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ON FUNCTIONS OF RECTANGLES  
AND THEIR APPLICATION TO ANALYTIC FUNCTIONS

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1. - BESICOVITCH has recently proved the following generalizations of the OSGOOD and the RIEMANN theorems <sup>(1)</sup>.

A) If a function  $f(z)$  of a complex variable, defined in an open simply connected domain  $D$ , is known to be continuous at all points of  $D$  and to be differentiable at all points, except, possibly, at the points of a set  $E$  of finite or of enumerably infinite linear measure <sup>(2)</sup>, then  $f(z)$  is also differentiable at the points of  $E$ , and thus is holomorphic in the domain  $D$ .

B) If a function  $f(z)$  of a complex variable, defined in an open simply connected domain  $D$  is known to be bounded in the domain and to be differentiable (i. e. to have a finite derivative) at all points of the domain, except, possibly, at the points of a set  $E$  of linear measure zero, then, for every point  $a$  of  $E$ , the limit of  $f(z)$ , as  $z$  tends to  $a$  through values of  $D-E$ , exists, and the function  $f(z)$ , defined at the points of  $E$  by the values of these limits, is also differentiable at the points of  $E$  and thus is holomorphic in the domain  $D$ .

In this paper we intend to give theorems A) and B) in a more abstract form, viz. in a form of theorems on additive functions of rectangles. In the case  $\alpha=0$ , where  $\alpha$  denotes the order of length (see below), the exceptional sets considered in Theorems 5.1 and 5.2 become enumerable and we re-find the well known theorems of LEBESGUE and DE LA VALLÉE POUSSIN. In the case  $\alpha=1$  we obtain the theorems of BESICOVITCH in a slightly more general form.

2. - The *Lebesgue measure* and the *diameter* of a point set  $E$  will be denoted respectively by  $|E|$  and  $\delta(E)$ . Given an enumerable family of sets  $\mathfrak{E}=\{E_i\}$  and a number  $\alpha \geq 0$ , we shall put

$$\delta_\alpha(\mathfrak{E}) = \delta_\alpha(\{E_i\}) = \text{upper bound}_{i=1, 2, \dots} [\delta(E_i)]^\alpha$$

$$\sigma_\alpha(\mathfrak{E}) = \sigma_\alpha(\{E_i\}) = \sum_i [\delta(E_i)]^\alpha.$$

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<sup>(1)</sup> BESICOVITCH [1], Theorems 2, 1. Cf. also TONELLI [1].

<sup>(2)</sup> A set  $E$  is said to be of enumerably infinite linear measure if it can be split into an enumerable set of sets of finite linear measure.

Let  $E$  be a point set. By  $\Lambda_\alpha^{(n)}(E)$  we denote the lower bound of all numbers  $\sigma_\alpha(\mathbf{C})$ , where  $\mathbf{C}$  is an arbitrary family of circles covering  $E$  and satisfying the condition  $\delta_\alpha(\mathbf{C}) < n^{-1}$ . The limit  $\Lambda_\alpha(E) = \lim_{n \rightarrow \infty} \Lambda_\alpha^{(n)}(E)$  will be called the *length of order  $\alpha$*  of the set  $E$  <sup>(3)</sup>. In the sequel it will be generally assumed that  $0 \leq \alpha \leq 2$ . It will be readily seen that  $\Lambda_2(E) = 4\pi^{-1} |E|$ .

A set  $E$  that is the sum of a sequence of sets of finite length of order  $\alpha$  is said to be of *enumerably infinite length of order  $\alpha$*  <sup>(4)</sup>. For the sake of brevity we shall term such sets the  $B_\alpha$ -sets.  $B_0$ -sets coincide, obviously, with the enumerable ones. In the case  $\alpha=1$  the expression « of order  $\alpha$  » and the index  $\alpha$  in the above notation will be usually omitted.

3. - We shall only consider the rectangles and squares with sides parallel to the axis. A function  $F(I)$  of rectangles is said to be additive if  $F(I_1 + I_2) = F(I_1) + F(I_2)$  for any pair of adjacent rectangles. It is said to be continuous if  $F(I) \rightarrow 0$  whenever  $\delta(I) \rightarrow 0$ .

Let now  $F(I)$  be a function of rectangles and  $x$  an arbitrary point. Consider the four expressions

$$\begin{aligned} \overline{F}(x) &= \overline{\lim}_{\delta(S) \rightarrow 0} \frac{F(S)}{|S|}, & \underline{F}(x) &= \underline{\lim}_{\delta(S) \rightarrow 0} \frac{F(S)}{|S|} \\ \overline{F}_\alpha(x) &= \overline{\lim}_{\delta(S) \rightarrow 0} \frac{F(S)}{[\delta(S)]^\alpha}, & \underline{F}_\alpha(x) &= \underline{\lim}_{\delta(S) \rightarrow 0} \frac{F(S)}{[\delta(S)]^\alpha}, \end{aligned}$$

$S$  denoting an arbitrary square containing  $x$ . The numbers  $\overline{F}(x)$  and  $\underline{F}(x)$  are called respectively the *upper* and the *lower derivatives* of  $F(x)$  at the point  $x$ . When  $\overline{F}(x) = \underline{F}(x)$  we shall call this common value the *differential coefficient* of  $F(x)$  at the point  $x$  and shall denote it by  $F'(x)$ .

We shall say  $F(I)$  has the property  $(L_\alpha^+)$  in a rectangle  $I_0$  if  $\overline{F}_\alpha(x) > -\infty$  everywhere in  $I_0$ ; if, moreover,  $\overline{F}_\alpha(x) \geq 0$  everywhere in  $I_0$ , we shall say that  $F(I)$  has the property  $(l_\alpha^+)$ . The analogous properties  $(L_\alpha^-)$  and  $(l_\alpha^-)$  correspond respectively to the inequalities  $\overline{F}_\alpha(x) < +\infty$ ,  $\overline{F}_\alpha(x) \leq 0$ . Finally, if a function has both the properties  $(l_\alpha^+)$  and  $(l_\alpha^-)$  (respectively  $(L_\alpha^+)$  and  $(L_\alpha^-)$ ) it will be said to have the property  $(l_\alpha)$  (respectively  $(L_\alpha)$ ).

4. -  $\mathfrak{D}_n$  will denote the  $n$ -th *net* on the plan, i. e. the enumerable set of squares into which the plan is divided by the two systems of parallel lines

$$x = k \cdot 2^{-n}, \quad y = k \cdot 2^{-n} \quad (k = 0, \pm 1, \pm 2, \dots).$$

The squares belonging to  $\mathfrak{D}_n$  will be called *meshes* of order  $n$ .

<sup>(3)</sup> See HAUSDORFF [1]; HAHN [1], pp. 459-461.

<sup>(4)</sup> See footnote <sup>(1)</sup>.

LEMMA 4.1. - Given a set  $E$  and non negative numbers  $N, \varepsilon > 0, \alpha \leq 2$ , there exists a sequence  $\mathfrak{S} = \{S_n\}$  of meshes of order  $> N$ , which satisfy the following conditions:

(i) 
$$\sigma_\alpha(\mathfrak{S}) \leq 32[\Lambda_\alpha(E) + \varepsilon],$$

(ii) to any point  $x$  of  $E$  there corresponds an integer  $n > 0$ , such that any mesh of order  $n$  that contains <sup>(5)</sup>  $x$  belongs to  $\mathfrak{S}$ .

*Proof.* - Let  $\mathfrak{C} = \{C_i\}$  be a sequence of circles such that

(4.1) 
$$E \subset \sum_{i=1}^{\infty} C_i, \quad \delta(\mathfrak{C}) < 2^{-N-1} \quad \text{and} \quad \sigma_\alpha(\mathfrak{C}) \leq \Lambda_\alpha(E) + \varepsilon.$$

Let, for every  $i, N_i$  denote the positive integer such that

(4.2) 
$$2^{-N_i} > \delta(C_i) \geq 2^{-N_i-1}.$$

It is easily seen that there exist at most four meshes of order  $N_i$  that have points in common with  $C_i$ . Let  $\mathfrak{S}$  be the set of all meshes of orders  $N_1, N_2, \dots, N_i, \dots$  that have points in common respectively with the circles  $C_1, C_2, \dots, C_i, \dots$ . The set  $\mathfrak{S}$  obviously satisfies the condition (ii). Next, it follows from (4.1) and (4.2) that

$$\sigma_\alpha(\mathfrak{S}) \leq 4 \cdot 2^{\frac{\alpha}{2}} \sum_{i=1}^{\infty} 2^{-\alpha N_i} \leq 4 \cdot 2^{\frac{3\alpha}{2}} \sum_{i=1}^{\infty} [\delta(C_i)]^\alpha \leq 32\sigma_\alpha(\mathfrak{C}) \leq 32[\Lambda_\alpha(E) + \varepsilon]$$

and so the condition (i) is also satisfied.

5. - LEMMA 5.1. - If an additive and continuous function  $F(I)$  has the property  $(I_\alpha^+)$ , where  $0 \leq \alpha \leq 2$ , in a rectangle  $I_0$ , and if  $F(x) \geq 0$  everywhere in  $I_0$ , except, perhaps, for  $x$  belonging to a  $B_\alpha$ -set  $D \subset I_0$ , then  $F(I_0) \geq 0$ .

*Proof.* - On account of the continuity of  $F(I)$  we may assume that  $I_0$  is a mesh, say of order  $N_0$ .

Let  $\varepsilon$  be an arbitrary positive number and let  $G(I) = F(I) + \varepsilon \cdot |I|$ . Put

$$D = \sum_i D_i, \quad \text{where} \quad \Lambda_\alpha(D_i) < +\infty \quad \text{for} \quad i=1, 2, \dots$$

Let  $R_{i,n}$  denote the set of points  $x$  in  $I_0$  such that

(5.1) 
$$G(S) > -\varepsilon \cdot 2^{-i} [\Lambda_\alpha(D_i) + 1]^{-1} \cdot [\delta(S)]^\alpha$$

for any mesh  $S$  of order  $\geq n$  containing  $x$ . Since  $F(I)$  possesses the property  $(I_\alpha^+)$ , we have

$$I_0 = \sum_{n=1}^{\infty} R_{i,n} \quad \text{for any} \quad i=1, 2, \dots$$

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<sup>(5)</sup> There exist at most four meshes that have this property.

and therefore, since the sets  $R_{i,n}$  are measurable ( $B$ ) (more exactly, any of them is the product of a sequence of open sets)

$$(5.2) \quad A_\alpha(D_i) = \sum_{n=1}^{\infty} A_\alpha(D_{i,n}),$$

where  $D_{i,n} = D_i \cdot (R_{i,n} - R_{i,n-1})$  for  $n > 1$ , and  $D_{i,1} = D_i \cdot R_{i,1}$  ( $i=1, 2, \dots$ ).

Now, by Lemma 4.1, there exists for any pair of positive integers  $n, i$ , an enumerable set  $\mathfrak{S}_{i,n}$  of meshes of order  $> n$  that are contained in  $I_0$  and satisfy the following conditions:

$$(5.3) \quad \sigma_\alpha(\mathfrak{S}_{i,n}) \leq 32 [A_\alpha(D_{i,n}) + 2^{-n}],$$

(5.4) for each point  $x$  in  $D_{i,n}$  there exists an integer  $n \geq N_0$ , such that any mesh of order  $n$  containing  $x$  belongs to  $\mathfrak{S}_{i,n}$ ,

(5.5) each mesh  $S$  that belongs to  $\mathfrak{S}_{i,n}$  has common points with  $D_{i,n}$ , and, consequently, satisfies the inequality (5.1).

Let us put  $\mathfrak{S} = \sum_{i,n=1}^{\infty} \mathfrak{S}_{i,n}$ . It easily follows from (5.5), (5.3) and (5.2) that for any sequence  $\{S_k\}$  of (different) meshes of  $\mathfrak{S}$  we have the inequality

$$(5.6) \quad \sum_{k=1}^{\infty} G(S_k) \geq -\varepsilon \cdot \sum_{i=1}^{\infty} \left\{ 2^{-i} [A_\alpha(D_i) + 1]^{-1} \cdot \sum_{n=1}^{\infty} \sigma_\alpha(\mathfrak{S}_{i,n}) \right\} \geq -32\varepsilon.$$

We shall say that a rectangle has the property ( $A$ ) if it is the sum of a finite number of meshes  $S$ , each of which either belongs to  $\mathfrak{S}$  or satisfies the inequality  $G(S) > 0$ . It follows from (5.6) that

$$(5.7) \quad G(I) \geq -32\varepsilon$$

for any rectangle  $I$  having the property ( $A$ ).

We are now going to prove that  $I_0$  has the property ( $A$ ). In fact, suppose that it does not possess this property. Then, by the well known argument, a decreasing sequence  $\{I_0 = S_1, S_2, \dots, S_k, \dots\}$  of meshes can be found, so that no  $S_k$  has the property ( $A$ ). Hence, each  $S_k$  neither belongs to  $\mathfrak{S}$  nor satisfies the inequality  $G(S_k) > 0$ . However this is impossible, for, if the limiting point  $x_0$  of the sequence  $\{S_k\}$  belonged to  $D$ , it would follow from (5.4) that at least one  $S_k$  belonged to  $\mathfrak{S}$ ; if, on the contrary,  $x_0 \in I_0 - D$ , then  $G(x_0) = F(x_0) + \varepsilon > 0$  and, consequently,  $G(S_k) > 0$  for all  $k$  sufficiently large. Hence  $I_0$  has the property ( $A$ ) and therefore the inequality (5.7) holds for  $I = I_0$ . Thus

$$F(I_0) = G(I_0) - \varepsilon |I_0| \geq -(32 + |I_0|)\varepsilon,$$

and, since  $\varepsilon$  may be chosen arbitrarily small,  $F(I_0) \geq 0$ .

**THEOREM 5.1.** - *If an additive and continuous function  $F(I)$  has the property ( $I_\alpha^+$ ) ( $0 \leq \alpha < 2$ ) in a rectangle  $I_0$ , and if the inequality  $-\infty \neq \underline{F}(x) \geq \psi(x)$*

where  $\psi(x)$  is a summable function, holds everywhere in  $I_0$ , except, perhaps, on a  $B_\alpha$ -set, then, for any rectangle  $I \subset I_0$ , we have

$$F(I) \geq \int_I \psi(x) dx.$$

*Proof.* - Let  $\Psi(I)$  be a minorant <sup>(6)</sup> of  $\psi(x)$ , i. e. an additive and continuous function of rectangles such that  $+\infty \neq \overline{\Psi}(x) \leq \psi(x)$  for every  $x$  in  $I_0$ . From the inequality  $\overline{\Psi}(x) \neq +\infty$  it follows that  $\Psi(I)$  has the property  $(l_\alpha^-)$  and, therefore, the difference  $\Delta(I) = F(I) - \Psi(I)$  has the property  $(l_\alpha^+)$ . Furthermore, everywhere in  $I_0$ , except, perhaps, on a  $B_\alpha$ -set, we have the inequality

$$\underline{\Delta}(x) \geq \underline{F}(x) - \overline{\Psi}(x) \geq \underline{F}(x) - \psi(x) \geq 0.$$

Hence, by the preceding lemma,  $\Delta(I) \geq 0$ , i. e.  $F(I) \geq \Psi(I)$ . Since the last inequality holds for any minorant  $\Psi(I)$  of  $\psi(x)$ , we have

$$F(I) \geq \int_I \psi(x) dx$$

and the theorem is established.

From theorem 5.1 we obtain at once

**THEOREM 5.2.** - *If an additive and continuous function  $F(I)$  has the property  $(l_\alpha)$  ( $0 \leq \alpha < 2$ ) in a rectangle  $I_0$  and if both derivatives  $\overline{F}(x)$  and  $\underline{F}(x)$  are summable over  $I_0$  and finite everywhere in  $I_0$ , with the exception at most of a  $B_\alpha$ -set, then  $F(I)$  is an absolutely continuous function in  $I_0$ , and, therefore*

$$F(I) = \int_I F'(x) dx$$

for any rectangle  $I \subset I_0$ .

6. - Now let  $f(z)$  be a (complex) continuous function of a complex variable. For any rectangle  $I$  consider the complex integral

$$(6.1) \quad \int_{(I)} f(z) dz = U(I) + iV(I)$$

taken along the boundary  $(I)$  of  $I$  in the positive sense. The real and imaginary parts,  $U(I)$  and  $V(I)$ , of this integral are both additive and continuous functions of  $I$ , and  $f(z)$  being continuous, they satisfy the condition  $(l_1)$ . Next, it is easily seen that  $U'(z) = V'(z) = 0$  at any point  $z$  at which  $f(z)$  has a differential coefficient. Moreover, the derivatives  $\overline{U}(z)$ ,  $\underline{U}(z)$ ,  $\overline{V}(z)$ ,  $\underline{V}(z)$  are finite, whenever

$$\overline{\lim}_{h \rightarrow 0} \left| \frac{f(z+h) - f(z)}{h} \right| < +\infty.$$

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<sup>(6)</sup> See DE LA VALLÉE-POUSSIN [1], pp. 74-76.

Hence, using the MORERA theorem, we deduce from Theorem 5.2 that:

*If a complex continuous function  $f(z)$  is differentiable almost everywhere in an open region  $R$  and if  $\overline{\lim}_{h \rightarrow 0} \left| \frac{f(z+h) - f(z)}{h} \right| < \infty$  everywhere, except, perhaps, on a set of enumerably infinite length, then  $f(z)$  is holomorphic in  $R$ .*

7. - Lemma 5.1 as well as Theorems 5.2 and 5.3 hold true if the conditions  $(l_a^+)$  and  $(l_a)$  are replaced respectively by  $(L_a^+)$  and  $(L_a)$ , provided that the exceptional  $B_a$ -sets are simultaneously replaced by *sets of length zero of order  $a$* . The proofs become even simpler. Consequently by the argument similar to that used in § 6, we get the second theorem of BESICOVITCH generalized as follows:

*If a complex function  $f(z)$ , bounded in an open region  $R$ , is differentiable almost everywhere in  $R$ , and if  $\overline{\lim}_{h \rightarrow 0} \left| \frac{f(z+h) - f(z)}{h} \right| < \infty$  everywhere in  $R$ , with the possible exception of a set of length zero, then  $f(z)$  is equivalent (i. e. almost everywhere equal) to a function holomorphic in  $R$  (<sup>7</sup>).*

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(<sup>7</sup>) It should be noticed that from the hypothesis of the theorem it follows that  $f(z)$  is measurable on any straight line in  $R$ , and consequently the complex integral (6.2) may be considered in the LEBESGUE sense. We also need the following analogue of the MORERA theorem: *if the complex integral (6.1) of a bounded and measurable function vanishes for any rectangle  $I$ , then  $f(z)$  is equivalent to a holomorphic function*. This follows at once from the MORERA theorem by the well known argument of integral means.

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