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EXISTENCE OF $C^{1,1}$ CRITICAL SUB-SOLUTIONS OF THE HAMILTON–JACOBI EQUATION ON COMPACT MANIFOLDS

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ABSTRACT. – We offer a simple proof of the existence of a $C^{1,1}$ solution of the Hamilton–Jacobi equation in the context of Mather theory. We derive some dynamical consequences of this result. We also prove that the solution can be obtained strict outside of the Aubry set.

RÉSUMÉ. – Nous donnons une preuve simple de l'existence d'une sous-solution $C^{1,1}$ de l'équation de Hamilton–Jacobi dans le contexte de la théorie de Mather. Nous donnons certaines conséquences dynamiques de ce résultat. Nous montrons que la solution peut être obtenue stricte en dehors de l'ensemble d'Aubry.

Let M be a compact manifold without boundary. A function $H(x,p):T^*M\to\mathbb{R}$ is called a Tonelli Hamiltonian if it is C^2 and if, for each $x\in M$, the function $p\mapsto H(x,p)$ is convex with positive definite Hessian and superlinear on the fibre T^*_xM. Each Tonelli Hamiltonian generates a complete C^1 flow ψ_t. We consider the Hamilton–Jacobi equation

$$H(x,du_x)=c, \tag{HJ}$$

with a special emphasis on sub-solutions. A function $u:M\to\mathbb{R}$ is called a sub-solution of (HJ) if it is Lipschitz and satisfies the inequality $H(x,du_x)\leqslant c$ at almost every point. Note that this definition is equivalent to the notion of viscosity sub-solutions, see [4]. We denote by $C^{1,1}(M,\mathbb{R})$ the set of differentiable functions with Lipschitz differential. The goal of the present paper is to present a short and selfcontained proof of:

THEOREM A. – *Let H be a Tonelli Hamiltonian. If the Hamilton–Jacobi equation* (HJ) *has a sub-solution, then it has a $C^{1,1}$ sub-solution. Moreover, the set of $C^{1,1}$ sub-solutions is dense for the uniform topology in the set of sub-solutions.*

Fathi and Siconolfi recently proved the existence of a C^1 sub-solution in [5], see [8] for the non-autonomous case. Our result is optimal in the sense that examples are known where a $C^{1,1}$ sub-solution exists, but no C^2 sub-solution, see Appendix A. There exists a real number $\alpha(H)$, called the Mañé critical value in the literature, such that the equation (HJ) has sub-solutions if and only if $c\geqslant\alpha(H)$. One can prove the existence of smooth sub-solutions for $c>\alpha(H)$ by standard regularization, see [3]. As a consequence, our theorem is relevant for the sub-solutions of the critical equation $H(x,du_x)=\alpha(H)$, which are called the critical sub-solutions of (HJ).

The study of the critical Hamilton–Jacobi equation $H(x,du_x)=\alpha(H)$ is the core of Fathi's weak KAM theory.

A sub-solution u is called strict on the open set $U\subset M$ if there exists a smooth non-negative function $V:M\to\mathbb{R}$ which is positive on U and such that u is also a sub-solution of the equation $H(x,du_x)+V(x)=c$. By applying the theorem to the Hamiltonian $H+V$, we obtain:

ADDENDUM. – *If there exists a sub-solution of* (HJ) *which is strict on the open set U, then there exists a $C^{1,1}$ sub-solution which is strict on U.*

We now expose some dynamical consequences of the main result, which lead to a very short proof of the existence of invariant sets contained in Lipschitz graphs:

THEOREM B. – *There exists a unique non-empty compact set $\tilde{\mathcal{A}}(H)\subset T^*M$ with the following properties*:

(1) *$\tilde{\mathcal{A}}(H)$ is invariant for the Hamiltonian flow, and*

$$\tilde{\mathcal{A}}(H)\subset H^{-1}\big(\alpha(H)\big).$$

(2) *For each C^1 critical sub-solution u of* (*HJ*)*, we have*

$$\tilde{\mathcal{A}}(H)\subset\Gamma_u:=\big\{(x,du_x)\mid x\in M\big\}.$$

(3) *There exists a critical $C^{1,1}$ sub-solution u of* (*HJ*) *which is strict on the complement of the projection $\mathcal{A}(H)$ of $\tilde{\mathcal{A}}(H)$ onto M.*

It is an easy consequence of Theorem B that the set $\tilde{\mathcal{A}}(H)$ is a Lipschitz graph above $\mathcal{A}(H)$ and is not empty. We explain in the course of the proof of Theorem B in Section 2 that $\tilde{\mathcal{A}}(H)$ is the set usually called the Aubry set in the literature (although it was introduced by John Mather).

Let us quote explicitly the following:

COROLLARY. – *There exists a critical $C^{1,1}$ sub-solution which is strict outside of the projected Aubry set.*

We give some examples in Appendix A, which explain why $C^{1,1}$ regularity is optimal. Theorem A is proved in Section 1, with the use of some properties of semi-concave functions which are recalled in Appendix B. Theorem B is proved in Section 2.

1. The Lax–Oleinik semi-groups and sub-solutions

We prove Theorem A. It is necessary to start with more definitions. We define the Lagrangian $L:TM\to\mathbb{R}$ associated to H by the relation

$$L(x,v)=\max_{p\in T_x^*M}p(v)-H(x,p).$$

Then we define, for each $t\geqslant 0$, the function $A_t:M\times M\to\mathbb{R}$ by

$$A_t(x,y):=\min_{\gamma}\int_0^t c+L\big(\gamma(s),\dot\gamma(s)\big)\,ds$$

where the minimum is taken on the set of curves $\gamma \in C^2([0,t], M)$ which satisfy $\gamma(0) = x$ and $\gamma(t) = y$. Following Fathi, we define the Lax–Oleinik semi-groups T_t and $\breve{T}_t$ on $C^0(M,\mathbb{R})$ by

$$T_t u(x) = \min_{y\in M}\big(u(y) + A_t(y,x)\big) \quad \text{and} \quad \breve{T}_t u(x) = \max_{y\in M}\big(u(y) - A_t(x,y)\big).$$

The following useful lemma is proved in Fathi's book:

LEMMA 1. – *Given a Lipschitz function $u : M \to \mathbb{R}$, the following properties are equivalent*:
– *u is a sub-solution of* (HJ).
– *The inequality $u(y) - u(x) \leqslant A_t(x,y)$ holds for each $t > 0$ and each $(x,y) \in M \times M$.*
– *The function $[0,\infty[\, \ni t \mapsto T_t u(x)$ is non-decreasing for each $x \in M$.*
– *The function $[0,\infty[\, \ni t \mapsto \breve{T}_t u(x)$ is non-increasing for each $x \in M$.*

An important consequence is that the semi-groups T_t and $\breve{T}_t$ preserve the set of sub-solutions. Another important property of these semigroups is that, for each $t > 0$ and each continuous function u, the function $T_t u$ is semi-concave and the function $\breve{T}_t u$ semi-convex, see [1,4] and Appendix B for the definitions. Recall that a function is $C^{1,1}$ if and only if it is both semi-concave and semi-convex.

If u is a sub-solution of (HJ), then for each $s > 0$ and $t > 0$, the function $T_s\breve{T}_t u$ is a sub-solution. We shall prove that, for each fixed $t > 0$, this function is $C^{1,1}$ when s is small enough. Ludovic Rifford has pointed out to the author that this is a kind of Lasry–Lions regularization, see [7]. Since $\breve{T}_t u$ is semi-convex, Theorem A follows from the following result, which may have other applications.

PROPOSITION 2. – *Let H be a Tonelli Hamiltonian. For each semi-convex function v, the function $T_s v$ is $C^{1,1}$ for each sufficiently small $s > 0$.*

Proof. – In order to prove this proposition, it is enough to prove that the function $T_s v$ is semi-convex for small s, since we already know that it is semi-concave for all $s > 0$. This follows from two lemmas:

LEMMA 3. – *For each bounded subset $F \subset C^2(M,\mathbb{R})$ there exists a time $s_0 > 0$ such that, for each $s \in [0, s_0]$, the image $T_s(F)$ is a bounded subset of $C^2(M,\mathbb{R})$ and the following relation holds for all functions $f \in F$ and all $x \in M$*

$$T_s f\big(x(s)\big) = f(x) + \int_0^s c + L\big(x(t), \dot{x}(t)\big)\, dt, \tag{1}$$

where $x(t)$ is the curve $\pi \circ \psi_t(x, df(x))$ ($\pi : T^*M \to M$ is the projection and ψ_t is the Hamiltonian flow).

Proof. – Let us consider a C^2 function f and the graph $\Gamma_f \subset T^*M$ of its differential. This graph is a C^1 Lagrangian manifold transversal to the fibers. It is known that, for $s \geqslant 0$ small enough, the Lagrangian manifold $\psi_s(\Gamma_f)$ is the graph of a C^2 function, and that this C^2 function is $T_s f$. Then, we have (1). The maximum time s_0 such that these properties hold is uniform for families of functions which are bounded in C^2 norm (for then the associated graphs are contained in a given compact set, and are uniformly transversal to the verticals). In addition, one can choose s_0 in such a way that the set $\{T_s f, s \in [0, s_0], f \in F\}$ is bounded in the C^2 topology, (which amounts to say that the manifolds $\psi_s(\Gamma_f)$ are uniformly transversal to the fibers). □

LEMMA 4. – *Let v be a semi-convex function. Then there exists a bounded subset $F \subset C^2(M,\mathbb{R})$ and a time $s_0 > 0$ such that*

$$T_s v = \max_{f \in F} T_s f$$

for all $s \in [0, s_0]$, hence $T_s v$ is a semi-convex function for $s \geqslant 0$ small enough.

Proof. – If v is semi-convex, then there exists a bounded subset $F \subset C^2(M,\mathbb{R})$ such that $v = \max_F f$ and such that for each x and each $p \in \partial^- v(x)$ (the set of proximal sub-differentials of v at point x, see Appendix B), there exists a function $f \in F$ satisfying $(f(x), df(x)) = (v(x), p)$, see Appendix B. Let us fix from now on such a family F of functions, and consider the time s_0 associated to this family by the first lemma. Notice that

$$T_s v \geqslant \sup_{f \in F} T_s f$$

for all s, because for each $f \in F$ we have $f \leqslant v$ hence $T_s f \leqslant T_s v$. In order to prove that the equality holds for $s \in [0, s_0]$, let us fix a point $x \in M$. There exists a point y such that

$$T_s v(x) = v(y) + A_s(y,x).$$

Now let $(x(t), p(t)) : [0,s] \to T^*M$ be a Hamiltonian trajectory which is optimal for $A_s(y,x)$. We mean that $x(0) = y$, $x(s) = x$, and

$$A_s(y,x) = \int_0^s c + L\big(x(t), \dot{x}(t)\big)\, dt.$$

It is known (see [4,1]) that $-p(0)$ is then a proximal super-differential of the function $z \mapsto A_s(z,x)$ at point y. Since the function $z \mapsto u(z) + A_s(z,x)$ is minimal at y, the linear form $p(0)$ is a proximal sub-differential of the function u at point y. Let us consider a function $f \in F$ such that $(f(y), df(y)) = (u(y), p(0))$. Then we have $(x(t), p(t)) = \psi_t(y, df(y))$ and, by the first lemma,

$$T_s f(x) = T_s f\big(x(s)\big) = f(y) + \int_0^s c + L\big(x(t), \dot{x}(t)\big)\, dt = u(y) + A_s(y,x) = T_s u(x).$$

We have proved that, for each point $x \in M$, there exists a function $f \in F$ such that $T_s f(x) = T_s u(x)$. This ends the proof. □

The proof also implies:

COROLLARY 5. – *If u is a $C^{1,1}$ sub-solution, then there exists $\epsilon > 0$ such that $T_t u$ and $\breve{T}_t u$ are $C^{1,1}$ sub-solutions when $t \in [0, \epsilon]$. In addition, we have, for these values of t,*

$$\Gamma_u = \psi_t(\Gamma_{\breve{T}_t u}) = \psi_{-t}(\Gamma_{T_t u})$$

where Γ_f is the graph of the differential of f. □

2. The Aubry set

In this section, we consider only the critical case $c = \alpha(H)$, and prove Theorem B. Let us first define the projected Aubry set $\mathcal{A}(H) \subset M$. This is the set of points $x \in M$ such that $H(x, du_x) = \alpha(H)$ for each C^1 sub-solution u. A similar definition is given in [6].

LEMMA 6. – *If u_1 and u_2 are two critical C^1 sub-solutions, then $du_1 = du_2$ on $\mathcal{A}(H)$.*

Proof. – If $du_1(x) \neq du_2(x)$, then, by the strict convexity of H, the function $(u_1 + u_2)/2$ is a C^1 critical sub-solution which is strict at x. This implies that x does not belong to $\mathcal{A}(H)$. □

As a consequence, we can define in a natural way the set

$$\tilde{\mathcal{A}}(H) := \{(x, du_x) \mid x \in \mathcal{A}(H)\}$$

where u is any C^1 critical sub-solution.

LEMMA 7. – *There exists a $C^{1,1}$ critical sub-solution u which is strict outside of $\mathcal{A}(H)$.*

Proof. – By the Addendum of Theorem A, it is enough to prove that there exists a critical sub-solution which is strict outside of $\mathcal{A}(H)$. Since $C^1(M, \mathbb{R})$ is separable, the set of critical C^1 sub-solutions of (HJ) endowed with the C^1 norm is separable. As a consequence, there exists a dense sequence u_n of C^1 critical sub-solutions. The C^1 function

$$u(x) := \sum_{n=1}^{\infty} \frac{u_n(x)}{2^n}$$

is a C^1 critical sub-solution of (HJ) which is strict outside of $\mathcal{A}(H)$. Indeed, for each point $x \notin \mathcal{A}(H)$, there exists a C^1 critical sub-solution v such that $H(x, dv_x) < \alpha(H)$. Since the sequence u_n is dense for the C^1 topology, we conclude that $H(x, du_n(x)) < \alpha(H)$ for some n. The desired conclusion follows by the convexity of H. □

This proposition implies that $\mathcal{A}(H)$ is not empty. Otherwise, there would exist a critical sub-solution strict on M, which is a contradiction.

PROPOSITION 8. – *The set $\tilde{\mathcal{A}}(H)$ is invariant.*

Proof. – Let us choose a $C^{1,1}$ critical sub-solution u which is strict outside of the projected Aubry set. We have $\tilde{\mathcal{A}}(H) = \Gamma_u \cap H^{-1}(\alpha(H))$. Let ϵ be given by Corollary 5. We claim that $\psi_t(\tilde{\mathcal{A}}(H)) = \tilde{\mathcal{A}}(H)$ for all $t \in [-\epsilon, \epsilon]$, where ψ_t is the Hamiltonian flow. This claim clearly implies the desired result. Let (x, du_x) be a point of $\tilde{\mathcal{A}}(H)$ and $t \in [-\epsilon, \epsilon]$. Let us denote by (y, dv_y) the point $\psi_t(x, du_x)$, where $v := T_t u$. Since v is a critical sub-solution, we have $(y, dv_y) \in \tilde{\mathcal{A}}(H)$ provided $y \in \mathcal{A}(H)$. In order to prove this inclusion, we denote by w the function $w := \breve{T}_t u$, which is a $C^{1,1}$ critical sub-solution. Since $x \in \mathcal{A}(H)$, we have $du_x = dw_x$. This implies that $\psi_t(x, dw_x) = \psi_t(x, du_x) = (y, dv_y)$. Since $\psi_t(\Gamma_w) = \Gamma_u$, this implies that $dv_y = du_y$, and, by energy conservation, that $H(y, du_y) = \alpha(H)$. Since the sub-solution u is strict outside of $\mathcal{A}(H)$, we conclude that $y \in \mathcal{A}(H)$. □

PROPOSITION 9. – *If u is a critical sub-solution (not necessarily C^1), then $T_t u(x) = \breve{T}_t u(x) = u(x)$ for all $t \geqslant 0$ and $x \in \mathcal{A}(H)$. Therefore, if u is a critical sub-solution, there exists a $C^{1,1}$ sub-solution which coincides with u on $\mathcal{A}(H)$.*

Proof. – The second part of the statement clearly follows from the first part: just take $T_\epsilon \breve{T}_t u$, which is equal to u on $\mathcal{A}(H)$. So we have to prove the first part of the statement. It is enough to prove the statement for $C^{1,1}$ critical sub-solutions, since these sub-solutions are dense for the C^0 topology. It is also enough to prove it for $t \leqslant \epsilon$. In order to do so, we observe, in the proof above, that

$$v(y) = u(x) + \int_0^t \alpha(H) + L\big(\gamma(s), \dot\gamma(s)\big)\, ds \geqslant u(y)$$

(the last inequality holds because u is a critical sub-solution) and

$$\begin{aligned} u(x) \geqslant w(x) &= u(y) - \int_0^t \alpha(H) + L\big(\gamma(s), \dot\gamma(s)\big)\, ds \\ &\geqslant v(y) - \int_0^t \alpha(H) + L\big(\gamma(s), \dot\gamma(s)\big)\, ds = u(x). \end{aligned}$$

It follows that $u(x) = w(x) = \breve{T}_t u(x)$. As a consequence, all the inequalities involved are equalities, hence $u(y) = v(y) = T_t u(y)$. This equality can be proved at any point $y \in \mathcal{A}(H)$ by taking $x = \pi \circ \psi_{-t}(y, du_y)$, where $\pi : T^*M \to M$ is the projection. Indeed, in this case, we have $\psi_t(x, du_x) = (y, dv_y) = (y, du_y)$. □

If $(\gamma(t), p(t)) : \mathbb{R} \to T^*M$ is a Hamiltonian trajectory contained in $\tilde{\mathcal{A}}(H)$. It follows from the remarks above that the curve $\gamma(t)$ is calibrated by all critical sub-solutions u in the sense that the equality

$$u\big(\gamma(t)\big) - u\big(\gamma(s)\big) = \int_s^t \alpha(H) + L\big(\gamma(\sigma), \dot\gamma(\sigma)\big)\, d\sigma$$

holds for all $[s,t] \subset \mathbb{R}$. This implies that our definition of the Aubry set is the same as the one given in Fathi's book.

Appendix A. Examples

A.1. Mechanical Hamiltonian system

Let us consider the case

$$H(x,p) = \frac{1}{2}\|p\|_x^2 + V(x)$$

where $\|.\|_x$ is a Riemannian metric on M and V is a smooth function on M. Then it is easy to see that $\alpha(H) = \max V$, and that there exists a smooth sub-solution to (HJ): any constant function is such a sub-solution!

A.2. Non-existence of a C^2 sub-solution

Let us now specialise to $M = \mathbb{T}$, and consider the Hamiltonian

$$H_P(x,p) = \frac{1}{2}(p+P)^2 - \sin^2(\pi x)$$

depending on the real parameter P. For $P = 0$ this is a mechanical system as discussed above, and the constants are sub-solutions of (HJ). Let $X(x) : \mathbb{T} \to \mathbb{R}$ be the function such that

$X(x)=\sin(\pi x)$ for $x\in[0,1]$. Let us set

$$a=\frac{2}{\pi}=\int_{\mathbb{T}} X(x)\,dx.$$

The reader can check easily that $\alpha(H_P)=0$ for $P\in[-a,a]$. For each $P\in]-a,a[$, the equation (HJ) has smooth sub-solutions. For these values of P, the Aubry set is the fixed point $(0,-P)$.

However, for $P=a$, there is one and only one critical sub-solution of (HJ), which turns out to be a solution. It is given by the primitive of the function $X-a$. This function is $C^{1,1}$ but not C^2. Note that the Aubry set, then, is not reduced to the hyperbolic fixed point $(0,-a)$ but is the whole graph of $X-a$.

Appendix B. Semi-concave functions

We recall some useful facts on semi-concave functions, see for example [2,4] for more material. In all this section, M is a compact manifold of dimension d. It is useful to fix once and for all a finite atlas Φ of M composed of charts $\varphi: B_3\to M$, where B_r is the open ball of radius r centered at zero in $\mathbb{R}^d$. We assume that the sets $\varphi(B_1), \varphi\in\Phi$ cover M. A family F of C^2 functions is said bounded if there exists a constant $C>0$ such that

$$\left\|d^2(u\circ\varphi)_x\right\|\leqslant C$$

for all $x\in B_1,\ \varphi\in\Phi,\ u\in F$. Note that a bounded family is not required to be bounded in C^0 norm, but will automatically be bounded in C^1 norm and thus equi-Lipschitz. The notion of bounded family of functions does not depend on the atlas Φ.

A function $u: M\to\mathbb{R}$ is called semi-concave if there exists a bounded subset F_u of the set $C^2(M,\mathbb{R})$ such that

$$u=\inf_{f\in F_u} f.$$

A semi-concave function is Lipschitz. We say that the linear form $p\in T_xM$ is a proximal super-differential of the function u at point x if there exists a C^2 function f such that $f-u$ has a minimum at x and $df_x=p$. We denote by $\partial^+u(x)$ the set of proximal super-differentials of u at x. We say that a linear form $p\in T_xM$ is a K-super-differential of the function u at point x if for each chart $\varphi\in\Phi$ and each point $y\in B_2$ satisfying $\varphi(y)=x$, the inequality

$$u\circ\varphi(z)-u\circ\varphi(y)\leqslant p\circ d\varphi_y(z-y)+K\|z-y\|^2$$

holds for each $z\in B_2$. A function u on M is called K-semi-concave if it has a K-super-differential at each point. It is equivalent to require that, for each $\varphi\in\Phi$, the function

$$u\circ\varphi(y)-K\|y\|^2$$

is concave on B_2. As a consequence, if u is K-semi-concave and if p is a proximal super-differential of u at x, then p is a K-super-differential of u at x.

PROPOSITION 10. – *A function u is semi-concave if and only if there exists a number $K>0$ such that u is K-semi-concave. Then, there exists a bounded subset $F\subset C^2(M,\mathbb{R})$ such that*

$$u=\min_{f\in F} f$$

and, for each point $x \in M$ and each super-differential p of u at x, there exists a function $f \in F$ such that $(f(x), df(x)) = (u(x), p)$.

Proof. – Let us consider a smooth function $g:\mathbb{R}^d \to \mathbb{R}$ such that $0 \leqslant g \leqslant 1$, and such that $g = 0$ outside of B_2 and $g = 1$ inside B_1. Let us associate, to each chart $\varphi \in \Phi$, and each point $(x,p) \in T_xM$ satisfying $x \in \varphi(B_1)$, the function $f_{x,p,\varphi} : M \to \mathbb{R}$ defined by

$$f_{x,p,\varphi} \circ \varphi(z) := g(z)\big(u(x) + p \circ d\varphi_y(z-y) + K\|z-y\|^2\big) + \big(1-g(z)\big)\max u$$

for $z \in B_2$, where $y = \varphi^{-1}(x)$, and $f_{x,p,\varphi} = \max u$ outside of $\varphi(B_2)$. The functions $f_{x,p,\varphi}$, $\varphi \in \Phi$, $x \in \varphi(B_1)$, $p \in \partial^+ u(x)$ form a bounded subset F of $C^2(M,\mathbb{R})$. It is easy to check that $f = \min_{f\in F} f$. $\square$

A function u is called semi-convex if $-u$ is semi-concave.

PROPOSITION 11. – *A function is $C^{1,1}$ if and only if it is semi-concave and semi-convex.*

A very elementary proof of this statement is given in the book of Fathi. Another proof is given in [2], Corollary 3.3.8.

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