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"NOTE ON CERTAIN REMARKABLE TYPES OF PLANE COLLINEATIONS"

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INTRODUCTION

The object of the present investigation is twofold. The *primary* object is to examine whether, for a given convex n-sided (plane) figure Γ, *viz.*, $A_1 A_2 \ldots A_n$, $(n \geq 3)$, there can exist a collineation Ξ, which shall shift the positions of the angular points $A_1, A_2, \ldots, A_n$ according to one or other of the two cyclic (1) orders:

$$[A_1 \to A_2,\ A_2 \to A_3,\ \ldots\ldots,\ A_{n-1} \to A_n,\ A_n \to A_1], \quad \ldots \quad \ldots \text{(I)}$$

$$[A_1 \to A_n,\ A_n \to A_{n-1},\ \ldots\ldots,\ A_3 \to A_2,\ A_2 \to A_1]. \quad \ldots \quad \ldots \text{(II)}$$

The *secondary* object is to characterise any of these collineations, if and when it exists. For the sake of brevity, the collineations (I) and (II) will, when existent, be symbolised respectively as:

$$[A_1 A_2 \ldots A_{n-1} A_n] \text{ and } [A_1 A_n \ldots A_3 A_2]. \quad \ldots \text{(I)}', \text{(II)}'$$

It is scarcely necessary to remark that any of these collineations (supposed existent) remains *essentially unaltered* by an arbitrary *cyclical rearrangement*

(1) For the sake of precision, the cyclic order of the n vertices, as contemplated in (I) or (I)′, will be considered *positive*, so that « the point *consecutive to* A_1 » will be understood to mean A_2 and not A_n. Similarly the point *consecutive to* A_2 will mean A_3 and *not* A_1; and so on It goes without saying that the reverse cyclic order, as indicated in (II) or (II)′, will be regarded as *negative*.

of its elements. Thus, for instance, (I)′, (assumed to exist), is representable in any of the *equivalent* forms:

$$[A_2 A_3 \ldots A_n A_1], \quad [A_3 A_4 \ldots A_1 A_2], \quad \ldots$$

Similarly, (II)′, (assumed to exist) can be alternatively exhibited as:

$$[A_n A_{n-1} \ldots A_2 A_1], \quad [A_{n-1} A_{n-2} \ldots A_1 A_n], \quad \ldots$$

Evidently the existence of any one of the two collineations (I)′, (II)′ implies that of the other and the two collineations are *inverse* to each other. For felicity of expression, a convex n-gon Γ, qualified by the condition that one — and therefore both — of the collineations (I)′ (II)′. . . exist, will be designated as « anharmonic » throughout this paper.

We mention below certain other special conventions, which have been adopted here for the purpose of providing for a compact and clear-cut phraseology:

(*i*) that a « range » of *collinear* points $A, B, C, D, \ldots$ is symbolised simply as $(A\,B\,C\,D \ldots)$ and that, in the particular case of *four* (collinear) points A, B, C and D, the cross-ratio of the points — taken in order — is symbolised as $\{A\,B\,C\,D\}$;

(*ii*) that a « pencil » of *concurrent* lines OP, OQ, OR, $OS, \ldots$ is symbolised as $(OP, OQ, OR, OS, \ldots)$ or $O.(P\,Q\,R\,S \ldots)$ and that, in the particular case of *four* (concurrent) lines OP, OQ, OR, OS, the cross-ratio of the four lines — taken in order — is symbolised as:

$$\{OP, OQ, OR, OS\} \quad \text{or} \quad O.\{P\,Q\,R\,S\};$$

(*iii*) that, in an n-gon $A_1 A_2 A_3 \ldots A_{n-1} A_n$, the pencil of $(n-1)$ (concurrent) lines, joining the vertex A_1 to the series of consecutive vertices — taken in the proper sequence — *viz.* $A_2, A_3, \ldots, A_n$, is designated simply as the A_1 — pencil and symbolised as Ω_{A_1},

i. e., $\Omega_{A_1} \equiv$ the pencil $A_1 . (A_2 A_3 A_4 \ldots A_{n-2} A_{n-1} A_n)$;

similarly, $\Omega_{A_2} \equiv$ » » $A_2 . (A_3 A_4 A_5 \ldots A_{n-1} A_n A_1)$,

$\Omega_{A_3} \equiv$ » » $A_3 . (A_4 A_5 A_6 \ldots A_n A_1 A_2)$,

. . . .

. . . .

. . . .

and $\Omega_{A_n} \equiv$ » » $A_n . (A_1 A_2 A_3 \ldots A_{n-3} A_{n-2} A_{n-1})$;

and (*iv*) that the pencils $(\Omega_{A_r}, \Omega_{A_{r+1}})$, associated with two *contiguous* (or consecutive) vertices A_r, A_{r+1} of Γ, will be said to be « contiguous » to each other.

We are not aware whether the main topic of this paper has been discussed at length by any previous writer. All the same we must frankly admit that, with a view to emphasise the salient features of the *more general* problem under discussion, it has been deemed *necessary* to touch briefly upon certain *known* lemmas, bearing on the two trivial cases $(n = 3)$ and $(n = 4)$.

ART. 1 — As is well known, the general type of plane collineation (Ξ), containing, as it does, *eight* arbitrary (or disposable) parameters in its analytic structure, can ordinarily be made to satisfy *eight* imposed conditions. Inasmuch as the assignment of a single pair of corresponding points in a plane collineation Ξ gives rise to *two* scalar conditions, the n point-to-point correspondences (implied in I or II), *viz.*,

$$(A_1 \rightarrow A_2 \, , \; A_2 \rightarrow A_3 \quad , \, . \, . \, . \, . \, . \, . \, , \; A_{n-1} \rightarrow A_n \, , \; A_n \rightarrow A_1 \, ,) \quad \ldots \qquad \ldots \text{(I)}$$

$$(A_1 \rightarrow A_n \, , \; A_n \rightarrow A_{n-1} \, , \, . \, . \, . \, . \, . \, . \, , \; A_3 \quad \rightarrow A_2 \, , \; A_2 \rightarrow A_1 \, ,) \quad \ldots \qquad \ldots \text{(II)}$$

give rise to $2n$ scalar conditions. That is to say, for a given n-gon Γ, *viz.* $A_1 A_2 \ldots A_n$ to be « anharmonic », it must be possible to choose the 8 parameters of Ξ, consistently with $2n$ imposed scalar conditions. Hence follow automatically the under — mentioned conclusions:

(*i*) that, when $n = 4$, the figure Γ — which is no other than an (*unrestricted*) quadrilateral or quadrangle — is always « anharmonic » and that consequently the two (inverse) collineations $[A_1 A_2 A_3 A_4]$ and $[A_1 A_4 A_3 A_2]$ are perfectly determinate,

(*ii*) that, when $n = 3$, the figure Γ — which is an (*unrestricted*) triangle — is « anharmonic » in a ∞^2 of ways

and (*iii*) that, when $n \geq 5$, the figure Γ can be « anharmonic », when and only when a set of special scalar conditions, numbering $2(n - 4)$, is fulfilled.

Evidently in Case (*iii*), $(n \geq 5)$, the existence of any of the collineations $[A_1 A_2 \ldots A_n]$, $[A_1 A_n \ldots A_2]$ is contingent upon the fulfilment of the aforementioned $2(n - 4)$ conditions. Thus whereas the convex rectilineal figure Γ is *unconditionally* « anharmonic », when $n = 3$ or 4, it is « *conditionally* anharmonic », when $n > 4$. Indeed when $n > 4$, the n-gon Γ of the *unrestricted* type cannot ordinarily, be « anharmonic », in fact Γ can be « anharmonic », only when $2(n - 4)$ conditions, — obviously restricting the relative positions or configurations of the vertices $A_1, A_2, \ldots, A_n$ — are satisfied.

In order to contrast the *general* case ($n \geq 5$) to the two trivial cases ($n = 3$) and ($n = 4$), we propose to hurry over the two latter cases in the first instance.

ART. 2 — When $n = 3$, Γ is a plane triangle (Δ), which is « anharmonic » in infinitely many ways. The general forms of (∞^2 of) requisite collineations of the two types (I)$'$, $\widehat{\text{(II)}}'$ can be presented as follows:

$$\text{coll}^n\ [A_1 A_2 A_3] \ldots\ldots (\varrho x' = \lambda z, \quad \varrho y' = \mu x, \quad \varrho z' = y), \qquad \ldots(1)$$

and $$\text{coll}^n\ [A_1 A_3 A_2] \ldots\ldots (\varrho x' = \lambda y, \quad \varrho y' = \mu z, \quad \varrho z' = x), \qquad \ldots(2)$$

it being understood that Δ is the triangle of reference (as shewn in Fig. 1) — so that the sides $A_2 A_3$, $A_3 A_1$, $A_1 A_2$ are respectively $x = 0$, $y = 0$, $z = 0$ — and that (x, y, z), (x', y', z') are the triads of projective (or homo-

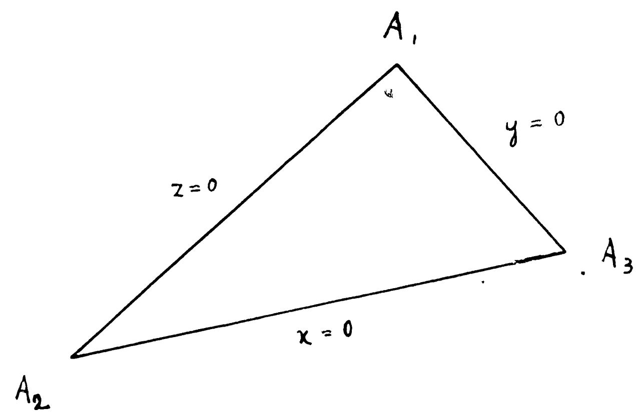

Fig. 1

geneons) coordinates of the initial and final positions of an arbitrary point, ϱ is a factor of proportionality and λ, μ are two indepenent *disposable* parameters.

Next, when $n = 4$, Γ is a plane quadrangle $A_1 A_2 A_3 A_4$, the projective coordinates of whose vertices may be taken as:

$$A_1(-\alpha, \beta, \gamma) \quad , \quad A_2(\alpha, -\beta, \gamma) \quad , \quad A_3(\alpha, \beta, -\gamma) \text{ and } A_4(\alpha, \beta, \gamma),$$

provided that the triangle, formed by the three *centres* (X, Y, Z) of the quadrangle is chosen as the triangle of reference. There is no difficulty in showing that the collineations $[A_1 A_2 A_3 A_4]$ and $[A_1 A_4 A_3 A_2]$, — which are inverse to each other — are both existent and representable analytically

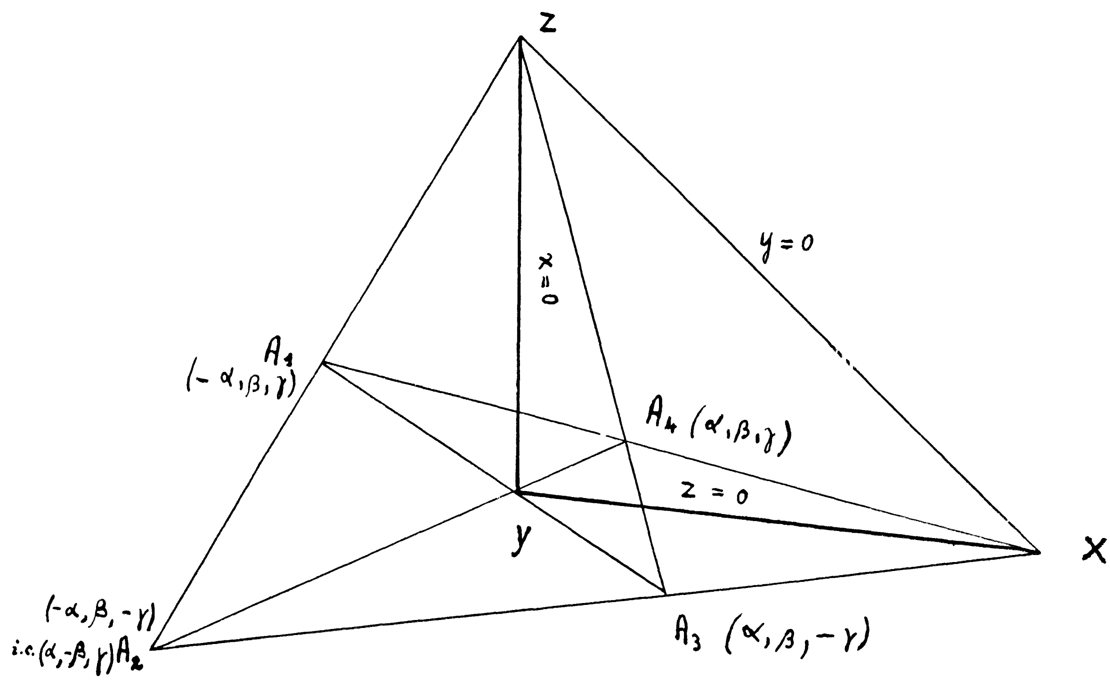

Fig. 2

by the two transforming schemes:

$$\left(\varrho\, x' = -\frac{\alpha}{\gamma}.z\,, \qquad \varrho\, y' = y\,, \qquad \varrho\, z' = \frac{\gamma}{\alpha}\, x\right), \qquad \ldots(3)$$

and

$$\left(\varrho\, x' = \frac{\alpha}{\gamma}\, z\,, \qquad \varrho\, y' = y\,, \qquad \varrho\, z' = -\frac{\gamma}{\alpha}.x\right), \qquad \ldots(4)$$

ϱ being, as usual, a factor of proportionality.

ART. 3 — When, however, $n > 4$, the figure Γ in its *unrestricted* form cannot be « anharmonic », — a fact taken notice of heretofore. As a preliminary to the discussion of the conditions under which $\Gamma(n > 4)$ can be « anharmonic », we deem it necessary to quote below — for ready reference — three simple lemmas on homography and collineation:

LEMMA 1 ($L - 1$) — In any plane collineation, a pencil of lines drawn through an arbitrary point O is trasformed into a homographic pencil (of lines), drawn through the corresponding point O'.

LEMMA 2 ($L - 2$) — In two homographies (or in the same homography), the cross-ratio of four (collinear) points or of four (concurrent) lines is the same as that of the four corresponding points or lines.

LEMMA 3 ($L - 3$) — In a plane collineation, the cross-ratio of four concurrent lines through a point O is the same as that of the four corresponding lines, passing (of course) through the corresponding point O'.

For obvious reasons $L - 3$ follows at once from $L - 1$ coupled with $L - 2$.

Let ws now proceed to determine a set of *necessary* conditions for a

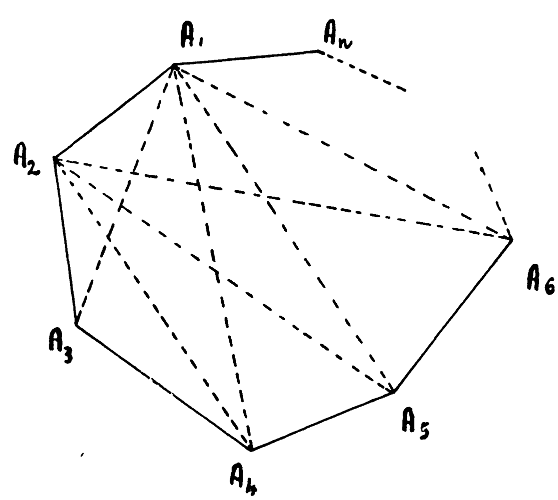

Fig. 3

convex n - gon Γ ($n \geq 5$) to be « anharmonic » (Fig. 3), so that

$$A_1 \to A_2 , A_2 \to A_3 , A_3 \to A_4 \ldots \ldots , A_{n-1} \to A_n , A_n \to A_1 .$$

Consequently we must have

$$A_1 A_2 \to A_2 A_3 , . A_1 A_3 \to A_2 A_4 , A_1 A_4 \to A_2 A_5 , \ldots$$

$$\ldots , A_1 A_{n-1} \to A_2 A_n , A_1 A_n \to A_2 A_1 .$$

This shews that the two pencils, designated as Ω_{A_1} and Ω_{A_2} in Introduction (*iii*), — *viz.*,

$$\left.\begin{aligned}\Omega_{A_1} &\equiv \text{the pencil } A_1.(A_2\, A_3\, A_4 \ldots A_{n-1}\, A_n)\\ &\equiv \text{the pencil } (A_1 A_2, A_1 A_3, A_1 A_4, \ldots, A_1 A_{n-2}, A_1 A_{n-1}, A_1 A_n)\end{aligned}\right\}\ldots(5)$$

$$\left.\begin{aligned}\& \ \Omega_{A_2} &\equiv \text{the pencil } A_2.(A_3\, A_4\, A_5 \ldots A_n\, A_1)\\ &\equiv \text{the pencil } (A_2 A_3, A_2 A_4, A_2 A_5, \ldots, A_2 A_{n-1}, A_2 A_n, A_2 A_1)\end{aligned}\right\}\ldots(6)$$

are *homographic* to each other, the lines occupying the *same* position in the two rows (5) and (6) — taken in the above sequence — corresponding to each other. For a similar reason, the pencils Ω_{A_2} and Ω_{A_3} are homographic to each other. Inasmuch as the relation of homography between different pairs of pencils obeys the « law of transitivity », it is crystal-clear that the series of pencils $\Omega_{A_1}, \Omega_{A_2}, \ldots, \Omega_{A_n}$ are all homographic to one another.

We thus arrive at the under mentioned proposition on an « anharmonic » (convex) polygon:

PROP. A. — *In an « anharmonic » n — gon Γ, viz.* $A_1 A_2 A_3 \ldots A_n$, $(n \geq 5)$, *the set of pencils:*

$$\Omega_{A_1}, \Omega_{A_2}, \Omega_{A_3}, \ldots\ldots, \Omega_{A_n}$$

are homographic to one another, tbe lines occupying the same ordinal position in any two of the pencils corresponding to one another.

It is scarcely necessary to point out that Prop. A enumerates the *necessary* conditions, whieh must be fulfilled in order that the n - gon Γ may be « anharmonic ». The *sufficiency* of the same set of conditions will be established in the succeeding articles. In disposing of the *converse* proposition we deem it expedient for the sake of clearness to go into some detail in the two special cases $(n = 5)$ and $(n = 6)$ and *then* to generalise the corresponding results.

ART. 4. — Take $n = 5$, and assume that the resulting pentagon Γ (Fig. 4) is qualified by the property that three « contiguous » pencils, say $\Omega_{A_1}, \Omega_{A_2}, \Omega_{A_3}$, defined as before by

$$\Omega_{A_1} \equiv (A_1 A_2,\ A_1 A_3,\ A_1 A_4,\ A_1 A_5), \ldots \ldots \ldots (7)$$

$$\Omega_{A_2} \equiv (A_2 A_3,\ A_2 A_4,\ A_2 A_5,\ A_2 A_1), \ldots \ldots \ldots (8)$$

and

$$\Omega_{A_3} \equiv (A_3 A_4,\ A_3 A_5,\ A_3 A_1,\ A_3 A_2), \ldots \ldots \ldots (9)$$

are homographic to one another.

Now introduce [2] a collineation Ξ by imposing the conditions:

$$A_1 \to A_2, \quad A_2 \to A_3, \quad A_3 \to A_4, \quad A_4 \to A_5 \quad \ldots \qquad \ldots (10)$$

and suppose that this Ξ converts A_5 into A_5'. (Fig. 4).

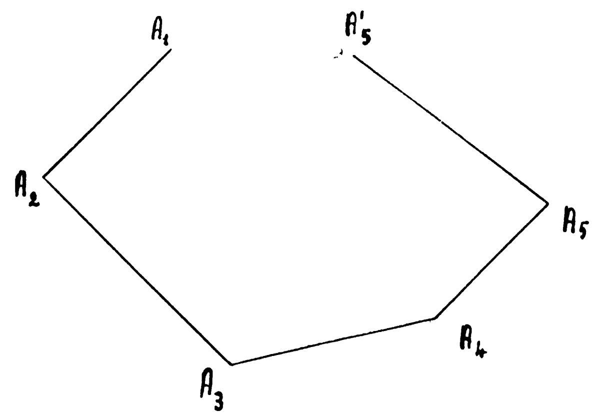

Fig. 4

Then by $L-1$ the two pencils $(A_1 A_2, A_1 A_3, A_1 A_4, A_1 A_5)$ and $(A_2 A_3, A_2 A_4, A_2 A_5, A_2 A_5')$ are homographic; so that by L—2 or $L-3$,

$$\{A_1 A_2, A_1 A_3, A_1 A_4, A_1 A_5\} = \{A_2 A_3, A_2 A_4, A_2 A_5, A_2 A_5'\}. \ldots (11)$$

Also $\because \Omega_{A_1}$ and Ω_{A_2} are homographic we have, by $L-2$,

$$\{A_1 A_2, A_1 A_3, A_1 A_4, A_1 A_5\} = \{A_2 A_3, A_2 A_4, A_2 A_5, A_2 A_1\}. \ldots (12)$$

Comparing (11) and (12), we infer that the *fourth* legs of the two right-hand-side pencils, *viz.*, $A_2 A_5'$ and $A_2 A_1$ must coincide, so that the point A_5' must, if distinct from A_1, be collinear with A_1 and A_2.

Again by $L-1$, the two pencils:

$(A_2 A_3, A_2 A_4, A_2 A_5, A_2 A_1)$ and $(A_3 A_4, A_3 A_5, A_3 A_5', A_3 A_2)$

are homographic, so that by $L-3$

$$\{A_2 A_3, A_2 A_4, A_2 A_5, A_2 A_1\} = \{A_3 A_4, A_3 A_5, A_3 A_5', A_3 A_2\}. \quad \ldots (13)$$

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(2) VIDE W. C Graustein, « *Introduction to Higher Geometry* » (1946), [Chap., X, Theorem 1 (P. 167) & Ex. 1 (P. 638)].

Also $\because$ Ω_{A_2} and Ω_{A_3} are homographic, we have, by $L-3$,

$$\{A_2 A_3, A_2 A_4, A_2 A_5, A_2 A_1\} = \{A_3 A_4, A_3 A_5, A_3 A_1, A_3 A_2\}. \quad \ldots (14)$$

Then comparing the $R.S.$ expressions of (13) and (14), we gather that the *third* legs of the two pencils, viz. $A_3 A_5'$ and $A_3 A_1$ must be identical, shewing that A_5' must, if distinct from A_1, be collinear with A_1 and A_3.

Putting this and that together, we readily conclude that A_5' must coincide with A_1, for otherwise A_5 would have to lie on a line, containing *all* the three points A_1, A_2 and A_3. Inasmuch as no three vertices can be collinear, it follows that A_5' is bound to coincide with A_1, *i. e.*, $A_5 \rightarrow A_1$. Now joining this relation to (10), we realise that our original assumption that Ω_{A_1}, Ω_{A_2}, Ω_{A_3} are homographic virtually affirms the « anharmonic » character of the pentagon Γ, so that by Prop. A the other two pencils *viz.* Ω_{A_4} and Ω_{A_5} are also homographic with them.

It is hardly necessary to observe that the final conclusion would remain precisely the same if the assumed homography of the three « contiguous » pencils, $(\Omega_{A_1}, \Omega_{A_2}, \Omega_{A_3})$ were replaced by the homography of *any other* triad of « contiguous » pencils, *e. g.*,

$$(\Omega_{A_2}, \Omega_{A_3}, \Omega_{A_4}), \qquad (\Omega_{A_3}, \Omega_{A_4}, \Omega_{A_5}), \quad \ldots .$$

The inevitable conclusion is that the mutual homography of any three « contiguous » pencils, attaching to the polygon Γ, may be regarded as the necessary and sufficient conditions for Γ to be « anharmonic ». Thus the converse of Prop. A has been established in the case of a pentagon.

We may accordingly summarise our conclusions in the form of a proposition on an « anharmonic » pentagon:

PROP. B. — *The necessary and sufficient conditions for a convex pentagon Γ, viz., $A_1 A_2 A_3 A_4 A_5$ to be « anharmonic » are that any three « contiguous » line-pencils of the pentad*

$$\Omega_{A_1}, \Omega_{A_2}, \Omega_{A_3}, \Omega_{A_4}, \Omega_{A_5} \quad \ldots \ldots \ldots \ldots (15)$$

should be mutually homographic or projective. These conditions being fulfilled, the remaining two pencils will be homographic with the first three, and at the same time the pentagon is « anharmonic » ; and the collineation $[A_1 A_2 A_3 A_4 A_5]$ or its inverse $[A_1 A_5 A_4 A_3 A_2]$ is uniquely determinate.

ART. 5 — The converse of Prop. A being verified for $n = 5$, we now proceed similarly with the case $n = 6$. To be precise, suppose that Γ is a convex hexagon *viz.*, $A_1 A_2 \ldots A_6$ (Fig. 5), for which the three « contiguous » pencils Ω_{A_1}, Ω_{A_2}, and Ω_{A_3}, *viz.,*

$$\Omega_{A_1} \equiv [A_1 A_2, \quad A_1 A_3, \quad A_1 A_4, \quad A_1 A_5, \quad A_1 A_6], \quad \dots \quad \dots (16)$$

$$\Omega_{A_2} \equiv [A_2 A_3, \quad A_2 A_4, \quad A_2 A_5, \quad A_2 A_6, \quad A_2 A_1] \quad \dots \quad \dots (17)$$

and $$\Omega_{A_3} \equiv [A_3 A_4, \quad A_3 A_5, \quad A_3 A_6, \quad A_3 A_1, \quad A_3 A_2] \quad \dots \quad \dots (18)$$

are homographic to one another.

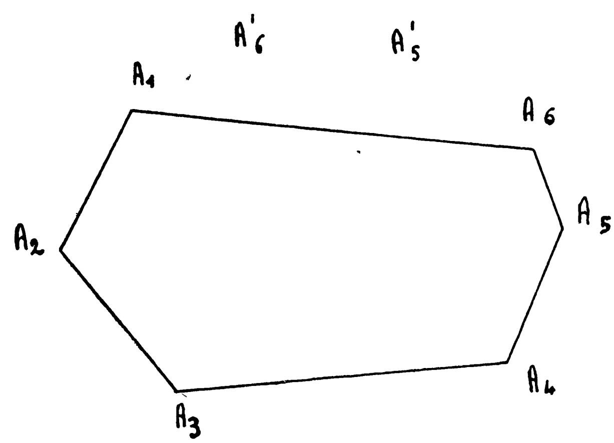

Fig. 5

As in Art. 4, define a collineation (3) Ξ by imposing the conditions:

$$A_1 \to A_2, \quad A_2 \to A_3, \quad A_3 \to A_4, \quad A_4 \to A_5. \quad \dots \quad \dots \quad \dots (10)$$

Let this Ξ convert A_5 and A_6 into A_5' and A_6' respectively, so that we may write:

$$A_5 \to A_5' \quad \text{and} \quad A_6 \to A_6'.$$

Then applying $L - 2$ to the two projective pencils of Ξ $(L - 1)$ with A_1 and A_2 as vertices, we have:

$$\{A_1 A_2, \ A_1 A_3, \ A_1 A_4, \ A_1 A_5\} = \{A_2 A_3, \ A_2 A_4, \ A_2 A_5, \ A_2 A_5'\}. \ \dots (19)$$

Also $\because$ Ω_{A_1} and Ω_{A_2} are homographic,

$$\{A_1 A_2, \ A_1 A_3, \ A_1 A_4, \ A_1 A_5\} = \{A_2 A_3, \ A_2 A_4, \ A_2 A_5, \ A_2 A_6\}. \ \dots (20)$$

(3) W. C. GRAUSTEIN, *loc. cit.*

Comparing (19) and (20), we conclude that the *fourth* legs of the two righthand-side pencils must be identical, *i. e.*,

$$A_2 A_5' \equiv A_2 A_6 . \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (21)$$

Again, applying $L-2$ to the two pencils of Ξ with A_2 and A_3 as vertices,

$$\{A_2 A_3, \; A_2 A_4, \; A_2 A_5, \; A_2 A_1\} = \{A_3 A_4, \; A_3 A_5, \; A_3 A_5', \; A_3 A_2\} \quad \dots (22)$$

Moreover Ω_{A_2} and Ω_{A_3} being homographic, we must have

$$\{A_2 A_3, \; A_2 A_4, \; A_2 A_5, \; A_2 A_1\} = \{A_3 A_4, \; A_3 A_5, \; A_3 A_6, \; A_3 A_2\} . \; \dots \; \dots (23)$$

Comparison of the R. S. of (22) and (23) shows that the two lines $A_3 A_5'$ and $A_3 A_6$ must be identical, *i. e.*,

$$A_3 A_5' \equiv A_3 A_6 . \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots (24)$$

Coupling (21) with (24), we observe that the point A_5' must either coincide with A_6 or else be collinear with the three points A_2, A_3, A_6. The second alternative is plainly untenable, for no three vertices of the *non-degenerate* polygon Γ can lie on a straight line. Hence by negation the first alternative is valid, *i. e.*,

$$A_5' \equiv A_6 . \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots (25)$$

Now applying $L-2$ to two other pencils (of Ξ) through A_1 and A_2, we have:

$$\{A_1 A_2, \; A_1 A_3, \; A_1 A_5, \; A_1 A_6\} = \{A_2 A_3, \; A_2 A_4, \; A_2 A_5', \; A_2 A_6'\}$$
$$= \{A_2 A_3, \; A_2 A_4, \; A_2 A_6, \; A_2 A_6'\}, \text{ by (25)}$$
$$= \{A_2 A_3, \; A_2 A_4, \; A_2 A_6, \; A_2 A_1\},$$

for Ω_{A_1} and Ω_{A_2} are homographic.

Hence the *fourth* legs of the last two pencils must coincide; *i. e.*,

$$A_2 A_6' \equiv A_2 A_1 . \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots (27)$$

Again applying $L-2$ to two pencils (of Ξ) through A_2 and A_3, we get:

$$\{A_2 A_3, \; A_2 A_4, \; A_2 A_6, \; A_2 A_1\} = \{A_3 A_4, \; A_3 A_5, \; A_3 A_6', \; A_3 A_2\}$$
$$= \{A_3 A_4, \; A_3 A_5, \; A_3 A_1, \; A_3 A_2\},$$

for Ω_{A_2} and Ω_{A_3} are homographic.

So the *third* legs of the last two pencils must coincide, *i. e.*,

$$A_3 A_6' = A_3 A_1 . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (28)$$

Combining (27) and (28), we readily perceive that A_6' must either coincide with A_1 or else lie on a right line, containing *all* the three points A_1, A_2, A_3. Since the second alternative is absurd for a *non-degenerate* polygon, we must have:

$$A_6' \equiv A_1 ,$$

shewing that the converse of Prop. A is true for $n = 6$. Inasmuch as the final result remains intact, whichever triad of « consecutive » pencils of the set $(\Omega_{A_1}, \Omega_{A_2}, \ldots, \Omega_{A_6})$ be assumed to be mutually homographic, our final conclusions may be summarised in the form of a substantive proposition, which reads as follows:

PROP. C. — *The necessary and sufficient conditions for a convex hexagon Γ, viz. $A_1 A_2 \ldots A_6$ to be « anharmonic » are that any three « consecutive » pencils of the hexad :*

$$(\Omega_{A_1} \Omega_{A_2}, \ldots, \Omega_{A_6})$$

should be mutually homographic or projective. In point of fact, the mutual homography of any three « consecutive » pencils « of the set induces the homography of all of them and this can happen, when and only when Γ is anharmonic ».

ART. 6 — Employing methods similar to those of Art. 5, one can without much difficulty substantiate the converse of Prop. A for the more general case $(n > 6)$. As is to be expected from general considerations, the detailed geometrical demonstration will be more and more tedious, as the number of sides (n) of Γ grows larger and larger. A moment's reflection shows that the generalised theorem admits of the following form:

PROP. D. — *A convex rectilineal figure Γ, viz. $A_1 A_2 \ldots A_n$, $(n \geq 5)$ is « anharmonic », if and only if three « contiguous » linepen-cils of the series*

$$\Omega_{A_1}, \Omega_{A_2}, \Omega_{A_3}, \ldots, \Omega_{A_n} \quad \ldots \quad \ldots (29)$$

are homographic or projective with one another. Furthermore, the mutual homography of any three « contiguous » line-pencils of set (29) *at once connotes similar relations among all the pencils (without exception) and at the same time the n-gon Γ becomes « anharmonic ».*

Regard being had to the fact that a homography between two line-pencils is determined uniquely by *three* pairs of corresponding lines, it follows that the mutual homography betwen two pencils, each consisting of m lines, $(m > 3)$, implies $(m - 3)$ scalar conditions. Because each of thè n pencils of (29) consists of only $(n - 1)$ lines, it is plain that the mutual homography of any two of them demands the fulfilment of $(\overline{n - 1} - 3)$ or $(n - 4)$ conditions. Consequently the mutual homography of three « contiguous » pencils, which have been enumerated in (29) of Prop. D as being the necessary and sufficient conditions for the n - gon Γ to be « anharmonic », must be contingent upon $2(n - 4)$ scalar conditions. This is in complete accord with the result, established from other considerations in Art. 1 (III).

When, however, $n = 5$, the number of conditions to be satisfied is $= 2(5 - 4) = 2$: and so only *two* conditions have to be fulfilled in order that a pentagon $A_1 A_2 \ldots A_5$ may be « anharmonic ». Hence remembering that the position of a point in a plane depends upon *two* parameters, we gather that, when *four* consecutive vertices (say, A_1, A_2, A_3, A_4) have pre assigned positions, the position of the fifth vertex A_5 compatible with the « anharmonic » character of the resulting pentagon, is determinate. More generally, an « anharmonic » n — gon is perfectly determinate, when only *four* consecutive vertices are given in position. Inquisitive readers may propose to frame actual geometrical construction for the $(n - 4)$ vertices $(A_5, A_6, A_7, \ldots, A_n)$ of an « anharmonic » n — gon, when the position of *four* consecutive vertices (*viz.*, A_1, A_2, A_3, A_4) are assigned beforehand.

Interested researchers may also set to themselves the task of ascertaining what modifications must be made in the final forms of Props. A, B, C, D, when the n — gon $A_1 A_2 A_3 \ldots A_n$ is *not* resricted to be convex, so that two or more of its non-adjacent sides may cross one another and one or more of the interior angles may be *re-entrant* (*i. e.*, $> 180^0$).