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SOME FORMULAS IN A RIEMANNIAN SPACE

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In this paper a number of theorems and formulas involving two arbitrary affine connections in a Riemannian space V_n have been established by imposing certain conditions on the affine connections. In section I it has been assumed that the covariant derivatives of the metric tensor of the V_n with respect to the affine connections are the same while in section 2 the torsions of the affine connections have been taken to be the same.

1. Let Γ^i_{jk} and L^i_{jk} be the coefficients of two arbitrary affine connections in a Riemannian space V_n with metric tensor g_{ij} and let a comma and a semicolon denote the covariant derivatives of the g_{ij}'s with respect to the two connections. Then

$$g_{ij,k} - g_{ij;k} = - g_{is} (\Gamma^s_{jk} - L^s_{jk}) - g_{js} (\Gamma^s_{ik} - L^s_{ik})$$

Putting $T^i_{jk} = \Gamma^i_{jk} \sim L^i_{jk}$, it follows that

$$g_{ij,k} = g_{ij;k}$$

if and only if

$$g_{is}\, T^s_{jk} + g_{js}\, T^s_{ik} = 0 \tag{I. 1}$$

Hence we have the following theorem:

THEOREM 1. The covariant derivatives of the g_{ij}'s with respect to two affine connections with coefficients Γ^i_{jk} and $\Gamma^i_{jk} + T^i_{jk}$ are the same if and only if the tensor T^i_{jk} satisfies (I. 1).

As an example it is easy to verify that the above result holds with respect to the coefficients of affine connections

$$\Gamma^t_{jk} \qquad \text{and} \qquad \Gamma^t_{jk} \pm g^{st} (g_{jk,s} - g_{ks,j})$$

This example is an application of Sen's sequence (3) which is defined as follows:

$$\text{Put } a = \Gamma_{ij}^t, \ a^* = \Gamma_{ij}^t + g^{mt} g_{im,j}, \qquad a' = \Gamma_{ji}^t$$

Then it is known that for every affine connection a there exist uniquely two others a^* and a' which are respectively called the associate and the conjugate of a having the property

$$a^{**} = a'' = a$$

In particular, a is self-associate if $a = a^*$ and self-conjugate if $a = a'$. Now if we construct the sequence

$$\text{(I. 2)} \qquad a_1 = a, a_2 = a^*, a_3 = a^{*\prime}, a_4 = a^{*\prime *}, a_5 = a^{*\prime *\prime}, \ldots\ldots$$

then the sequence is a finite cyclic sequence of twelve terms and it is Sen's sequence. In the sequence if we put

$$\alpha = g^{mt} g_{im,j}, \qquad \alpha_c = g^{mt} g_{jm,i}, \qquad \gamma = g^{mt} g_{ij,m} = \gamma_c$$
$$\beta = g^{mt} g_{is} (\Gamma_{mj}^s - \Gamma_{jm}^s), \qquad \beta_c = g^{mt} g_{js} (\Gamma_{mi}^s - \Gamma_{im}^s)$$

and suppose that a is self-conjugate, then we have

$$a_1 = a_{12} = a, \qquad a_2 = a_{11} = a + \alpha, \qquad a_3 = a_{10} = a + \alpha_c$$
$$a_4 = a_9 = a + \alpha - \gamma, \qquad a_5 = a_8 = a + \alpha_c - \gamma, \qquad a_6 = a_7 = a + \alpha + \alpha_c - \gamma$$

It follows that

$$\text{(I. 3)} \qquad a_1 - a_5 = a_2 - a_6 = g^{st} (g_{jk,s} - g_{ks,j})$$

Further, if the covariant derivatives of the $g_{ij}'s$ with respect to two affine connections are the same and if one of them is self-associate then the other is also self-associate, because the covariant derivatives of the $g_{ij}'s$ must vanish. Now when $a_1 = a_2$ is self-associate, then $a_7 = a_8$ is also self-associate. Hence we have the following theorem:

THEOREM 2. If a_1 be coefficients of a self-conjugate affine connection, then the $g_{ij}'s$ have the same covariant derivatives with respect to the pairs of affine connections (a_1, a_5) and (a_2, a_6) of Sen's sequence. And if a_1 be self-associate then the $g_{ij}'s$ have the same covariant derivatives with respect to the pair (a_1, a_7).

Further, let Γ_{ij}^{t} and $\Gamma_{ij}^{t} + T_{ij}^{t}$ be the coefficients of two affine connections. Then their associates are

$$\Gamma_{ij}^{*t} \qquad \text{and } \Gamma_{ij}^{*t} - g^{mt} g_{is} T_{mj}^{s}$$

and their conjugates are

$$\Gamma_{ji}^{t} \qquad \text{and } \Gamma_{ji}^{t} + T_{ji}^{t}$$

It follows immediately from Theorem 1 that

THEOREM 3. If the g_{ij}'s have the same covariant derivatives with respect to two arbitrary affine connections, then the same is true with respect to their associates and conjugates.

Now, let Γ_{ij}^{t} and L_{ij}^{t} be the coefficients of two affine connections and

$$\Delta_{ij}^{t} = \frac{1}{2}(\Gamma_{ij}^{t} + L_{ij}^{t}), \; T_{ij}^{t} = \Gamma_{ij}^{t} - L_{ij}^{t}$$

Also, let Γ_{ijk}^{t}, L_{ijk}^{t} and Δ_{ijk}^{t} denote curvature tensors formed with Γ_{ij}^{t}, L_{ij}^{t} and Δ_{ij}^{t} respectively.

Then it is known that

$$\text{(I. 4)} \qquad \Delta_{ijk}^{t} - \frac{1}{2}(\Gamma_{ijk}^{t} + L_{ijk}^{t}) = \frac{1}{4}(T_{sk}^{t} T_{ij}^{s} - T_{sj}^{t} T_{ik}^{s})$$

Now if the g_{ij}'s have the same covariant derivatives with respect to Γ_{ij}^{t} and L_{ij}^{t} then by (I. 1)

$$T_{sk}^{t} T_{ij}^{s} = g^{mt} g_{sp} T_{mk}^{p} g^{ns} g_{iq} T_{nj}^{q} = g^{mt} g_{iq} T_{mk}^{n} T_{nj}^{q}$$

Therefore

$$g_{ht}(T_{sk}^{t} T_{ij}^{s} - T_{sj}^{t} T_{ik}^{s}) = g_{ht} g^{mt} g_{iq}(T_{mk}^{n} T_{nj}^{q} - T_{mj}^{n} T_{nk}^{q}) = g_{it}(T_{sj}^{t} T_{hk}^{s} - T_{sk}^{t} T_{hj}^{s})$$

Or

$$\Delta_{hijk} - \frac{1}{2}(\Gamma_{hijk} + L_{hijk}) = \frac{1}{4} g_{ht}(T_{sk}^{t} T_{ij}^{s} - T_{sj}^{t} T_{ik}^{s}) = \frac{1}{4} g_{it}(T_{sj}^{t} T_{hk}^{s} - T_{sk}^{t} T_{hj}^{s})$$

Hence

$$\text{(I. 5)} \qquad \Delta_{hijk} - \frac{1}{2}(\Gamma_{hijk} + L_{hijk}) = -\left[\Delta_{ihjk} - \frac{1}{2}(\Gamma_{ihjk} + L_{ihjk})\right]$$

Thus we have the following theorem:

THEOREM 4. If the g_{ij}'s have the same covariant derivatives with respect to Γ^t_{ij} and L^t_{ij} and if $\Delta^t_{ij} = \frac{1}{2}(\Gamma^t_{ij} + L^t_{ij})$, then the curvature tensors formed with them satisfy the relation (I. 5).

This result is easily verified in the case when Γ^t_{ij} and therefore L^t_{ij}, Δ^t_{ij} are self-associate. For in this case the curvature tensors are skew in the first two indices (4).

As before, suppose that the covariant derivatives of the g_{ij}'s with respect to Γ^t_{ij} and L^t_{ij} are the same. Forming the second covariant derivatives it is seen that

$$g_{ij,kl} - g_{ij;kl} = -[g_{sj,k}\, T^s_{il} + g_{is,k}\, T^s_{jl} + g_{ij,s}\, T^s_{kl}] \tag{I. 6}$$

where

$$\Gamma^t_{ij} - L^t_{ij} = T^t_{ij} = -g^{mt} g_{i\,n} T^n_{mj}$$

Therefore

by (I. 6) $g^{ij}(g_{ij,kl} - g_{ij;kl}) = g^{ms} g_{sj,k} T^j_{ml} + g^{ms} g_{is,k} T^i_{ml} - g^{ij} g_{ij,s} T^s_{kl} =$

$$= g^{ij}[g_{sj,k}\, T^s_{il} + g_{si,k}\, T^s_{jl} + g_{ij,s}\, T^s_{kl} - 2\, g_{ij,s}\, T^s_{kl}] = -g^{ij}(g_{ij,kl} - g_{ij;kl}) - 2\, g^{ij} g_{ij,s} T^s_{kl}$$

Therefore

$$g^{ij}(g_{ij,kl} - g_{ij;kl}) = -g^{ij} g_{ij,s} T^s_{kl}$$

Interchanging k and l and subtracting

$$g^{ij}[(g_{ij,kl} - g_{ij,lk}) - (g_{ij;kl} - g_{ij;lk})] = g^{ij} g_{ij,s}(T^s_{lk} - T^s_{kl})$$

Finally using Ricci's identity

$$g^{ij}[(g_{it}\Gamma^t_{jkl} + g_{jt}\Gamma^t_{ikl}) - (g_{it} L^t_{jkl} + g_{jt} L^t_{ikl})] = g^{ij} g_{ij,s}[(\Gamma^s_{lk} - L^s_{lk}) - (\Gamma^s_{kl} - L^s_{kl})] \tag{I. 7}$$

Let us further suppose that Γ^t_{ij} and L^t_{ij} are both self-associate or both self-conjugate. Then the right hand side of (I. 7) vanishes. We have therefore

$$g^{ij}[(\Gamma_{ijkl} + \Gamma_{jikl}) - (L_{ijkl} + L_{jikl})] = 0$$

whence

$$g^{ij}(\Gamma_{jikl} - L_{ijkl}) = 0 \tag{I. 8}$$

As said before, this result is obvious when both the affine connections are self-associate.

Hence we have the following theorem:

THEOREM 5. If the g_{ij}'s have the same covariant derivatives with respect to two self-conjugate affine connections with coefficients Γ_{ij}^t, L_{ij}^t and if Γ_{ijkl}, L_{ijkl} be the corresponding covariant curvature tensors, then (I. 8) holds.

2. The torsion of an affine connection with coefficients Γ_{ij}^t is defined to be the tensor $\frac{1}{2}(\Gamma_{ij}^t - \Gamma_{ji}^t)$ (2). It follows that two arbitrary affine connections with coefficients Γ_{ij}^t and $\Gamma_{ij}^t + T_{ij}^t$ have the same torsion if and only if T_{ij}^t is symmetric in i and j. It is now easy to see that if two affine connections have the same torsion the same is true of their conjugates. E. g., in Sen's sequence each of the pairs (a_1, a_6), (a_2, a_9), (a_4, a_{11}) and therefore their conjugates (a_{12}, a_7), (a_3, a_8), (a_5, a_{10}) have the same torsion.

Again, let $a = \Gamma_{ij}^t$, $b = L_{ij}^t$ be the cocfficients of two affine connections and $a - b = T_{ij}^t$. Their associates a^* and b^* will have the same torsion if the tensor

$$T_{ij}^t + g^{mt}(g_{im,j} - g_{im;j}) = -g^{mt} g_{is} T_{mj}^s$$

is symmetric in i,j i. e., if

$$g_{is} T_{mj}^s = g_{js} T_{mi}^s \tag{2. 1}$$

Putting $g_{is} T_{mj}^s = T_{imj}$ we have the following theorem:

THEOREM 6. Let a and b have the same torsion; then their associates will also have the same torsion if the tensor T_{ijk} is symmetric in all the indices.

Let $e_0 = (a\ , b\ ) = \frac{1}{2}(a + b) + \quad (a - b)$, $\bar{e}_0 = (b, a) = \frac{1}{2}(a + b) + (b - a)$

$$e_1 = (e_0, \bar{e}_0) = \frac{1}{2}(a + b) + 2(a - b), \ \bar{e}_1 = \frac{1}{2}(a + b) + 2(b - a)$$

Similarly for e_2, $\bar{e}_2$ etc.
Then

$$e_r = \frac{1}{2}(a + b) + 2^r(a - b), \ \bar{e}_r = \frac{1}{2}(a + b) + 2^r(b - a)$$

Therefore

$$e_r - \bar{e}_r = 2^r(a - b) \tag{2. 2}$$

It follows that if a and b have the same torsion then the same is true of e_r and $\bar{e}_r$.

Further, we have the following theorem:

THEOREM 7. If the associates of a and b have the same torsion, the same is true of the associates of e_r and $\overline{e}_r$.

Let Γ_{ij}^t and L_{ij}^t have the same torsion. Then applying the condition that T_{ij}^t is symmetric in i, j, we obtain from (I. 4) the cyclical property, namely

$$(2.3)\quad \Delta_{ijk}^t - \frac{1}{2}(\Gamma_{ijk}^t + L_{ijk}^t) + \Delta_{jki}^t - \frac{1}{2}(\Gamma_{jki}^t + L_{jki}^t) + \Delta_{kij}^t - \frac{1}{2}(\Gamma_{kij}^t + L_{kij}^t) = 0$$

This result is obvious if Γ_{ij}^t and therefore L_{ij}^t, Δ_{ij}^t are self-conjugate.

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