on the left by $(W_1\otimes\mathrm{Id})^{-1}$) to

$$T_{\xi''_1}\circ W'_2(s^*F_2\otimes Id_{\hat{\mathcal{B}}})W_2^{-1} - (-1)^{\partial\xi''_1}(\mathrm{Id}_{s^*\mathcal{E}_1}\otimes W_2)(s^*T\otimes\mathrm{Id}_{\hat{\mathcal{B}}})(\mathrm{Id}_{s^*\mathcal{E}_1}\otimes W_2)^{-1}T_{\xi''_1} \overset{?}{\in} \mathcal{K}.$$

Now (with the abuse of notation (41)),

$$T_{\xi''_1}\circ W_2 = W_2\otimes T_{\xi''_1},$$
$$(\mathrm{Id}_{s^*\mathcal{E}_1}\otimes W_2)\circ T_{\xi''_1} = T_{\xi''_1}\circ(\mathrm{Id}_{s^*\mathcal{E}_1}\otimes W_2).$$

We are finally reduced to showing that

$$T_{\xi''_1}(s^*F_2\otimes\mathrm{Id}_{\hat{\mathcal{B}}}) - (-1)^{\partial\xi''_1}(s^*T\otimes\mathrm{Id}_{\hat{\mathcal{B}}})T_{\xi''_1} \in \mathcal{K},$$

which is true since T is a F_2-connection.

This completes the proof of Eq. (42).

Let us now show (b). Using the fact that t^*T is a t^*F_2-connection and that

$$T^*_{a'}T_{a''\otimes\xi'_1} = T_{\langle a',a''\rangle\xi'_1},$$

we get

$$T^*_{a'}T_{a''\otimes\xi'_1}t^*F_2 - (-1)^{\partial a'+\partial a''+\partial\xi'_1}(t^*T)T^*_{a'}T_{a''\otimes\xi'_1} \in \mathcal{K}.$$

Thus we need to show that

$$T^*_{a'}\sigma(T)T_{a''\otimes\xi'_1} - (-1)^{\partial(a'\otimes\xi'_1)}T^*_{a'}T_{a''\otimes\xi'_1}(t^*F_2) \in \mathcal{K}.$$

But this is immediate from Eq. (42). Thus (b) is proved. □

Theorem 6.9 enables us to construct the Kasparov product

$$KK_\Gamma(A,D)\times KK_\Gamma(D,B)\to KK_\Gamma(A,B)$$

of $[(\mathcal{E}_1,F_1)]\in KK_\Gamma(A,D)$ and $[(\mathcal{E}_2,F_2)]\in KK_\Gamma(D,B)$ by $[(\mathcal{E},F)]\in KK_\Gamma(A,B)$, where $\mathcal{E}=\mathcal{E}_1\hat{\otimes}_D\mathcal{E}_1$ and $F\in F_1\hat{\#}_\Gamma F_2$. As in the case of C^*-algebras endowed with an action of Γ in the usual sense, the product is well-defined, bilinear, homotopy-invariant, associative, covariant with respect to B and contravariant with respect to A.

More generally, there is an associative product (for $i,j\in\{0,1\}$)

$$\begin{aligned}&KK^i_\Gamma(A,B_1\hat{\otimes}_{C_0(M)}D)\times KK^j_\Gamma(D\hat{\otimes}_{C_0(M)}A_1,B)\\ &\to KK_{\Gamma,i+j}(A\hat{\otimes}_{C_0(M)}A_1,B_1\hat{\otimes}_{C_0(M)}B_1).\end{aligned}\tag{43}$$

6.5. Twisted K-theory is a KK-group

Assume that $S^1\to R\to\Gamma\rightrightarrows M$ is an S^1-central extension of Lie groupoids. Recall that the line bundle $L=R\times_{S^1}\mathbb{C}$ can be considered as a Fell bundle over the groupoid Γ, and thus the C^*-algebra $C_0(M)$ is endowed with a generalized action of Γ. Denote by A_R this C^*-algebra. Our goal is to show

PROPOSITION 6.10. – *If $\Gamma\rightrightarrows M$ is a proper Lie groupoid and M/Γ is compact, then for $i=0,1$, $KK^i_\Gamma(C_0(M),A_R)$ is isomorphic to $K^i_\alpha(\Gamma^\bullet)$, where $\alpha\in H^2(\Gamma^\bullet,\mathcal{S}^1)$ denotes the class of the extension $S^1\to R\to\Gamma\rightrightarrows M$.*

We will show a more general proposition, which can be considered as a generalization of the Green–Julg theorem.

PROPOSITION 6.11. – *Let $\Gamma\rightrightarrows M$ be a proper locally compact groupoid with a Haar system such that M/Γ is compact. Let E be a u.s.c. Fell bundle over Γ and $A=C_0(M;E)$. Then $KK_\Gamma(C_0(M),A)$ and $K_0(C^*_r(\Gamma;E))$ are isomorphic.*

Note that Proposition 6.11 implies Proposition 6.10: take $E=L$ if $i=0$ and $E=L\otimes C_0(\mathbb{R})$ if $i=1$.

Proof. – Let us construct a map $\Phi:KK_\Gamma(C_0(M),A)\to K_0(C^*_r(\Gamma;E))$. Consider $(\mathcal{E},F)\in\mathbf{E}_\Gamma(C_0(M),A)$. Recalling Theorem 4.6, we have to construct a generalized Fredholm operator $T\in\mathcal{F}^0(\Gamma,E)$. By the stabilization theorem (Proposition 5.25), we may assume that $\mathcal{E}=L^2(\Gamma;E)\otimes\mathbb{H}\oplus L^2(\Gamma;E)\otimes\mathbb{H}$ with the obvious $\mathbb{Z}_2$-grading. Then, replacing F by $\frac{1}{2}(F+F^*)$, and then by F^Γ (see notation (30)), we may assume that F is self-adjoint and Γ-invariant. Thus, F can be represented as a matrix

$$F=\begin{pmatrix}0 & T^*\\ T & 0\end{pmatrix}.$$

Put $\Phi([\mathcal{E},F])=[T]$. It is routine to check that Φ is a well-defined group homomorphism. The only slightly tricky point is to check that T is invertible modulo $\mathcal{K}_\Gamma(L^2(\Gamma;E)\otimes\mathbb{H})$. To see this, note that Condition (ii) in Definition 6.6 implies that $TT^*-\mathrm{Id}$ and $T^*T-\mathrm{Id}$ belong to $\mathcal{C}(L^2(\Gamma;E)\otimes\mathbb{H})$. By the compactness assumption on M/Γ and the fact that T is Γ-invariant, we find that $TT^*-\mathrm{Id}$ and $T^*T-\mathrm{Id}$ are in $\mathcal{K}_\Gamma(L^2(\Gamma;E)\otimes\mathbb{H})$.

We now construct a map in the other direction $\Psi\colon K_0(C^*_r(\Gamma;E))\to KK_\Gamma(C_0(M,A))$. Let $T\in\mathcal{F}^0(\Gamma,E)$. Let $B=\mathcal{L}(L^2(\Gamma;E)\otimes\mathbb{H})^\Gamma$ and $J=\mathcal{K}_\Gamma(L^2(\Gamma;E)\otimes\mathbb{H})$. By definition, T is an element in B whose image τ in B/J is invertible. Write the polar decomposition $\tau=u(\tau^*\tau)^{-1/2}$ and lift u to an element $T'\in B$. One easily proves by a standard argument that T' is homotopic to T. Therefore, replacing T by T', we may assume that T is unitary modulo J. That is, $T^*T-\mathrm{Id}$ and $TT^*-\mathrm{Id}$ belong to $\mathcal{K}_\Gamma(\mathcal{L}^2(\Gamma;E)\otimes\mathbb{H})$. Let $F=\left(\begin{smallmatrix}0 & T^*\\ T & 0\end{smallmatrix}\right)$, which acts on the $\mathbb{Z}_2$-graded Hilbert module $\mathcal{E}=L^2(\Gamma;E)\otimes\mathbb{H}\oplus L^2(\Gamma;E)\otimes\mathbb{H}$. It is not hard to check that $(\mathcal{E},F)\in\mathbf{E}_\Gamma(C_0(M),A)$. Define $\Psi([T])=[(\mathcal{E},F)]$. One can verify that Φ and Ψ are inverse from each other. □

6.6. The product $K^i_\alpha(\Gamma^\bullet)\otimes K^j_\beta(\Gamma^\bullet)\to K^{i+j}_{\alpha+\beta}(\Gamma^\bullet)$

Suppose that $S^1\to R_1\to\Gamma\rightrightarrows M$ and $S^1\to R_2\to\Gamma\rightrightarrows M$ are S^1-central extensions of a Lie groupoid Γ. Denote by α and β their classes in $H^2(\Gamma^\bullet,\mathcal{S}^1)$. Using the general Kasparov product (43), we get a product

$$KK^i_\Gamma\big(C_0(M),A_R\big)\otimes KK^j_\Gamma\big(C_0(M),A_{R'}\big)\to KK^{i+j}_\Gamma\big(C_0(M),A_{R\otimes R'}\big).$$

If in addition Γ is proper and M/Γ is compact, then by Proposition 6.10, we obtain a product

$$K^i_\alpha(\Gamma^\bullet)\otimes K^j_\beta(\Gamma^\bullet)\to K^{i+j}_{\alpha+\beta}(\Gamma^\bullet). \tag{44}$$

From the general properties of the Kasparov product [11], the product defined by Eq. (44) is associative and graded commutative, where graded commutativity comes from commutativity of the diagram

$$\begin{tikzcd} A_R\otimes_{C_0(M)}A_{R'} \arrow[r] \arrow[d, "\text{flip}"] & A_{R\otimes R'} \arrow[d, equal] \\ A_{R'}\otimes_{C_0(M)}A_R \arrow[r] & A_{R\otimes R'}. \end{tikzcd}$$

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Appendix A. Fell bundles over groupoids

In this appendix, we recall the definition and some basic properties of a Fell bundle over a groupoid (Definition A.7) and its reduced C^*-algebra.

A.1. Fields of C^*-algebras

DEFINITION A.1. – Let X be a Hausdorff topological space. A continuous (resp. upper semicontinuous) field of Banach spaces E over X consists of a family $(E_x)_{x\in X}$ of Banach spaces together with a topology on $\widetilde{E} = \coprod_{x\in X} E_x$ such that

(i) the topology on E_x induced from that on $\widetilde{E}$ is the norm-topology;
(ii) the projection $\pi : \widetilde{E} \to X$ is continuous and open;
(iii) the operations $(e, e') \in \widetilde{E} \times_X \widetilde{E} \mapsto e + e' \in \widetilde{E}$ and $(\lambda, e) \in \mathbb{C} \times \widetilde{E} \to \lambda e \in \widetilde{E}$ are continuous;
(iv) the norm $\widetilde{E} \to \mathbb{R}_+$ is continuous (resp. u.s.c.);
(v) if $\|e_i\| \to 0$ and $\pi(e_i) \to x$, then $e_i \to 0_x$;
(vi) for all $e \in E_x$ there exists a continuous section ξ such that $\xi(x) = e$.

In [25,26] only continuous fields were studied and they are called Banach bundles. We will also use that terminology. In this paper we are mainly concerned with continuous fields, but most constructions and results only require the field to be u.s.c. In particular, a field of Banach spaces can be constructed in the following way [26]:

PROPOSITION A.2. – *Let X be a topological Hausdorff space. Assume that $(E_x)_{x\in X}$ is a family of Banach spaces and $\mathbb{E}$ is a $C(X)$-module of sections of $\widetilde{E} := \coprod_{x\in X} E_x \to X$ such that*

(i) *for every $\xi \in \mathbb{E}$, the function $x \mapsto \|\xi(x)\|$ is continuous* (*resp. u.s.c.*);
(ii) *for all $x \in X$, the set $\{\xi(x) \mid \xi \in \mathbb{E}\}$ is dense in E_x.*

Then there is a unique topology on $\widetilde{E}$ making $\widetilde{E} \to X$ into a continuous (*resp. u.s.c.*) *field of Banach spaces such that elements of $\mathbb{E}$ are exactly continuous sections.*

In the same way, one defines fields of Banach algebras and fields of C^*-algebras. For instance, if $f : Y \to X$ is a continuous map between two locally compact spaces, then $C_0(Y)$ may be considered as an u.s.c. field of C^*-algebras over X with the fiber $C_0(f^{-1}(x))$ at $x \in X$. Moreover, the field is continuous if and only if f is an open map.

We will use the following conventions. Denote by $C(X;E)$, $C_0(X;E)$ and $C_c(X;E)$ the space of continuous sections, the space of continuous sections vanishing at infinity, the space of compactly supported continuous sections of the bundle $\widetilde{E} \to X$, respectively. We also use the notations $C(X, \widetilde{E})$, $C_0(X, \widetilde{E})$ and $C_c(X, \widetilde{E})$.

Let us explain how pull-backs of fields are constructed. Let $\coprod_{x\in X} E_x \to X$ be an u.s.c. (resp. continuous) field of Banach spaces over X, and let $f : Y \to X$ be a continuous map. Then the u.s.c. (resp. continuous) field f^*E is the field with the fiber $E_{f(y)}$ at $y \in Y$, and whose total space is $Y \times_X \widetilde{E}$ with the induced topology from $Y \times \widetilde{E}$. If E is determined by a $C(X)$-module of sections $\mathbb{E} \subset C(X, \widetilde{E})$ as in Proposition A.2, then f^*E is determined by $f^*\mathbb{E} = \{\xi \circ f \mid \xi \in \mathbb{E}\}$.

Recall that if X is a locally compact space, then a $C_0(X)$-algebra is a C^*-algebra A together with a *-homomorphism $C_0(X) \to Z(M(A))$ (the center of the multiplier algebra of A) such that $C_0(X)A = A$. The proposition below indicates that there is a bijection between $C_0(X)$-algebras and u.s.c. fields of C^*-algebras over X.

For any $x \in X$, by $C_x(X)$, we denote the ideal of $C_0(X)$ consisting of functions that vanish at x.

PROPOSITION A.3. – *Let X be a locally compact space, A a $C_0(X)$-algebra and*

$$A_x = A/\big(C_x(X)A\big).$$

Denote by $\pi_x : A \to A_x$ the projection. There is a unique u.s.c. field of C^*-algebras $\widetilde{A} := \coprod_{x\in X} A_x \to X$ such that the map

$$A \to C_0(X, \widetilde{A}),$$
$$a \mapsto \big(x \mapsto \pi_x(a)\big)$$

is an isomorphism of C^*-algebras.

Conversely, assume that $\widetilde{\mathcal{A}} = \coprod_{x\in X} \mathcal{A}_x \to X$ is a u.s.c. field of C^*-algebras over X, and $A = C_0(X, \widetilde{\mathcal{A}})$ is the space of continuous sections vanishing at infinity. Then A is obviously a $C_0(X)$-algebra, and the evaluation map $A \to \mathcal{A}_x$ induces a $*$-isomorphism $A_x \to \mathcal{A}_x$.

Proof. – This is immediate from [12, Proposition 2.12 a)]. □

Assume that $\Gamma \rightrightarrows X$ is a topological groupoid. Recall [44, Definition 3.3] that a Γ-action on a $C_0(X)$-algebra A is an isomorphism of $C_0(\Gamma)$-algebras $\alpha : s^*A \to t^*A$ such that $\alpha_{gh} = \alpha_g\alpha_h$ for all $(g,h) \in \Gamma^{(2)}$, where $\alpha_g : (s^*A)_g \cong A_{s(g)} \to (r^*A)_g \cong A_{t(g)}$ is the induced isomorphism.

Let $\widetilde{\mathcal{A}} = \coprod_{x\in X} \mathcal{A}_x \to X$ be an u.s.c. field of C^*-algebras. We say that the groupoid Γ acts on $\widetilde{\mathcal{A}} \to X$ if there is an isomorphism $\alpha : s^*\mathcal{A} \to t^*\mathcal{A}$ of fields of C^*-algebras over Γ such that $\alpha_{gh} = \alpha_g\alpha_h$ for all $(g,h) \in \Gamma^{(2)}$. It is clear that using the dictionary above (Proposition A.3), Γ-actions on $C_0(X)$-algebras are in bijective correspondence with Γ-actions on fields of C^*-algebras over X.

Now, let us explain how C^*-modules over a $C_0(X)$-algebra can be considered as u.s.c. fields of Banach spaces over X.

PROPOSITION A.4. – *Let $\widetilde{\mathcal{A}} = \coprod_{x\in X} \mathcal{A}_x \to X$ be an u.s.c. field of C^*-algebras over X and $A = C_0(X, \widetilde{\mathcal{A}})$. Assume that $\mathcal{E}$ is an A-Hilbert module. Let $\mathcal{E}_x := \mathcal{E} \otimes_A A_x$ and denote by $\pi_x : \mathcal{E} \to \mathcal{E}_x$ the canonical map. Then there is an unique u.s.c. field of Banach spaces $\widetilde{\mathcal{E}} := \coprod_{x\in X} \mathcal{E}_x \to X$ such that the map*

$$\mathcal{E} \to C_0(X, \widetilde{\mathcal{E}}),$$
$$\xi \mapsto \big(x \mapsto \pi_x(\xi)\big)$$

is an isomorphism. Moreover, if the field $\widetilde{\mathcal{A}} \to X$ is a continuous field, then $\widetilde{\mathcal{E}} \to X$ is a continuous field as well.

The proof, which uses Proposition A.2, is straightforward and is left to the reader. In particular, any $C_0(X)$-module is the space of continuous sections vanishing at infinity of a continuous field of Hilbert spaces.

Consider an u.s.c. field of C^*-algebras

$$\widetilde{\mathcal{A}} = \coprod_{x\in X} \mathcal{A}_x \to X.$$

Let $A = C_0(X, \widetilde{\mathcal{A}})$. Assume that $\mathcal{E}$ is an A-Hilbert module. It is simple to show that there is a unique topology on $\mathcal{L}(\widetilde{\mathcal{E}}) := \coprod_{x\in X} \mathcal{L}(\mathcal{E}_x)$ such that for every net $T_i \in \mathcal{L}(\mathcal{E}_{x_i})$ and $T \in \mathcal{L}(\mathcal{E}_x)$, T_i converges to T if and only if for every $\xi \in C(X, \widetilde{\mathcal{E}})$,

(i) $x_i \to x$;
(ii) $T_i\xi(x_i) \to T\xi(x)$; and
(iii) $T_i^*\xi(x_i) \to T^*\xi(x)$.

Then the bundle $\mathcal{L}(\widetilde{\mathcal{E}}) \to X$ satisfies all the properties of Definition A.1, except that the norm is not necessarily u.s.c. (in fact, one can show that it is *lower* semi-continuous if $\widetilde{\mathcal{E}} \to X$ is a continuous field), and the induced topology on $\mathcal{L}(\mathcal{E}_x)$ is not the norm-topology.

We say that a section $x \mapsto T_x$ of $\mathcal{L}(\widetilde{\mathcal{E}})$ is strongly continuous if for every $\xi \in C(X,\widetilde{\mathcal{A}})$, $x \mapsto T_x\xi(x)$ belongs to $C(X,\widetilde{\mathcal{A}})$, and a section $x \mapsto T_x$ is $*$-strongly continuous if both $x \mapsto T_x$ and $x \mapsto T_x^*$ are strongly continuous. It is not hard to show that a section is $*$-strongly continuous if and only if it is a continuous section of the bundle defined above. Denote by $C_b(X,\mathcal{L}(\widetilde{\mathcal{E}}))$ the space of continuous and norm-bounded sections.

PROPOSITION A.5. – *There is an isomorphism*

$$\mathcal{L}(\mathcal{E}) \to C_b\big(X,\mathcal{L}(\widetilde{\mathcal{E}})\big),$$
$$T \mapsto (x \mapsto T_x),$$

where $T_x = T \otimes_A \mathrm{Id} \in \mathcal{L}(\mathcal{E} \otimes_A A_x) = \mathcal{L}(\mathcal{E}_x)$.

Proof. – This follows directly from Proposition A.4 and the fact that $\mathcal{L}(\mathcal{E})$ is, by definition, the space of maps from $\mathcal{E}$ to $\mathcal{E}$ admitting an adjoint. □

The analogue of the above proposition for $\mathcal{K}(\mathcal{E})$ is less simple. However since we do not need it in full generality in this paper, we only consider a particular case below.

PROPOSITION A.6. – *Let* $\widetilde{\mathcal{H}} = \coprod_{x\in X}\mathcal{H}_x \to X$ *be a continuous field of Hilbert spaces, and* $\mathcal{H} = C_0(X,\widetilde{\mathcal{H}})$ *be the associated* $C_0(X)$*-Hilbert module. Then there exists a unique topology on* $\mathcal{K}(\widetilde{\mathcal{H}}) := \coprod_{x\in X}\mathcal{K}(\mathcal{H}_x)$ *such that*

(i) *the field* $\mathcal{K}(\widetilde{\mathcal{H}}) \to X$ *is a continuous field of* C^**-algebras*;

(ii) *for every* $\xi, \eta \in C_0(X,\widetilde{\mathcal{H}})$, *we have* $(x \mapsto T_{\xi(x),\eta(x)}) \in C_0(X,\mathcal{K}(\widetilde{\mathcal{H}}))$.

Moreover, the map

$$T_{\xi,\eta} \mapsto (x \mapsto T_{\xi(x),\eta(x)})$$

extends uniquely to an isomorphism of C^**-algebras* $\mathcal{K}(\mathcal{H}) \xrightarrow{\sim} C_0(X,\mathcal{K}(\widetilde{\mathcal{H}}))$.

Proof. – We sketch the proof in the case that the field is countably generated. In this case, by the stabilization theorem (either by Dixmier and Douady [22] or Kasparov [72]), we may assume that the field is trivial: $\widetilde{\mathcal{H}} \cong X \times \mathbb{H}$. It is well-known that $\mathcal{K}(\mathcal{H})$ is isomorphic to $C_0(X,\mathcal{K}(\mathbb{H}))$, where $\mathcal{K}(\mathbb{H})$ is endowed with the norm-topology. Thus it follows from Proposition A.2 that $C_0(X,\mathcal{K}(\mathbb{H}))$ is the space of continuous sections vanishing at infinity of a continuous field of C^*-algebras over X with fibers isomorphic to $\mathcal{K}(\mathbb{H})$. □

A.2. Fell bundles over groupoids: definition and first properties

DEFINITION A.7. – Let $\Gamma \rightrightarrows M$ be a locally compact groupoid and denote by $m : \Gamma^{(2)} \to \Gamma$ the multiplication map. A continuous (resp. u.s.c.) Fell bundle over Γ is a continuous (resp. u.s.c.) field of Banach spaces $(E_g)_{g\in\Gamma}$ over Γ together with an associative bilinear product $(\xi,\eta) \in E_g \times E_h \mapsto \xi\eta \in E_{gh}$, whenever $(g,h) \in \Gamma^{(2)}$, and an antilinear involution $\xi \in E_g \mapsto \xi^* \in E_{g^{-1}}$ such that for any $(g,h) \in \Gamma^{(2)}$, and $(e_1,e_2) \in E_g \times E_h$,

(i) $\|e_1e_2\| \leqslant \|e_1\|\|e_2\|$;

(ii) $(e_1e_2)^* = e_2^*e_1^*$;

(iii) $\|e_1^*e_1\| = \|e_1\|^2$;

(iv) $e_1^*e_1$ is a positive element of the C^*-algebra $E_{s(g)}$;

(v) the product $(e,e') \in m^*(\widetilde{E}) \mapsto ee' \in \widetilde{E}$, and the involution $e \in \widetilde{E} \mapsto e^*$ are continuous;

(vi) for all $(g,h) \in \Gamma^{(2)}$, the image of the product $E_g \times E_h \to E_{gh}$ spans a dense subspace of E_{gh}.

Remark A.8. – Note that (i)–(iii) imply that E_x, $x \in M$, is a C^*-algebra, so (iv) makes sense.

Continuous Fell bundles were first defined by Yamagami in [71], and were called C^*-algebras over groupoids. Since continuous Fell bundles are simply called "Fell bundles" in the literature [28,42,55], we will follow this convention. In the literature one also finds the terminology "full" Fell bundle: this refers to condition (vi). Note that $A := C_0(M;E)$, the restriction of $C_0(\Gamma;E)$ to M, is a $C_0(M)$-algebra, and $A_x = E_x$ for all $x \in M$, by Proposition A.3.

Example A.9. – Let Γ be a locally compact groupoid acting on a $C_0(M)$-algebra A, $\mathcal{A}$ the associated u.s.c. field of C^*-algebras (Proposition A.3). There is an isomorphism $\alpha : s^*\widetilde{\mathcal{A}} \to t^*\widetilde{\mathcal{A}}$ such that $\alpha_{gh} = \alpha_g \circ \alpha_h$ for all $(g,h) \in \Gamma^{(2)}$. Then $E = s^*\mathcal{A}$ is a u.s.c. Fell bundle over Γ with the product $(a,b) \in E_g \times E_h = \mathcal{A}_{s(g)} \times \mathcal{A}_{s(h)} \mapsto \alpha_{h^{-1}}(a)b \in E_{gh} = \mathcal{A}_{sh}$ and the involution $a \in E_g \mapsto \alpha_g(a^*) \in E_{g^{-1}}$.

Therefore the notion of u.s.c. Fell bundles over Γ generalizes that of actions of Γ on C^*-algebras. In fact, u.s.c. Fell bundles over Γ can be viewed as "actions of Γ on C^*-algebras by Morita equivalences" (see [55]).

Now we return to the discussion on a general u.s.c. Fell bundle E. Define an $A_{s(g)}$-valued scalar product on E_g by $\langle e, e'\rangle = e^*e'$. Then E_g becomes an $A_{s(g)}$-Hilbert module, and the left multiplication by elements of $A_{t(g)}$ defines a *-homomorphism $A_{t(g)} \to \mathcal{L}(E_g)$. In other words, E_g is an $A_{t(g)}$-$A_{s(g)}$-correspondence.

Note also that the product $E_g \times E_h \to E_{gh}$ induces an isomorphism of $A_{t(g)} - A_{s(h)}$ bimodules $E_g \otimes_{A_{s(g)}} E_h \to E_{gh}$. Indeed, to check that this map is isometric, we note that $\forall \xi_i \in E_g$, $\eta_i \in E_h$,

$$\Big\langle \sum_i \xi_i \otimes \eta_i, \sum_i \xi_i \otimes \eta_i \Big\rangle = \sum_{i,j} \big\langle \eta_i, \langle \xi_i, \xi_j\rangle \eta_j \big\rangle = \sum_{i,j} (\xi_i\eta_i)^*(\xi_j\eta_j) = \Big\langle \sum_i \xi_i\eta_i, \sum_i \xi_i\eta_i \Big\rangle.$$

The surjectivity of $E_g \times E_h \to E_{gh}$ follows from condition (vi) of Definition A.7.

The following proposition justifies the reason that we require the field to be u.s.c.:

PROPOSITION A.10. – *If E is an u.s.c. Fell bundle over the groupoid Γ, then sections of the form $(g,h) \mapsto \sum_i \xi_i(g)\eta_i(h)$, where ξ_i, $\eta_i \in C_0(\Gamma;E)$, are dense in $C_0(\Gamma^{(2)}, m^*E)$.*

To prove the proposition, we need the following:

LEMMA A.11. – *Let K and L be two compact spaces, (Ω_k) an open cover of $K \times L$. Then there exist finite covers U_i and V_j of K and L respectively such that $(U_i \times V_j)$ is a refinement of (Ω_k).*

Proof. – For every $(x,y) \in K \times L$, there exist $K_{x,y}$, $L_{x,y}$ compact and $U^1_{x,y}$, $V^1_{x,y}$ open such that $(x,y) \in \mathrm{Int}(K_{x,y}) \times \mathrm{Int}(L_{x,y}) \subset K_{x,y} \times L_{x,y} \subset U^1_{x,y} \times V^1_{x,y} \subset \Omega_k$ for some k. Let $U^2_{x,y} = K - K_{x,y}$ and $V^2_{x,y} = L - L_{x,y}$. By compactness, there exists a finite family $(x_i,y_i)_{i\in I}$ such that $\bigcup_{i\in I} \mathrm{Int}(K_{x_i,y_i}) \times \mathrm{Int}(L_{x_i,y_i})$ covers $K \times L$.

For any $\alpha = (\alpha_i)_{i\in I} \in \{1,2\}^I$, let $U^\alpha = \bigcap U^{\alpha_i}_{x_i,y_i}$ and $V^\alpha = \bigcap V^{\alpha_i}_{x_i,y_i}$. It is not hard to check that (U_α) and (V_α) are, respectively, covers of K and L that satisfy the required properties. □

Proof. – Let $\zeta \in C_0(\Gamma^{(2)}, m^*E)$. We can assume that ζ is compactly supported. There exist K and $L \subset \Gamma$ compact such that the support of ζ is in the interior of

$$KL = \big\{ gh \mid (g,h) \in K * L = (K \times L) \cap \Gamma^{(2)} \big\}.$$

By the definition of m^*E, for every $(g,h) \in K * L$ there exist $\xi^i_{g,h}$, $\eta^i_{g,h} \in C_0(\Gamma;E)$ such that

$$\Big\| \sum_i \xi^i_{g,h}(g)\eta^i_{g,h}(h) - \zeta(g,h) \Big\| < \varepsilon.$$

Since the field m^*E is u.s.c., there exists a neighborhood $\Omega_{g,h}$ of (g,h) such that

$$\left\|\sum_i \xi^i_{g,h}(g')\eta^i_{g,h}(h') - \zeta(g',h')\right\| < \varepsilon$$

for all $(g',h') \in \Omega_{g,h}$.

Now, by Lemma A.11, there exist compactly supported nonnegative continuous functions φ_k and ψ_l on Γ such that $\sum_k \varphi_k = 1$ on K, $\sum_l \psi_l = 1$ on L, $0 \leqslant \sum_k \varphi_k \leqslant 1$, $0 \leqslant \sum_l \psi_l \leqslant 1$, and $(g,h) \mapsto \varphi_k(g)\psi_l(h)$ is supported in some Ω_{g_k,h_l} or in $(K \times L) - K * L$ for all k,l.

Thus,

$$\left\|\zeta(g,h) - \sum_{k,l,i} \varphi_k(g)\psi_l(h)\xi^i_{g_k,h_l}(g)\eta^i_{g_k,h_l}(h)\right\| < \varepsilon,$$

for all $(g,h) \in K * L$. Now, choose two compact sets K' and L' whose interior contain K and L respectively. Applying the above to K' and L' instead of K and L, there exist $\xi_i, \eta_i \in E$ such that

$$\left\|\zeta(g,h) - \sum_i \xi_i(g)\eta_i(h)\right\| < \varepsilon \tag{45}$$

for $(g,h) \in K' * L'$. Replacing ξ_i by $\varphi\xi$, where $\varphi \in C_c(\Gamma)_+$ has the support $\subset \mathrm{Int}(K')$, $0 \leqslant \varphi \leqslant 1$ and $\varphi = 1$ on K, and replacing η_i by $\psi\eta_i$, where $\psi \in C_c(\Gamma)_+$ has the support $\subset \mathrm{Int}(L')$, $0 \leqslant \psi \leqslant 1$ and $\psi = 1$ on L, we may assume that Eq. (45) holds for all $(g,h) \in \Gamma^{(2)}$.

A.3. The reduced C^*-algebra

In this subsection, we recall the definition of the reduced C^*-algebra associated to an u.s.c. Fell bundle over a groupoid. See [55], or [58, Section 7.7] for the definition of the crossed-product algebra by a locally compact group, or [61, Chapter 2] for the C^*-algebra of a groupoid.

Assume that Γ is a locally compact groupoid with a Haar system, and E is an u.s.c. Fell bundle over Γ. Let $C_c(\Gamma; E)$ denote the space of compactly supported continuous sections. For $\xi, \eta \in C_c(\Gamma; E)$, define the convolution by

$$(\xi * \eta)(g) = \int_{h \in \Gamma^{t(g)}} \xi(h)\eta(h^{-1}g)\, \lambda^{t(g)}(dh)$$

and the involution by $\xi^*(g) = \xi(g^{-1})^*$.

Let us check that $\xi * \eta$ belongs to $C_c(\Gamma; E)$. By (v) in Definition A.7, $(g,h) \mapsto \xi(h)\eta(h^{-1}g)$ is the uniform limit of maps of the form $\sum_i f_i(h, h^{-1}g)\zeta_i(h(h^{-1}g))$, where $f_i \in C_c(\Gamma^{(2)})$ and $\zeta_i \in C_c(\Gamma; E)$, and hence of sums of the form $f(h)f'(g)\zeta(g)$, where f, $f' \in C_c(\Gamma)$ and $\zeta \in C_c(\Gamma; E)$. Moreover, the function $f'(g)$ can be assumed to be supported on a fixed compact subset of Γ. Now,

$$\int_{h \in \Gamma^{t(g)}} f(h)\, \lambda^{t(g)}(dh) f'(g)\zeta(g)$$

is the product of ζ by an element of $C_c(\Gamma)$, and hence belongs to $C_c(\Gamma; E)$. Therefore $\xi * \eta$ can be uniformly approximated by elements in $C_c(\Gamma; E)$.

Let $\|\xi\|_1 = \sup_{x\in M} \int_{\Gamma^x} \|\xi(g)\|\,\lambda^x(dg)$, and $\|\xi\|_I = \max(\|\xi\|_1, \|\xi^*\|_1)$. Then the completion of $C_c(\Gamma; E)$ with respect to the norm $\|\cdot\|_I$ is a Banach $*$-algebra, and is denoted by $L^1(\Gamma; E)$. Its enveloping C^*-algebra is denoted by $C^*(\Gamma; E)$, and is called the C^*-algebra of the field E.

Let $L^2(\Gamma; E)$ be the A-Hilbert module obtained by completing $C_c(\Gamma; E)$ with respect to the A-valued scalar product:

$$\langle \xi, \eta\rangle(x) = \int\limits_{g\in\Gamma_x} \big\langle \xi(g), \eta(g)\big\rangle\,\lambda_x(dg) \in A_x.$$

Then for every $\xi \in C_c(\Gamma; E)$, the map $\pi_l(\xi) : \eta \mapsto \xi * \eta$ belongs to $\mathcal{L}(L^2(\Gamma; E))$, and $\xi \mapsto \lambda(\xi)$ extends to a representation of $L^1(\Gamma; E)$, called the left regular representation. Its image

$$C_r^*(\Gamma; E) = \overline{\pi_l\big(L^1(\Gamma; E)\big)} = \overline{\pi_l\big(C_c(\Gamma; E)\big)} \subset \mathcal{L}\big(L^2(\Gamma; E)\big)$$

is called the reduced C^*-algebra of the field E.

The A-Hilbert module $L^2(\Gamma; E)$ can be considered, by Proposition A.4, as a field of Banach spaces over M with fiber $L^2(\Gamma; E) \otimes_A A_x$ at $x \in M$. Denote the total space of this bundle by $\widetilde{L}^2(\Gamma; E)$ and the fiber by $L^2(\Gamma_x; E)$. To justify our notation, let $i_x : \Gamma_x \to \Gamma$ be the inclusion. Then $L^2(\Gamma_x; E)$ is the completion of $C_c(\Gamma_x; i_x^* E)$ with respect to the A_x-valued scalar product $\langle \xi, \eta\rangle = \int_{g\in\Gamma_x} \langle \xi(g), \eta(g)\rangle\,\lambda_x(dg)$. Thus $L^2(\Gamma_x; E)$ is an A_x-Hilbert module.

The algebra of compact operators $\mathcal{K}(L^2(\Gamma; E))$ is a field of C^*-algebras over M whose total space is denoted by $\mathcal{K}(\widetilde{L}^2(\Gamma; E))$ (see Proposition A.6). Its fiber at $x \in M$ is $\mathcal{K}(L^2(\Gamma_x; E))$. If E is a continuous Fell bundle, then $\mathcal{K}(L^2(\Gamma; E))$ is a continuous field of C^*-algebras [55, pp. 76–77].

The C^*-algebra $\mathcal{K}(L^2(\Gamma; E))$ is endowed with a continuous action of Γ: for every $\gamma \in \Gamma_x^y$, the map

$$\alpha_\gamma : \mathcal{K}\big(L^2(\Gamma_x; E)\big) \xrightarrow{\sim} \mathcal{K}\big(L^2(\Gamma_y; E)\big)$$

is obtained as follows: let $R_{\gamma^{-1}}$ be the right multiplication by γ^{-1}, and let $E' = (R_{\gamma^{-1}})^*(E_{|\Gamma_y})$, i.e. $E'_g = E_{g\gamma^{-1}} \cong E_g \otimes E_{\gamma^{-1}}$. Then there is an isomorphism from $\mathcal{K}(L^2(\Gamma_x; E))$ to $\mathcal{K}(L^2(\Gamma_x; E'))$ given by $T \mapsto T \otimes 1$. However, $L^2(\Gamma_x; E')$ and $L^2(\Gamma_y; E)$ are isomorphic under the map $\xi \mapsto \eta$, where $\eta(g) = \xi(g\gamma)$. See [55, pp. 76–77] for further details.

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