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NOTE ON CERTAIN FUNCTIONAL EQUATIONS, CONNECTED WITH HERMITE AND WEBER FUNCTIONS

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INTRODUCTION

The main object of the present paper is to investigate (in its *general* analytic form) the *common* solution of the triad of simultaneous equations, *viz*,

(I) $$f_{n+1}(z) - \lambda z f_n'(z) + \mu n f_{n-1}(z) = 0\,,$$

(II) $$f_n'(z) + k z f_n(z) - \lambda n f_{n-1}(z) = 0\,,$$

and (A)

$$\frac{d^2 w}{d z^2} + \left(2k - \frac{\lambda^2}{\mu}\right) z \frac{d w}{d z} + \left(n \frac{\lambda^2}{\mu} + k + \frac{k^2 \mu - k \lambda^2}{\mu} z^2\right) w = 0\,, \qquad (w \equiv f_n(z))\,,$$

it being understood that λ, μ, k are certain *fixed constants* and n is a *non-negative integral parameter*. The paper consists of four articles, of which the first deals with the inter-relations between the three equations, and the second disposes of their *most general* solution. Applications of these results to the classical functions of Hermite and Weber are outlined in Arts. 3 and 4; in this connection it has been deemed necessary to introduce Hermite and Weber's functions *of the second kind*.

We are not aware whether the main results discussed in this paper have been dealt with by any previons writer.

ART. 1. — As a preliminary to the investigation of *common* solutions of the three equations (I), (II) and (A), it is necessary to examine the interrelations among them. There are three cases to consider.

Case I. — *Firstly*, assuming that $\{f_n(z)\}$ satisfies both (I) and (II), we may eliminate $f_{n-1}(z)$ *linearly* from them so as to derive

$$(1) \qquad \lambda f_{n+1}(z) - (\lambda^2 - \mu k) z f_n(z) + \mu f'_n(z) = 0 ,$$

which can, on n being replaced by $(n-1)$, be written as

$$(2) \qquad \lambda f_n(z) - (\lambda^2 - \mu k) z f_{n-1}(z) + \mu f'_{n-1}(z) = 0 .$$

If we now eliminate $f'_{n-1}(z)$ *linearly* from (2) and the equation, obtained from (II) by differentiation, we readily obtain

$$(3) \qquad \mu f''_n(z) + k \mu z f'_n(z) + (\lambda^2 n + k \mu) f_n(z) - \lambda n (\lambda^2 - \mu k) z f_{n-1}(z) = 0.$$

If we again eliminate $f_{n-1}(z)$ *linearly* from (3) and (II), we find, after easy reductions,

$$f''_n(z) + \left(2k - \frac{\lambda^2}{\mu}\right) z f'_n(z) + \\ + \left(n \frac{\lambda^2}{\mu} + k + \frac{k^2 \mu - k \lambda^2}{\mu} z^2\right) f_n(z) = 0,$$

shewing that $w \equiv f_n(z)$ satisfies (A). Thus (I), coupled with (II), leads to (A).

Case II. — *Secondly*, assuming that $\{f_n(z)\}$ satisfies both (II) and (A), we have, on differentiating (II),

$$(4) \qquad f''_n(z) + k z f'_n(z) + k f_n(z) - \lambda n f'_{n-1}(z) = 0 .$$

Exhibiting (A) in the equivalent form :

$$f''_n(z) = \left(\frac{\lambda^2}{\mu} - 2k\right) z f'_n(z) - \left(\frac{n \lambda^2}{\mu} + k + \frac{k^2 \mu - k \lambda^2}{\mu} z^2\right) f_n(z),$$

and eliminating $f''_n(z)$ from this equation and (4), we get

$$(5) \qquad \left(\frac{\lambda^2}{\mu} - k\right) z f'_n(z) - \left(n \frac{\lambda^2}{\mu} + \frac{k^2 \mu - k \lambda^2}{\mu} z^2\right) f_n(z) - \lambda n f'_{n-1}(z) = 0.$$

If we now eliminate $f'_n(z)$ and $f'_{n-1}(z)$ linearly from (5) and the two relations

$$f'_n(z) = -k z f_n(z) + \lambda n f_{n-1}(z)$$

and

$$f'_{n-1}(z) = -k z f_{n-1}(z) + \lambda (n-1) f_{n-2}(z),$$

(which are virtually implied in (II)), we obtain, after elementary manipulations,

$$f_n(z) - \lambda z f_{n-1}(z) + \mu (n-1) f_{n-2}(z) = 0. \tag{6}$$

Since (6) coincides with (I) as soon as n is changed into $(n+1)$, it is manifest that the combination of (II) and (A) leads to (I).

Case III. — *Thirdly*, assuming that $\{f_n(z)\}$ satisfies both (I) and (A), we have, on differentiating (I) twice,

$$f'_{n+1}(z) - \lambda z f'_n(z) - \lambda f_n(z) + \mu n f'_{n-1}(z) = 0, \tag{7}$$

and

$$f''_{n+1}(z) - \lambda z f''_n(z) - 2\lambda f'_n(z) + \mu n f''_{n-1}(z) = 0. \tag{8}$$

Adding (8) to (7), multiplied by $\left(2k - \frac{\lambda^2}{\mu}\right) z$ and then to (I), multiplied by

$$\frac{(n+1)\lambda^2}{\mu} + k + \frac{k^2 \mu - k \lambda^2}{\mu} z^2,$$

and attending to the three differential equations, inherent in (A), *viz.*,

$$f''_\nu(z) + \left(2k - \frac{\lambda^2}{\mu}\right) z f'_\nu(z) +$$
$$+ \left(\frac{\nu \lambda^2}{\mu} + k + \frac{k^2 \mu - k \lambda^2}{\mu} z^2\right) f_\nu(z) = 0, \qquad (\nu = n+1, n, n-1),$$

we find, without much difficulty,

$$f'_n(z) + k z f_n(z) - n \lambda f_{n-1}(z) = 0.$$

Thus the two relations (I) and (A) ultimately lead to (II).

Summarising the results of the three cases, we can assert that *the three equations (I), (II) and (A) virtually count as two effective equations, any one of them following as a matter of course, from the combination of the other two.*

ART. 2. — Plainly the difference-equation (I) being linear and homogeneous and of the *second* order, its general solution must be expressible in the form:

(9) $$f_n(z) = \alpha_n(z)\, g_n(z) + \beta_n(z)\, h_n(z),$$

where $\alpha_n(z)$ and $\beta_n(z)$ are two linearly independent particular solutions, and $g_n(z)$ and $h_n(z)$ are two *arbitrary* functions of z, which are periodic in n *with unit period.* As n is here restricted to be an integer ≥ 0, it is manifest that $g_n(z)$ and $h_n(z)$ are practically *independent of* n and are as such representable simply as $g(z)$ and $h(z)$. So the *general* solution of (I) can be presented in the form:

(10) $$f_n(z) = \alpha_n(z)\, g(z) + \beta_n(z)\, h(z),$$

where $g(z)$ and $h(z)$ are *arbitrary* functions of z.

If we now suppose that $\alpha_n(z)$ and $\beta_n(z)$ are *particular* solutions not only of (I) but also of (II), it is possible to adjust the two *arbitrary* (or *disposable*) functions $g(z)$ and $h(z)$ in such a way that (10) may represent the *most general* common solution of (I) and (II).

For, if we start with (10) and impose the condition that (10) may satisfy (II), we readily obtain:

$$g(z)\{\alpha_n'(z) + k z\, \alpha_n(z) - \lambda n\, \alpha_{n-1}(z)\} +$$

$$+ h(z)\{\beta_n'(z) + k z\, \beta_n(z) - \lambda n\, \beta_{n-1}(z)\} + \alpha_n(z)\, g'(z) + \beta_n(z)\, h'(z) = 0,$$

which simplifies to

(11) $$\alpha_n(z)\, g'(z) + \beta_n(z)\, h'(z) = 0,$$

because both $\alpha_n(z)$ and $\beta_n(z)$ satisfy (II).

Now if (11) is to hold for *every* value of z and for *every* non-negative integer n, it is easy to see in the first place that the vanishing of any one of the two functions $g'(z)$ and $h'(z)$ entails that of the other.

For, if $g'(z) \equiv 0$ (identically), (11) gives

$$\beta_n(z)\, h'(z) \equiv 0,$$

shewing that $h'(z) \equiv 0$, (for $\beta_n(z) \neq 0$).

Thus there are two possible cases to consider:

Case i. — $g'(z)$ and $h'(z)$ each $\equiv 0$.

Case ii. — $g'(z) \neq 0$ and $h'(z) \neq 0$.

For obvions reasons, Case *ii* is untenable; for otherwise (11) would give rise to:

$$\frac{\alpha_n(z)}{\beta_n(z)} = -\frac{h'(z)}{g'(z)} = \text{a function of } z, \text{ (being independent of } n),$$

leading ultimately to the chain of equalities:

$$\frac{\alpha_1(z)}{\beta_1(z)} = \frac{\alpha_2(z)}{\beta_2(z)} = \ldots = \frac{\alpha_n(z)}{\beta_n(z)} = \ldots \tag{12}$$

Relations like (12) are clearly *unthinkable*, for the two *independent* particular solutions of (I), *viz* $\alpha_n(z)$ and $\beta_n(z)$ are *arbitrarily* chosen. Thus the possibility implied in Case *ii* is negatived and we have to reckon only with Case *i*, whence we derive by simple integration

$$g(z) = a \qquad \text{and} \qquad h(z) = b,$$

where a and b are *arbitrary* constants, *which are certainly independent of* n.

Hence recollecting (Art. 1) that any common solution of (I) and (II) is a solution of (A) as well, we arrive at the under-mentioned proposition:

PROP. A. — *The three equations (I), (II) and (A) are tantamount to two independent equations and the most general form of their common solution is representable in the analytic form:*

$$f_n(z) = a\,\alpha_n(z) + b\,\beta_n(z), \tag{13}$$

where $\alpha_n(z)$ *and* $\beta_n(z)$ *are two particular common solutions of (I) and (II) and* a *and* b *are two numerical constants, independent of* n.

In the two succeeding articles we shall consider two interesting applications of Prop. A.

ART. 3. — Let us now put

$$\lambda = \mu = 2 \qquad \text{and} \qquad k = 0$$

in (I), (II) and (A). Then these equations assume the respective forms:

$$f_{n+1}(z) - 2z f_n(z) + 2n f_{n-1}(z) = 0, \tag{I)'$$

$$f'_n(z) = 2n f_{n-1}(z), \tag{II)'$$

and

$$\frac{d^2 w}{dz^2} - 2z\frac{dw}{dz} + 2nw = 0, \quad (w \equiv f_n(z)). \tag{A)'$$

Recognising that (I)′, (II)′ and (A)′ are the three equations, usually associated with Hermite's polynomial $H_n(z)$, we have a confirmation of the results proved elsewhere (¹) that the three equations (I)′, (II)′ and (A)′ are equivalent to only *two* effective equations.

In order to derive the common solution of the three equations last written, we have to choose, in accordance with Prop. A, two linearly independent solutions of (I)′ and (II)′. Certainly we are entitled to set $\alpha_n(z) \equiv H_n(z)$; as for $\beta_n(z)$, we may utilise a subsidiary function, introduced by G. PALAMA (²).

To be precise, we may set

$$(14) \qquad \beta_n(z) \equiv (-1)^{\frac{3n}{2}+\frac{1+3(-1)^n}{4}} \cdot \frac{n!}{2^{\frac{1}{2}}} \cdot \sum_{m=0}^{\infty} \frac{\Gamma\left\{-m-\frac{(-1)^n}{2}\right\} z^{2m+\frac{1+(-1)^n}{2}}}{\Gamma\left\{\frac{n}{2}-m+\frac{3-(-1)^n}{4}\right\} m!}$$

$$\equiv h_n(z), \qquad \text{(say)}.$$

We propose to designate this function $h_n(z)$ as *Hermite's function of the second kind* in contra-distinction to $H_n(z)$, which will now be called *Hermite's function of the first kind* (³).

(¹) See « *Note on Hermite's function* $H^n(z)$ *and associated equations* (*functional and differential*) » by H. D. Bagchi & P. C. Chatterjee [Vide '*Journal of the Royal Asiatic Society of Bengal*, (1950)] and Copson: « Theory of Functions of a Complex Variable » (1935), P 271, Ex 32.

(²) See « *Funzioni di Laguerre di 2a specie* » by Giuseppe Palama, [Vide *Bott. Un. mat. Ital.*, III, S. 5, 72-77 (1950)]

(³) Strictly speaking, the function $h_n(z)$, as introduced by G. PALAMA (loc. cit.) is defined (for even and odd positive integral values of n) by the two equations:

$$(14)' \quad \left. \begin{aligned} h_{2k}(z) &\equiv (-1)^k \cdot 2^{2k+\frac{1}{2}} k! \, G\left(-k+\frac{1}{2}, \frac{3}{2}, z^2\right), \\ &= \frac{(-1)^{k+1}(2k)!}{2^{\frac{1}{2}}} \sum_{m=0}^{\infty} \frac{\Gamma\left(-\frac{1}{2}-m\right)}{\Gamma\left(k-m+\frac{1}{2}\right) \cdot m!} z^{2m+1}, \\ \text{and } h_{2k+1}(z) &\equiv (-1)^k \cdot k! \cdot 2^{2k+\frac{1}{2}} \cdot G\left(-k-\frac{1}{2}, \frac{1}{2}, z^2\right), \\ &= \frac{(-1)^k \cdot (2k+1)!}{2^{\frac{1}{2}}} \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}-m\right)}{\Gamma\left(k-m+\frac{3}{2}\right) \cdot m!} \cdot z^{2m}, \end{aligned} \right]$$

Evidently (14) gives rise to.

$$h'_n(z) = (-1)^{\frac{3n}{2}+\frac{1+3.(-1)^n}{4}} . \tag{15}$$
$$. \frac{n!}{2^{\frac{1}{2}}} . \sum_{m=0}^{\infty} \frac{\left\{2m + \frac{1+(-1)^n}{2}\right\} \Gamma\left\{-m - \frac{(-1)^n}{2}\right\}}{\Gamma\left\{\frac{n}{2} - m + \frac{3-(-1)^n}{4}\right\} m!} z^{2m - \frac{1-(-1)^n}{2}}$$

and
$$h_{n-1}(z) = (-1)^{\frac{3n}{2} - \frac{5+3(-1)^n}{4}} . \tag{16}$$
$$. \frac{(n-1)!}{2^{\frac{1}{2}}} . \sum_{m=0}^{\infty} \frac{\Gamma\left\{-m + \frac{(-1)^n}{2}\right\} . z^{2m + \frac{1-(-1)^n}{2}}}{\Gamma\left\{\frac{n}{2} - m + \frac{1+(-1)^n}{4}\right\} m!} .$$

Comparing the two expressions (15) and (16) in the two cases ($n =$ even) and ($n =$ odd) and making use of the familiar lemma on the Gamma-function, (*viz.* $\Gamma(p+1) = p\,\Gamma(p)$), one can without much difficulty substantiate the relation:

$$h'_n(z) = z\, n\, h_{n-1}(z),$$

shewing that $h_n(z)$ is but a particular solution of (II)$'$. Consequently $h_n(z)$ must satisfy also (I), for the three equations (I)$'$, (II)$'$ and (A)$'$ being equivalent to *two* independent equations, any common solution of (II)$'$ and (A)$'$ must be a solution of (I)$'$ as well.

Thus in the present context we may set

$$\alpha_n(z) = H_n(z) \qquad \text{and} \qquad \beta_n(z) = h_n(z)$$

in Prop. A and deduce immediately the following proposition:

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it being understood that G is the well known function of Kummer. Comparison of (14) and (14)$'$ at once reveals the fact that the function $h_n(z)$, as defined by (14)$'$, differs only by the factor $(-1)^n$ from the function $h_n(z)$, as defined by (14). Inasmuch as Palama's function (14)$'$ satisfies the differential equation (A)$'$ it follows that the same is true also of the function $h_n(z)$, as defined above by (14). The absolute necessity for introducing the multiplying factor $(-1)^n$ arose from the fact that, whereas the function $h_n(z)$, as defined by (14)$'$, satisfies *only* the differential equation (A)$'$ but *not* the functional equations (I)$'$ (II$'$), the function $h_n(z)$, as defined by (14), satisfies not only (A)$'$ bot also both (I)$'$ and (II)$'$, a fact which will be amply put in evidence in the concluding portion of Art. 3.

PROB. B. — *The three equations* (I)$'$, (II)$'$, *and* (A)$'$ *count as two independent equations and the most comprehensive form of their common solution is*

$$f_n(z) = a\, H_n(z) + b\, h_n(z)\,,$$

where $H_n(z)$ *and* $h_n(z)$ *are Hermite's functions of the first and second kinds and* a , b *are numerical constants, independent of* n.

ART. 4. — As a further application of Prop. A, let us put $\mu = \lambda = 1$ and $k = \dfrac{1}{2}$. Then the equation (I), (II) and (A) may, by a slight change of notation [4] be written as:

$$\left.\begin{aligned} &\Phi_{n+1}(x) - x\,\Phi_n(x) + n\,\Phi_{n-1}(x) = 0\,, && \text{(I)}'' \\ &\Phi_n'(x) + \frac{1}{2}\,x\,\Phi_n(x) - n\,\Phi_{n-1}(x) = 0\,, && \text{(II)}'' \\ \text{and}\quad &\frac{d^2 v}{d x^2} + \left(n + \frac{1}{2} - \frac{1}{4}\,x^2\right) v = 0\,, \qquad (v \equiv \Phi_n(x))\,. && (A)'' \end{aligned}\right\}$$

Remembering that (I)$''$, (II)$''$ and (A)$''$ are the three equations [5], associated with the parabolic cylinder function $D_n(z)$, we readily perceive that in any application of Prop. A, we are entitled to take

$$\alpha_n(z) = D_n(z)\ i\,.\,e\,.\ \alpha_n(x) = D_n(x)\,.$$

Before we choose a *second* particular solution $\beta_n(z)$, *i. e.* $\beta_n(x)$ we may remark that the two functional equations (I)$'$ and (II)$'$, associated with Hermite function $H_n(z)$ and considered already in Art. 3, are converted into the two functional equations (I)$''$ and (II)$''$ by means of the transforming scheme:

$$f_n(z) = 2^{\frac{n}{2}}\, e^{\frac{x^2}{4}}\, \Phi_n(x)\,. \tag{17}$$

Furthermore the differential equation (A)$'$ of Art 3 is carried over into (A)$''$ by the relations:

$$\left.\begin{aligned} w &= 2^{\frac{n}{2}}\, e^{\frac{x^2}{4}}\, v\,, \\ z &= \frac{x}{\sqrt{2}}\,. \end{aligned}\right\}. \tag{18}$$

[4] The reason for the change of notation will be obvious from the latter part of Art. 4.

[5] See « Note on Weber function $D_n(z)$ and its associated equations (functional and differential) » by H. D BAGCHI & P. C. CHATTERJI [Vide Annali della R. Scuola Normale Superiore Palazzo dei Cava, (1951) (in the press. (Pisa, Italy)].

The relation (17) states distinctly that to *every* common solution $\{ f_n(z) \}$ of the triad of simultaneous equations (I),′ (II)′ and (A)′, there answers a *uniquely determinate* common solution $\{ \Phi_n(x) \}$ of the triad of equations (I)″, (II)″ and (A)″. In point of fact, we have it on the authority of Bailey (6) that the particular solution $\Phi_n(x)$ of (I)″, (II)″ and (A)″, which corresponds to the solution:

$$f_n(z) = H_n(z)$$

of (I)′, (II)′, (A)′ is that given by .

$$\Phi_n(x) = D_n(x) .$$

Noticing that in view of (18) the relation (17) is equivalent to each of the relations:

$$\left. \begin{aligned} f_n(z) &= 2^{\frac{n}{2}} e^{\frac{z^2}{2}} \Phi_n\left(\sqrt{2}\, z\right), \\ \Phi_n(x) &= 2^{-\frac{n}{2}} e^{-\frac{x^2}{4}} f_n\left(\frac{x}{\sqrt{2}}\right) \end{aligned} \right\},$$

we are in a position to introduce a *second* common solution $\beta_n(x)$ of (I)″, (II)″ and (A)″ in the form:

$$\beta_n(x) = 2^{-\frac{n}{2}} e^{-\frac{x^2}{4}} h_n\left(\frac{x}{\sqrt{2}}\right) \equiv d_n(x), \quad (\text{say}), \tag{19}$$

where $h_n(x)$ is *Hermite's function of the second kind.* (Art. 2).

Let us now designate $d_n(x)$ as *Weber function of the second kind* in contrast to the ordinary function $D_n(z)$, which may now be called *Weber function of the first kind.*

Thus setting $\alpha_n(z), \beta_n(z) \equiv D_n(z), d_n(z)$ respectively and repeating the sort of reasoning employed in Art. 2, we immediately deduce from Prop. A the following subsidiary proposition:

PROP. C — *The most general type of solution, common to the three equations (I)″, (II)″ and (A)″, is given by:*

$$\Phi_n(x) = a\, D_n(x) + b\, d_n(x) ,$$

where $D_n(x)$ and $d_n(x)$ are Weber functions of the first and second kinds respectively and a,b are constants, independent of the positive integral parameter n.

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(6) See W. N. Bailey's paper in « *Journal of the London Mathematical Society* », Vol. XIII (1933), P 202.