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A remark on the regularity at the boundary for solutions of elliptic equations


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A REMARK ON THE REGULARITY AT THE BOUNDARY
FOR SOLUTIONS OF ELLIPTIC EQUATIONS

by M. K. Venkatesha Murthy (Bombay).

§ 1. Introduction.

The object of this note is to prove the following result.

Let $A$ be a linear elliptic operator (of order $2m$) with infinitely differentiable coefficients in a domain $\Omega$, having a smooth boundary $\partial\Omega$, in a euclidean space and let $B_j(0 \leq j \leq 2m - 1)$ be differential operators, with infinitely differentiable coefficients on $\partial\Omega$. If $(A, \{B_j\})$ is an admissible system (see § 2), and $f$ and $g_j$ are functions in certain classes of infinitely differentiable functions (referred to as Friedman classes in the sequel), then any function $u$ infinitely differentiable in $\overline{\Omega}$ and satisfying $Au = f$ in $\Omega$, $B_ju = g_j$ in $\partial\Omega$, is itself in a Friedman class in $\overline{\Omega}$ when the coefficients of $A$ and of $B_j$ and the functions which define the boundary $\partial\Omega$ locally, are in certain Friedman classes.

When $f$, $g_j$, and the coefficients of $A$ and of $B_j$ are real analytic functions of their arguments the real analyticity of the solution $u$ (of the system) up to the boundary has been proved, using a method of Morrey and Nirenberg [4], by Magenes and Stampacchia [3] assuming that $(A, \{B_j\})$ is an admissible system and $\partial\Omega$ is analytic. Our result includes that of Magenes and Stampacchia. In the case of the Dirichlet problem this result was proved in the case of real analytic functions by Morrey and Nirenberg [4] and in the case of functions in Friedman classes by Friedman in [2].

The notation, necessary norms, and other preliminaries are introduced in § 2. In § 3 two lemmas, which lead to $L^2$-estimates for $u$ and its tangential derivatives of all orders and normal derivatives of order up to $2m$, are proved. In § 4 $L^2$-estimates for derivatives of all orders of $u$ are obtained and finally an application of Sobolev's lemma yields the result.

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§ 2. Notation and Preliminaries.

Let $\Omega$ denote a bounded domain in a $v$-dimensional Euclidean space and let $(x_1, \ldots, x_v)$ be a coordinate system in $\Omega$. First we define certain classes of infinitely differentiable ($C^\infty$) functions on $\Omega$. Let $\{M_n\}$ be a sequence of positive numbers satisfying the following condition: there exists a positive constant $C$, independent of $n$, such that

\[
\binom{n}{\lambda} M_1 M_{n-\lambda} \leq C M_n \quad \text{(for } \lambda = 1, 2, \ldots, n; n = 1, 2, \ldots)\]

Then if $p$ is any non-negative integer, we denote by $C(M_{n-p}; \Omega)$ the class of $C^\infty$-functions $f$ on $\Omega$ satisfying the following condition:

(QA) For every closed subdomain $\Omega_0$ of $\Omega$ there exist two constants $H_1$ and $H_2$, depending on $f$ and on $\Omega_0$, such that, for any $x \in \Omega_0$, we have

\[
\frac{\partial^{\mid k\mid} f(x)}{\partial x_1^{k_1} \cdots \partial x_v^{k_v}} \leq H_1 H_2^{|k|} M_{|k|-p}
\]

where $k = (k_1, \ldots, k_v)$ and $|k| = k_1 + \cdots + k_v$.

We call a class of the type $C(M_{n-p}; \Omega)$ a Friedman class.

Similarly we define the classes $C(M_{n-p}; \overline{\Omega})$ when the condition (QA) is satisfied in $\overline{\Omega}$.

It is clear that (1) implies

\[
(n + 1) M_n \leq C_1 M_{n+1}
\]

with a positive constant $C_1$ independent of $n$ (in fact we can take $C_1 = \frac{C}{M_1}$).

Now we make some remarks on the Friedman classes $C(M_n; \Omega)$ which will be of use in the sequel.

(i) If $f$ is a function in the class $C(M_n; \Omega)$ such that $f(x) \neq 0$ for $x \in \Omega$ then $\frac{1}{f}$ is itself in the class $C(M_n; \Omega)$. For, let $(x_1, \ldots, x_v)$ be a coordinate system in $\Omega$ and let $d_{i_1}$ denote a generic partial differentiation operator of order one. Therefore, a generic partial differentiation operator of order $k$ can be written in the form $d_{i_1} d_{i_2} \ldots d_{i_k}$. Then

\[
(d_{i_n} d_{i_{n-1}} \ldots d_{i_1}) \left( \frac{1}{f} \right) = \frac{1}{f^{n+1}} H_n(f)
\]
where $H_n(f)$ is a homogeneous polynomial of degree $n$ in the set of arguments $(f, \ldots, f^n, \ldots, d_{i_1}f, \ldots, d_{i_k}f, \ldots, d_{i_n}d_{i_r}f)$ and where the degree of $d_{i_1}d_{i_r}f$ is taken to be $r$. Any monomial of degree $k$ in these arguments is majorized on any closed subset $\Omega_0$ of $\Omega$ by $(H_0H_1)^k \Omega^{k-1} M_k \leq H_0(H_0H_1)^k M_k$. Hence one can easily see that

$$|H_n(f)| \leq H_0 (c H_0 H_1)^n M_n$$

with a suitable positive constant $c$ independent of $n$.

Let $\delta = \min_{x \in \Omega_0} |f(x)|$. Taking $K_0 = \frac{H_0}{\delta}$ and $K_1 = \frac{c H_0 H_1}{\delta}$ we see that

$$|d_{i_1}d_{i_{n-1}} \ldots d_{i_j} \left( \frac{1}{f} \right)| \leq K_0 K_1^n M_n$$
on $\Omega_0$ which establishes (i).

(ii) If $f, g$ are in $C(M_n, \Omega)$ then their product $fg$ is itself in $C(M_n, \Omega)$. In fact, we have for an $\alpha = (a_1, \ldots, a_r)$

$$D^\alpha (fg) = \sum_{\beta} \binom{\alpha}{\beta} D^\beta (f) \cdot D^{\alpha-\beta} (g)$$

by Leibniz formula, where $\beta = (\beta_1, \ldots, \beta_s)$ and $\binom{\alpha}{\beta} = \binom{a_1}{\beta_1} \ldots \binom{a_r}{\beta_s}$.

Then $|D^\alpha (fg)| \leq \sum \binom{\alpha}{\beta} H_1 H_2^\beta M_\beta H_1 H_2^{\alpha-\beta} M_{\alpha-\beta} \leq C' H_1^2 H_2^\alpha M_n$, with a suitable constant $C' > 0$.

The remarks (i) and (ii) together imply the following:

(iii) If $f$, $g$ belong to $C(M_n, \Omega)$ with $g$ non-vanishing in $\Omega$ then $f/g$ is itself in the class $C(M_n, \Omega)$.

Let $s$ denote a fixed positive real number. Let $H^s$ denote the space or all tempered distributions $\varphi$ such that its Fourier transform $\hat{\varphi}$ satisfies the condition that $(1 + |\xi|^2)^{s/2} \varphi$ is square integrable and we define the scalar product in $H^s$ by

$$(\varphi, \psi)_s = \int \varphi(\xi) \overline{\psi(\xi)} (1 + |\xi|^2)^s d\xi$$

for any two $\varphi, \psi \in H^s$ and the corresponding norm

$$||\varphi||_s = \left[ \int \varphi(\xi) \overline{\psi(\xi)} (1 + |\xi|^2)^s d\xi \right]^{1/2}$$

for any $\varphi \in H^s$.

(see [3]).
In view of the local nature of the problem it is enough to consider the solution of the problem in a hemi-sphere with the boundary conditions defined on the plane part of the boundary.

Throughout, the function \( u \) is assumed to be infinitely differentiable in the hemi-sphere together with the plane part of the boundary. This is so, for example, in the following cases:

Let the coefficients of \( A \) and \( B_j \), and \( f \), \( g_j \) be infinitely differentiable functions of their arguments. Then any solution \( u \) of \( A u = f, B_j u = g_j \) is infinitely differentiable either when \( (A, [B_j]) \) is an elliptic system in the sense defined by J. Peetre [5] or when the boundary operators \( B_j \) satisfy the complementing condition of Agmon, Douglis and Nirenberg [1] with respect to the elliptic operator \( A \).

Next we introduce the differential operators. Let \( \omega_r \) denote the hemi-sphere \( \{x_1^2 + \ldots + x_r^2 < r^2, x_r > 0\} \) and \( \partial_1 \omega_r \) the plane part \( \{x_r = 0\} \) of the boundary of \( \omega_r \). Let \( \pi_0 \) denote the \((r - 1)\)-dimensional subspace \((\{x_1, \ldots, x_{r-1}, 0\}\) and let \( x' \) denote either \((x_1, \ldots, x_{r-1}, 0)\) or \((x_1, \ldots, x_{r-1})\) inadvertently in the context. We adopt the following notation throughout:

If \( p = (p_1, \ldots, p_r) \) then \( a_p(x) \) denotes a function \( a_{p_1 \ldots p_r}(x) \) and \( D^p = \frac{\partial^{p_1} \ldots \partial^{p_r}}{\partial x_1^{p_1} \ldots \partial x_r^{p_r}} \).

Similar notation is used in \( \pi_0 \) also with \( x' \) in place of \( x \) and \( D^p_{x'} \) in place of \( D^p \).

All our functions are defined in \( \omega_{R_0} \), together with the plane part \( \partial_1 \omega_{R_0} \) of the boundary of \( \omega_{R_0} \), where \( R_0 \) is a fixed positive number. Let

\[
A = \sum_{|p| \leq 2m} a_p(x) D^p
\]

be an elliptic linear partial differential operator of order \( 2m \) on \( \omega_{R_0} \) with the coefficients \( a_p(x) \) \( C^\infty \) in \( \omega_{R_0} \) and let

\[
B_j = \sum_{|p| \leq 2m - 1 - j} b^j_p(x') D^p \ (0 \leq j \leq 2m - 1)
\]

be differential operators (boundary operators) where \( b^j_p(x') \) are \( C^\infty \) functions on \( \partial_1 \omega_{R_0} \).

Let \( \varrho (t) \) be a real valued \( C^\infty \) function of the variable \( t \) \((-\infty < t < \infty)\) such that

\[
\varrho (t) = 1 \text{ for } t \leq 0,
\]

\[
= 0 \text{ for } t \geq 1,
\]
then for any pair of positive numbers \( r \) and \( h \) with \( 0 < r < r + h < R_0 \), define

\[
\varphi_{r,h}(x) = q\left(\frac{x - r}{h}\right).
\]

Then clearly we have

\[
\varphi_{r,h}(x) = 1 \quad \text{for} \quad |x| \leq r
\]

\[
= 0 \quad \text{for} \quad |x| \geq r + h
\]

and further for any \( p = (p_1, \ldots, p_s) \) we have

\[
|D^p \varphi_{r,h}(x)| \leq C_2 h^{-|p|} \quad \text{(when } h < r),
\]

where \( C_2 \) is a positive constant depending on \( \nu, p \) and the bounds for the derivatives of \( q \).

**Definition.** The system \( (A, \{B_j\}) \) is said to be an admissible system if \( A \) and the boundary operators \( \{B_j\} \) satisfy the following condition: there exists a constant \( C_3 \) such that for any \( C^\infty \) function \( u \) and for any \( r \) with \( 0 < r < r + h < R_0 \), we have

\[
\sum_{q=2m} \left\| D^q \varphi_{r,h} u \right\|^2_{0,\partial \Omega} \leq C_3 \left\{ \left\| A(\varphi_{r,h} u) \right\|^2_{0,\partial \Omega} + \sum_{j=0}^{2m-1} \left\| B_j(\varphi_{r,h} u) \right\|^2_{j+\frac{1}{2},\partial \Omega} \right\}
\]

where if \( B_j(\varphi_{r,h} u) \) is considered as having its support contained in \( \partial_1 \omega_{r+h} \) then \( B_j(\varphi_{r,h} u) \) is extended to the whole of \( \Omega_0 \) by taking it to be equal to zero in \( \Omega_0 - \partial_1 \omega_{r+h} \). Here \( \|f\|_{0,\partial \Omega}^2 \) is defined by \( \|f\|_{0,\partial \Omega}^2 = \int_{\partial \Omega} |f(x)|^2 \, dx \).

**Remark.** (a) When \( (A, \{B_j\}) \) is an admissible system the analyticity of a solution \( u \) of \( Au = f \) and \( B_j u = g_j \) up to the boundary (the coefficients of \( A \) and \( B_j \) and \( f, g_j \) being real analytic functions of their arguments) was proved by Magenes and Stampacchia [3].

(b) The inequality (2) has been obtained by J. Peetre when \( (A, \{B_j\}) \) is an elliptic system in the sense defined in [5]. When \( A \) is elliptic and \( B_j \) satisfy the complementing condition with respect to \( A \) an analogous inequality has been proved by Agmon, Douglis and Nirenberg [1].

The following is the precise statement of our theorem.

**Theorem.** Let \( (A, \{B_j\}) \) be an admissible system, with \( A \) elliptic, such that the following conditions are satisfied:

(i) the coefficients \( a_p(x) \) of \( A \) are in \( C[M_n; \omega_{\Omega_0}] \);

and (ii) the coefficients \( b'_p(x') \) of \( B_j \) are in \( C[M_n; \partial_1 \omega_{\Omega_0}] \).
Then any function \( u \), \( C^\infty \) in \( \omega_{R_0} \) and satisfying the system

\[
Au = f \text{ in } \omega_{R_0},
\]

\[
B_j u = g_j \text{ in } \partial_1 \omega_{R_0} \quad (0 \leq j \leq 2m - 1),
\]

where \( f \) is in \( C \left[ M_n ; \omega_{R_0} \right] \) and \( g_j \) are in \( C \left[ M_n - 2m + \left\{ r \right\}_1 + 1 ; \omega_{R_0} \cup \partial_1 \omega_{R_0} \right] \) respectively, is a function in \( C \left[ M_n - 2m + \left\{ r \right\}_1 + 1 ; \omega_{R_0} \right] \).

In the course of the proof of the theorem we need the following norms (introduced in [3]):

\[
e_{k, \tau} (f) = \left( \sum_{|q| = k} \left\| \varphi_{r, h} D^q_m f \right\|_{0, \omega_{R_0} + h}^2 \right)^{\frac{1}{2}} \quad \text{with } h = \frac{R - r}{k + 1}, \quad k = 0, 1, 2, \ldots
\]

\[
e_{j, k, \tau} (g) = \left( \sum_{|q| = k} \left\| \varphi_{r, h} D^q_m g \right\|_{j + \frac{1}{2}, \omega_{R_0} + h}^2 \right)^{\frac{1}{2}} \quad \text{with } h = \frac{R - r}{k + 1}, \quad k = 0, 1, 2, \ldots
\]

\[\quad (0 \leq j \leq 2m - 1)\]

and

\[
d_{k, \tau} (u) = \left( \sum_{|q| = 2m} \sum_{|p| = k} \left\| D^q D^p u \right\|_{0, \omega_{R_0}}^2 \right)^{\frac{1}{2}} \quad \text{for } k = 0, 1, 2, \ldots
\]

\[= \left( \sum_{|q| = 2m + k} \left\| D^q u \right\|_{0, \omega_{R_0}}^2 \right)^{\frac{1}{2}} \quad \text{for } k = -2m, \ldots, 0.
\]

We make the convention that

\[
[M_k] = M_k \quad \text{if } k \geq 0 \quad \text{and} \quad [M_k] = 1 \quad \text{if } k < 0
\]

and introduce the following notation (in analogy with that introduced in [4])

\[
M_{R, k}(f) = \frac{1}{M_k} \sup_{R \leq \tau < R} \left( R - (r + h)^{2m + k} e_{k, \tau} (f) \right) \quad \text{for } k = 0, 1, 2, \ldots
\]

\[
M_{j, R, k}(g) = \frac{1}{M_k} \sup_{R \leq \tau < R} \left( R - (r + h)^{2m + k} e_{j, k, \tau} (g) \right)
\]

\[\quad \text{for } k = 0, 1, 2, \ldots; \quad 0 \leq j \leq 2m - 1\]

and

\[
N_{R, k}(u) = \frac{1}{[M_k]} \sup_{R \leq \tau < R} \left( R - r \right)^{2m + k} d_{k, \tau} (u)
\]

\[\quad \text{for } k = -2m, -2m + 1, \ldots, 0, 1, 2, \ldots
\]
§ 3. In this paragraph we present two lemmas leading to the proof of the main theorem stated in the previous paragraph. In principle we obtain an $L^q$-estimate for the derivatives, up to order $2m$ in the transverse direction and of all orders in the tangential direction, for a function satisfying the system. To begin with we have the following result due to Magenes and Stampacchia (see [3] p. 331).

If $u$ is any $C^\infty$ function and if $(A, \{B_j\})$ is an admissible system then there exists a constant $C_4$, independent of $u$, $r$ and $h$ such that for $0 < r < R + h < R_0$, $r > h$ we have

$$\sum_{|q|=2m} \left\| D^q u \right\|_{0,\alpha,r}^2 \leq C_4 \left\{ \left\| \varphi_{r,h} A u \right\|_{0,\alpha,r+h}^2 + \sum_{j=0}^{2m-1} \left\| \varphi_{r,h} B_j u \right\|_{j+1,\alpha,\gamma}^2 + \sum_{l=0}^{2m-1} \left\| u \right\|_{l,\alpha,r+h}^2 \right\}.$$

Now we observe that, for any positive integers $\lambda, \ell, k, h$, we have

$$\sum_{|\ell|=1} \sum_{|\ell|=k} \left\| D^\ell D^{\ell'} u \right\|^2 \leq \sum_{|\ell|=1+k} \left\| D^\ell u \right\|^2 \quad \text{always.}$$

It follows from this that, for $k \geq 0$, $0 \leq \lambda \leq 2m$ we have

$$\sum_{|\ell|=k} d_{-1,r}^2 (D^{\ell'} u) \leq d_{-1,r}^2 (u).$$

On the other hand we also have

$$\sum_{|\ell|=k} e_{0,r}^2 (D^{\ell'} f) = e_{k,r}^2 (f)$$

$$\sum_{|\ell|=k} e_{j,0,r}^2 (D^{\ell'} g) = e_{j,k,r}^2 (g).$$

Taking $h = \frac{R - r}{R + 1}$ ($R/2 < r < R$), (3) can now be written in the form

$$d_{-1,r}^2 (u) \leq C_4 \left\{ e_{0,r}^2 (u) + \sum_{j=0}^{2m-1} e_{j,0,r}^2 (B_j u) + \sum_{l=1}^{2m} d_{-1,l+r+h}^2 (u) \cdot h^{-2l} \right\}.$$

**Lemma 3.1.** If $u$ is any $C^\infty$ function and if $(A, \{B_j\})$ is an admissible system then there exists a positive constant $C_5$ independent of $u$, $R$ and of
k, such that for any $R < R_1$, $k > 0$ the following inequality holds:

$$N_{R,k}(u) \leq C_5 \left\{ M_{R,k}(Au) + \sum_{j=0}^{2m-1} M_{j,R,k}(B_j u) + \sum_{l=1}^{2m} N_{R,k-l}(u) + \right.$$ 
$$+ \sum_{r=1}^{2m-k} (H_R^s R)^r N_{R,k-r}(u) \right\}.$$

**PROOF.** Consider any one of the tangential derivatives $D_{a^k} u$ of $u$ with $|q| = k$ and apply (3) in the form (7) taking $R/2 \leq r < R$ and $h = \frac{R - k}{k + 1}$.

We obtain

$$d_{a^k}^2 (D_{a^k} u) \leq C_4 \left\{ c_{a^k} (A (D_{a^k} u)) + \sum_{j=0}^{2m-1} e_{j,a^k,r} (B_j (D_{a^k} u)) + \right.$$ 
$$+ \sum_{l=0}^{2m-k} d_{a^k,l+r,k} (D_{a^k} u) \cdot h^{-2l} \right\}.$$

Using Leibniz formula for the derivation of a product of two functions and the fact that

$$\left( \frac{q_1}{s_1} \frac{q_2}{s_2} \cdots \frac{q_r}{s_r} \right) \leq \left( \frac{k}{\mu} \right)$$

where $q_1$ and $s_i$ are non-negative integers such that $q_1 + \cdots + q_r = k$ and $s_1 + \cdots + s_r = \mu$ we have the inequalities

$$\sum_{|q| = k} |A (D_{a^k} u)| \leq \sum_{|q| = k} |D_{a^k}(Au)| + \sum_{|p| \leq m} \mu \left( \frac{k}{\mu} \right) \sum_{|q| = k} |D_{a^k} A p| \sum_{|q| = k} |D_{a^k} D^p u|$$

and

$$\sum_{|q| = k} |B_j (D_{a^k} u)| \leq \sum_{|q| = k} |D_{a^k}(B_j u)| + \sum_{|p| \leq m-1-j} \mu \left( \frac{k}{\mu} \right) \sum_{|q| = k} |D_{a^k} b_{p'}|$$

$$\cdots \sum_{|q| = k-\mu} |D_{a^k} D^p u|.$$

Summing over all $q$ with $|q| = k$ in (9), using the following majorizations

$$\left( \sum_{|q| = k} |D^a p(x)|^2 \right)^{1/2} \leq H_1 H_2^e M_k$$

and

$$\left( \sum_{|q| = k} |D_{a^k} b_{p'}(x')|^2 \right)^{1/2} \leq H_1 H_2^e M_k.$$
(with the constants $H_1, H_2$ suitably changed) and making use of (5), (6) we obtain

$$d_{k,r}(u) \leq C_0 \left\{ e_{k,r}(Au) + \sum_{j=0}^{2m-1} e_{j, k,r}(B_j u) + \sum_{l=1}^{2m} d_{k-l, r+h}(u) \cdot h^{-l} + \right.$$  

$$+ \sum_{\mu=1}^{k} \left( \begin{array}{l} k \\ \mu \end{array} \right) H_1 H_2^{\mu} M_\mu \left( \sum_{|p| \leq 2m} \sum_{|q| = \mu - 1} \sum_{|t| = \mu - 1} \| D^p_x D^q_t u \|_{0, \omega_{r+h}}^2 \right)^{1/2} + \right.$$  

$$+ \sum_{j=0}^{2m-1} \sum_{\mu=1}^{k} \left( \begin{array}{l} k \\ \mu \end{array} \right) H_1 H_2^{\mu} M_\mu \left( \sum_{|p| \leq 2m-1} \sum_{|q| = \mu - 1} \sum_{|t| = \mu - 1} \sum_{|v| = \mu - 1} \| \varphi_{r+h} J^p_x D^q_t u \|_{0, \omega_{r+h}}^2 \right)^{1/2} \right\}$$

where $C_8$ is a positive constant independent of $u, r, h, k, R$. Moreover we have $\| \varphi_{r+h} v \|_{j + \frac{1}{2}, \omega_{r+h}} \leq C_7 \| v \|_{j + \frac{1}{2}, \alpha_{r+h}}$ (see [3]) with $C_7$ independent of $v$. From this remark it is clear that the last term of the second member of the above inequality can be majorized by the last but one term. Hence

(10)  

$$d_{k,r}(u) \leq C_0 \left\{ e_{k,r}(Au) + \sum_{j=0}^{2m-1} e_{j, k,r}(B_j u) + \sum_{l=1}^{2m} d_{k-l, r+h}(u) \cdot h^{-l} + \right.$$  

$$+ C_8 \sum_{\mu=1}^{k} \left( \begin{array}{l} k \\ \mu \end{array} \right) H_2^{\mu} M_\mu \left( \sum_{|p| \leq 2m-1} \sum_{|q| = \mu - 1} \sum_{|t| = \mu - 1} \| D^p_x D^q_t u \|_{0, \omega_{r+h}}^2 \right)^{1/2} \right\}$$

where $C_8$ is a positive constant independent of $u, r, h$ and $k$. Applying the inequality (4) to the last term of the second member of (10) we obtain

(11)  

$$d_{k,r}(u) \leq C_0 \left\{ e_{k,r}(Au) + \sum_{j=0}^{2m-1} e_{j, k,r}(B_j u) + \sum_{l=1}^{2m} d_{k-l, r+h}(u) \cdot h^{-l} + \right.$$  

$$+ C_8 \sum_{\mu=1}^{k} \left( \begin{array}{l} k \\ \mu \end{array} \right) H_2^{\mu} M_\mu \sum_{l=0}^{2m} d_{k-l, r+h}(u) \right\}.$$  

Multiplying both sides of (11) by $\frac{1}{M_k} (R - r)^{2m+k}$ we have the following estimates:

$$\frac{1}{M_k} (R - r)^{2m+k} e_{k,r}(Au) = \frac{1}{M_k} \left[ \frac{R - r}{R - (r + h)} \right]^{2m+k} (R - (r + h))^{2m+k} e_{k,r}(Au) \leq$$

$$\leq \left( 1 + \frac{1}{k} \right)^{2m+k} M_{R,k}(Au)$$

$$\frac{1}{M_k} (R - r)^{2m+k} e_{j, k,r}(B_j u) \leq \left( 1 + \frac{1}{k} \right)^{2m+k} M_{j, R,k}(B_j u) \text{ for } 0 \leq j \leq 2m - 1.$$
Further since \( h = \frac{R - r}{k + 1} \)

\[
\frac{1}{M_k} (R - r)^{2m+k} d_{k-\mu-1, r+h}(u) h^{-1} \leq \frac{1}{M_k} (k + 1)^{+\frac{1}{k}} \left( 1 + \frac{1}{k} \right)^{2m+k-1} [M_{k-\mu-1}] N_{R,k-\mu-1}(u) \leq \left( \frac{k - \lambda}{k + 1} \right) \frac{(k - \mu)!}{(k - \lambda)!} \frac{c_1}{M_k} \left( 1 + \frac{1}{k} \right)^{2m+k-1} N_{R,k-\mu-1}(u)
\]

because \([M_{k-\mu-1}] \leq \frac{(k - \lambda)!}{k + 1} \frac{c_1}{M_k} M_k\). Similarly we have

\[
\left( \frac{k}{\mu} \right) \frac{M_{\mu}}{M_k} (R - r)^{2m+k} d_{k-\mu-1, r+h}(u) \leq \left( \frac{k}{\mu} \right) \frac{M_{\mu}}{M_k} \left( 1 + \frac{1}{k} \right)^{2m+k-\mu-1} \cdot R^{n+1} N_{R,k-\mu-1}(u).
\]

But by (1) \( M_{\mu} [M_{k-\mu-1}] \leq \left( \frac{k - \mu}{k} \right) \frac{(k - \mu)!}{(k - \lambda)!} \frac{c_1}{M_k} M_k \) and by (1') it follows that \( M_{k-\mu-1} \leq \frac{c_1}{k + 1} \frac{(k - \mu)!}{(k - \lambda)!} M_k\). Hence we have:

\[
\left( \frac{k}{\mu} \right) \frac{M_{\mu}}{M_k} (R - r)^{2m+k} d_{k-\mu-1, r+h}(u) \leq C_1 \left( \frac{k - \mu}{k - \lambda} \right) \frac{(k - \mu)!}{(k - \lambda)!} \left( 1 + \frac{1}{k} \right)^{2m+k-\mu-1} \cdot R^{n+1} N_{R,k-\mu-1}(u).
\]

Then the inequality (11) becomes

\[
\frac{1}{M_k} (R - r)^{2m+k} d_{k,r}(u) \leq C_0 \left( 1 + \frac{1}{k} \right)^{2m+k} \left\{ M_{R,k}(A u) + \sum_{j=0}^{2m-1} M_{R,k}(R_j A u) + \right.
\]

\[
+ \sum_{k=1}^{2m} \left( \frac{k - \lambda}{k + 1} \right) \frac{(k + 1)!}{(k - \lambda)!} C_1 \left( 1 + \frac{1}{k} \right)^{-1} N_{R,k-\mu-1}(u) +
\]

\[
+ C_0 \sum_{\mu=1}^{k} \sum_{l=0}^{2m} H^l \cdot R^{n+1} \cdot \frac{(k - \mu - l)!}{(k - \mu)!} \left[ \frac{(k - \mu - l)!}{(k - \mu)!} \right] \cdot N_{R,k-\mu-1}(u)
\]

Since \( k \frac{(k - \lambda)!}{k + 1} \leq \lambda^2 \leq 2m^2m \) and \( \frac{(k - \mu - l)!}{(k - \mu)!} \leq 1 \) it follows that there
exists a constant $C_5$ such that
\[
\frac{1}{M_k} (R - r)^{2m+k} d_{k,r}(u) \leq C_5 \left\{ M_{R,k} (A u) + \sum_{j=0}^{2m-1} M_{j,R,k} (B_j u) + \sum_{\lambda=1}^{2m} N_{R,k-\lambda}(u) + \sum_{\mu=1}^{k} \sum_{\ell=0}^{2m} (H_2 R)_{\mu+\ell} N_{R,k-\mu-\ell}(u) \right\}.
\]

Taking $\mu = l = r$ in the last term of the second member and the sup-remum for $R/2 \leq r < R$ of the first member we obtain (8) and this completes the proof of the lemma.

**Lemma 3.2.** Let $(A, \{B_j\})$ be an admissible system and $u$ be any $C^\infty$ function satisfying the system
\[
Au = f \quad \text{in} \quad \omega_{R_1},
\]
\[
B_j u = g_j \quad \text{in} \quad \partial_1 \omega_{R_1} \quad (0 \leq j \leq 2m - 1),
\]
with $f$ and $g_j$ respectively in the classes $C \{M_n; \omega_{R_1}\}$ and $C \{M_{n-j-1}; \partial_1 \omega_{R_1}\}$. Then there exist two positive constants $M$ and $\lambda$ such that
\[
N_{R,k}(u) \leq M \lambda^k \quad \text{for} \quad k = -2m, -2m + 1, ...
\]

**Proof.** We can suppose, if necessary after some modification that the constants $\lambda_1, H_2$ and $R_1$ are the same as before and are such that
\[
(\Sigma_{|\nu|=-k} |D\nu f(x)|^2)^{1/2} \leq H_1 H_2^k M_k \quad \text{for} \quad x \in \omega_{R_1}, \quad k = 0, 1, 2, ...
\]
and
\[
(\Sigma_{|\nu|=-k} |D_{\nu} g_j(x')|^2)^{1/2} \leq H_1 H_2^k M_k \quad \text{for} \quad x' \in \partial_1 \omega_{R_1}, \quad k = 0, 1, 2,...; \quad 0 \leq j \leq 2m - 1.
\]
Let $\beta_2^2$ denote the volume of the unit ball in the $v$-dimensional Euclidean space. Then for $R < R_1$ we have
\[
M_{R,k}(f) \leq \frac{1}{M_k} R^{2m+k} \left( \Sigma_{|\nu|=-k} \int_{\omega_{R_1}} |D_{\nu} f(x)|^2 \right)^{1/2} \leq H_1 H_2^k \beta_2^2.
\]
Similarly using $\| q_{r,h} W \|_{j, \frac{1}{2}, n_0} \leq \tilde{C} \| W \|_{j+1, \omega_{r+h}, \psi}$, with a positive constant $\tilde{C}$
independent of \( W \), we obtain

\[
M_{j,R,k}(g_j) \leq C \frac{1}{M_k} R^{2m+k} \left( \sum_{l=0}^{j+1} \sum_{|l|=-k-1} \left| \int \frac{D_{\omega}^k g_j}{\partial_\omega R} \right|^2 \right)^{1/2} \leq
\]

\[
\leq C \frac{1}{M_k} R^{2m+k} H_1^k \sum_{l=0}^{r-1} \left( \frac{M_k^{k+i-l-1}}{M_k} \right)^{1/2} \leq
\]

\[
\leq C_{10} R^{2m+k} H_1^k \frac{R^{r-1}}{M_k^{k+i-l-1}}
\]

using (1'), where \( C_9, C_{10} \) are positive constants independent of \( R \) and \( k \). Then the inequality (8) becomes, for any \( k \) and \( R < R_1 \),

\[
N_{R,k}(u) \leq C_9 \left( C_{11} (H_2 R)^k + \sum_{l=1}^{2m} (H_2 R)^{l+1} N_{R,k-l}(u) \right).
\]

Now proceeding, as in the proof of Magenes and Stampacchia, with the constants \( M \geq 3C_9 C_{11} \) and \( \lambda = (3C_9 + 1) (H_2 R_1 + 1) \) we obtain

\[
N_{R,k}(u) \leq M \lambda^k \text{ for } k = -2m, -2m+1, ...
\]

after using an induction argument on \( k \). This completes the proof of lemma 3.2.

\S 4. We complete the proof of the main theorem (see \S 2) in this paragraph. For this purpose it is necessary to obtain estimates of the type (12) for all derivatives, tangential as well as transversal, of \( u \). To obtain such estimates we follow a procedure used by Morrey and Nirenberg in [4]. We introduce the following norms analogous to those in \S 2.

For \( p \geq 0, q \geq -2m \) define

\[
N_{R,p,q}(u) = \frac{1}{[M_{p+q}] R^{2m+p+q}} \left( \sum_{|a|=p} \left( \int \frac{D_{\omega}^{2m+q} D_{\omega}^{l+1} u}{\partial_\omega R} \right)^2 \right)^{1/2}.
\]

Analogous to (4) we have

\[
\sum_{|a|=p} \frac{1}{|l|} \left( \frac{D_{\omega}^{2m+q} D_{\omega}^{l+1} u}{\partial_\omega R} \right) \leq \sum_{|a|=q} \frac{1}{|l|} \left( \frac{D^{2m+q} D_{\omega}^{l+1} u}{\partial_\omega R} \right) \leq
\]

\[
\sum_{|a|=-2m} \left( \frac{D^{2m} D_{\omega}^{l+1} u}{\partial_\omega R} \right) \leq \sum_{|a|=-2m} \left( \frac{D^{2m} D_{\omega}^{l+1} u}{\partial_\omega R} \right) \leq \sum_{|a|=-p+q} \left( \frac{D^{2m} D_{\omega}^{l+1} u}{\partial_\omega R} \right) \leq \sum_{|a|=-p+q} \left( \frac{D^{2m} D_{\omega}^{l+1} u}{\partial_\omega R} \right)
\]

in all cases.
This implies that

\[ N_{R,p,q}(u) \leq N_{R,p+q}(u) \text{ if } p \geq 0, \quad q \leq 0. \]

We now prove the following extension of the estimation (12): if \( R \) is smaller than or equal to a fixed number depending only on the given differential equation, then

\[ N_{R,p,q}(u) \leq \overline{M} \lambda^{p+q} \Theta^p, \quad (p \geq 0, q \geq -2m) \]

with \( \overline{M}, \lambda \geq 1 \) and \( \theta \leq \frac{1}{2} \) fixed constants, \( \lambda \) and \( \theta \) depending only on the equation.

The following is a sketch of the derivation of the estimate (15). Let us denote \( x \) by \( y \) for convenience. By assumption \( y = 0 \) is not a characteristic surface for the given equation \( Au = f \). Hence one can solve for the normal derivative \( \partial^m u \) of \( u \) in terms of the derivatives involving normal derivatives of \( u \) of orders less than \( 2m \):

\[ \partial^{2m} u = g + \sum_{l=1}^{2m} b_l \partial^{2m-l} \partial^l u \]

where in view of the remarks on the classes \( C_{\{M_n; \Omega \}} \), made in § 2, \( g \) and \( b_l \) are functions belonging to the class \( C_{\{M_n; \omega_R \}} \). This implies that both

\[ \sum_{|l| \neq p} \partial_{y}^{q} \partial_{x}^{p} g(x) \leq H_{1} H_{2}^{p+q} M_{p+q} \]

for suitable constants \( H_1, H_2 \) and \( R_0 \leq 1 \). We can assume these constants to be the same as before by suitable choice. Then we have from (16)

\[ \sum_{|l| \neq p} \frac{\partial_{y}^{q} \partial_{x}^{p} D_{x}^{l} u}{D_{y}^{m} D_{x}^{l} u} \leq H_{1} H_{2}^{p+q} M_{p+q} \]

Hence

\[ \sum_{|l| \neq p} \frac{\partial_{y}^{q} \partial_{x}^{p} D_{x}^{l} u}{D_{y}^{m} D_{x}^{l} u} \leq \sum_{|l| \neq p} \frac{\partial_{y}^{q} \partial_{x}^{p} D_{x}^{l} u}{D_{y}^{m} D_{x}^{l} u} + \sum_{|l| \neq p} \sum_{l=1}^{2m} \sum_{\beta \neq 0} \left( \alpha \right)_{\beta} H_{1} H_{2}^{p+q} M_{p+q} \frac{\partial_{y}^{q} \partial_{x}^{p} D_{x}^{l} u}{D_{y}^{m} D_{x}^{l} u}. \]

It is clear from (12) and (14) that (15) follows for \( -2m \leq q \leq 0 \) and all \( p \geq 0 \) provided that \( R < R_1 < R_0 \) (\( R_1 \) chosen suitably) and

\[ \overline{M} \lambda^{p+q} \Theta^p \geq M \lambda^{p+q} \text{ for } -2m \leq q \leq 0, \quad p \geq 0. \]
We prove (15) for $q > 0$, $p > 0$ by induction on $q$. Let us assume that (15) holds for all values of $q$ less than a certain positive integer which we again denote by $q$. Squaring both sides of (18) and integrating over $\omega_r$ we obtain

\[
\left( \sum_{|\lambda| < p} \left| \int \frac{D_{|a|}^{2m+q} D_{|y|}^p u \, dx}{\omega_r} \right|^2 \right)^{1/2} \leq H_1^{q+q} M_{p+q} \beta_r r^{\gamma/2} + \sum_{l=1}^{2m} \sum_{|a|=0}^{p} \left( \begin{array}{c} p \\ \alpha \end{array} \right) \left( \begin{array}{c} q \\ \beta \end{array} \right) H_1^{q+q} M_{|a|+\beta} \cdot (\beta_r r)^{q+q} + \left( \sum_{|\lambda| > p} \int \frac{D_{|a|}^{2m+q-\beta-\gamma} D_{|y|}^{q+q} u \, dx}{\omega_r} \right)^{1/2} .
\]

Multiplying both sides of this inequality by

\[
\frac{ (R-r)^{2m+q} }{ [M_{p+q}] } \bar{M}^{-1} \lambda^{-(p+q)} \theta^{-p}
\]

for $R < R_1$, taking the supremum over all $r$ with $R/2 < r < R$ and using the induction assumption we obtain

\[
(20) \quad \bar{M}^{-1} \bar{\lambda}^{-(p+q)} \theta^{-p} N_{R,p,q} \leq \frac{ K H_1 }{ \bar{M} } \left( \frac{ H_2 R }{ \lambda } \right)^{p+q} \theta^{-p} + \sum_{l=1}^{2m} \theta^\gamma \sum_{|a|=0}^{p} \left( \begin{array}{c} p \\ \alpha \end{array} \right) \left( \begin{array}{c} q \\ \beta \end{array} \right) \left[ \frac{ (H_2 R)_{|a|} }{ \theta \lambda } \right] \left( \frac{ (H_2 R)^\beta }{ [M_{p+q}] } \right) [M_{|a|+\beta}],
\]

where $K$ is a suitable constant.

But by (1) we have the inequality

\[
[M_{p+q-|a|+\beta}] \leq C \left[ \left( \frac{ p + q }{ \alpha + \beta } \right) \right]^{-1} [M_{p+q}],
\]

Then the inequality (20) becomes

\[
\bar{M}^{-1} \bar{\lambda}^{-(p+q)} \theta^{-p} N_{R,p,q} \leq \frac{ K H_1 }{ \bar{M} } \left( \frac{ H_2 R }{ \lambda } \right)^{p+q} \theta^{-p} + \sum_{l=0}^{2m} \theta^\gamma \sum_{|a|=0}^{p} \left( \begin{array}{c} q \\ \alpha \end{array} \right) \left( \begin{array}{c} p + q \end{array} \right) \left[ \left( \frac{ p + q }{ \alpha + \beta } \right) \right]^{-1} \left( \frac{ H_2 R }{ \theta \lambda } \right) \left( \frac{ H_2 R }{ \lambda } \right)^{\beta}
\]
Here all the terms in the summation over $a, \beta$ are less than unity. Taking $\frac{H_k R}{\theta \lambda} \leq \frac{1}{2}$ and $\theta \leq \frac{1}{2}$ the second member does not exceed $\frac{K H_1}{M} + 8 C H_1 \theta$ which is again less than unity if $M \geq 2 K H_1$ and $\theta \leq \frac{1}{16 C H_1}$. Thus we have proved that

$$N_{R, p, q}(u) \leq \bar{M} \bar{\lambda}^{p+q} \theta^p$$

holds for all $q$ with $q \geq -2m$ and for $R < R_1$ if we take $\theta = \frac{1}{16 C H_1}$, $\bar{\lambda} = \max \left(\frac{2 H_k R_1}{\theta}, \frac{\lambda}{\theta}\right)$ and $\bar{M} \geq 2 K H_1$.

As we have already said in the introduction the result is deduced by applying Sobolev’s lemma to the $L^q$-norms of the derivatives of $u$. For this we need estimates for the square integrals of the type

$$\tilde{a}_p^2 (u, \omega_r) = \sum_{|\gamma| \leq p} \left\| D^\gamma u \right\|^2 dx.$$

These are easily obtained from (15) as follows:

$$\tilde{a}_p^2 (u, \omega_r) = \sum_{l=0}^p \sum_{|\gamma| = p-l} \int_{\omega_r} \left\| D^l_y D^\gamma_x u \right\|^2 dx \leq \sum_{l=0}^p \left[ \frac{\bar{M} \bar{\lambda}^{p-2m} \theta^{p-l} M_{p-2m}}{(R - r)^p} \right]^2.$$ 

Hence

$$\tilde{a}_p (u, \omega_r) \leq \frac{\bar{M} \bar{\lambda}^p}{(R - r)^p} M_{p-2m} \left( \sum_{l=0}^p \theta^{2l} \right)^{1/2}.$$ 

Thus we obtain

$$\tilde{a}_p (u, \omega_r) \leq \frac{2 \bar{M} (\bar{\lambda})^p M_{p-2m}}{(R - r)^p}. \tag{21}$$

Now we apply Sobolev’s lemma in the form used in [4], namely, for $x \in \omega \cup \partial \omega$,

$$\left| D^p u (x) \right| \leq C' \left[ \sum_{l=0}^{[r/2]+1} r^{2l-r} \tilde{a}_l^2 (D^p u, \omega_r) \right]^{1/2} \leq C' \left[ \sum_{l=0}^{[r/2]+1} r^{2l-r} \left( \frac{2 \bar{M} (\bar{\lambda})^{p+1}}{(R - r)^{p+1}} M_{p+1-2m} \right)^{1/2} \right].$$
Since $R - r \geq r$ and $R \leq 1$ we obtain the following inequality
\[
|D^\alpha u(x)| \leq \frac{K'(r/2)^{p+1}}{(R - r)^{p+1}} M_{p+|r/2|+1-2m}
\]
after using (1') ($K'$ being a positive constant independent of $p$). This proves the fact that $u \in C [M_{n-2m+|r/2|+1}; \omega_{R_0} \cup \partial_1 \omega_{R_0}]$ thus completing the proof of the theorem.

BIBLIOGRAPHY