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Generalized subharmonic functions : monotonic approximations and an improved maximum principle

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GENERALIZED SUBHARMONIC FUNCTIONS: MONOTONIC APPROXIMATIONS AND AN IMPROVED MAXIMUM PRINCIPLE

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1. Introduction

Let D be a bounded domain in Euclidean n-space; let the partial differential operator L defined by (1)

$$Lu = (b_{ij}(x)\, u)_{x_i x_j} - (b_i(x)\, u)_{x_i} + b(x)\, u \tag{1.1}$$

be uniformly elliptic there, and let us denote by L^* the formal adjoint of L, i. e.

$$L^* v = b_{ij}(x)\, v_{x_i x_j} + b_i(x)\, v_{x_i} + b(x)\, v. \tag{1.2}$$

We say that

$$Lu(x) \geq 0 \qquad \text{(weakly)} \tag{1.3}$$

in D if u is locally integrable in D and if the inequality

$$\int u(x)\, L^* v(x)\, dx \geq 0 \tag{1.4}$$

holds for all non-negative v in $C^2(D)$ with compact support in D. Such fun-

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(1) We agree to use the summation convention.

ctions u were introduced in [2] where they were called *weakly L-subharmonic.* Let us remark that if the coefficients appearing in L are sufficiently smooth, then L can also be written in the more usual form

$$Lu = a_{ij} u_{x_i x_j} + a_i u_{x_i} + au \tag{1.5}$$

The main result of this paper is to show, roughly speaking, that if $Lu \geq 0$ (weakly) then u is the limit of a monotonically increasing sequence of *continuous* functions u_h satisfying $Lu_h \geq 0$ (weakly) ($h \to \infty$). If the coefficients in (1.1) are smooth then so is u_h and $Lu_h \geq 0$ in the strict sense. As a biproduct of our investigation we will obtain (in theorem B) a strengthened form of the maximum principle proved in [2].

To a certain extent, the object of this paper (and also of [2]) may be described as the study of *weak solutions of partial differential inequalities,* as opposed to weak solutions to partial differential *equations* (which have been studied extensively). If the coefficients of L are constant, theorem A becomes very easy: all we need do then is to mollify the function u in the usual manner. Indeed this has been done for the Laplacian (see. for ex. [4]). If the coefficients are variable, however, the mollification kernel must be made to depend on the coefficients in a very specific way. That is the main point of this paper.

At this point it may be appropriate to observe that although the three ways of writing a second order linear operator Lu

$$a_{ij} u_{x_i x_j} + a_i u_{x_i} + au \qquad \text{(standard form)}$$

$$(b_{ij} u)_{x_i x_j} - (b_i u)_{x_i} + bu \qquad \text{(completely integrated form)}$$

$$(c_{ij} u_{x_i})_{x_j} + (c_i u)_{x_i} + c_i' u_{x_i} + cu \qquad \text{(half way integrated form)}$$

are equivalent if the coefficients are sufficiently smooth, this is not the case if the coefficients are merely Hölder continuous, for example. In the latter instance, each of the three cases must be considered separately. For operators in the standard form, it must be assumed that the functions u on which they operate have second derivatives in some sense. Thus, for Hopf's maximum principle to hold, it is required that u have pointwise second derivatives, and the coefficients be bounded. For a maximum principle to hold in the half-way integrated form u is required to posses generalized first derivatives. Finally, for the completely integrated form, the case treated in this paper, we make no assumptions on the function u other than local integrability.

2. Statement of Results.

The following « basic » assumptions will be made throughout the paper unless the contrary is indicated. We let D be a bounded Dirichlet domain (with respect to L^*) in R^n. The coefficients $b_{ij}(x)$, $b_i(x)$, $b(x)$ used in (1.1) and (1.2) to define $L \equiv L_x$ and $L^* \equiv L^*_x$ respectively [2] are assumed to be uniformly Hölder continuous in D and L (and therefore also L^*) is to be uniformly elliptic in D, in the sense that

$$b_{ij}(x)\, t_i\, t_j \geq m \, \Sigma \, t_i^2 , \tag{2.1}$$

m being independent of x and t.

THEOREM A : If, under the foregoing assumptions,

$$Lu \geq 0 \qquad \text{(weakly)},$$

then there exists a one parameter family of functions $u_h(x)$ with the following properties: For every compact subdomain $\bar{D}_\lambda \subset D$

a) u_h is continuous in D_λ for h sufficiently large.

b) $Lu_h \geq 0$ (weakly) in D_λ;

c) for fixed x in D_λ $u_h(x)$ is nonincreasing with increasing h.

d) $\int_{D_\lambda} | u(x) - u_h(x) | \, dx \longrightarrow 0$ as $h \longrightarrow \infty$.

e) u equals almost everywhere in D an upper semi-continuous function u^* which is the pointwise limit of u_h in D_λ. We allow u to assume the value $-\infty$ on a set of measure zero.

f) If, in addition to the Hölder continuity assumed, we assume that $b_{ij} \in C^{2+\alpha}(\bar{D})$, $b_i \in C^{1+\alpha}(\bar{D})$, $b \in C^{\alpha}(\bar{D})$, then $u_h \in C^{2+\alpha}$ and $Lu_h \geq 0$ (in the strict sense).

g) u_h is given explicitly as the result of an integral operator acting on u:

$$u_h(y) = \int K_h(x, y)\, u(x)\, dx,$$

where the kernel $K_h(x, y)$ is constructed explicitly from the Green's function of L and D.

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(2) The subscripts x in L_x and L^*_x refer to the fact that the coefficients as well as the functions on which they act are functions of x. This notation will be of importance later when we shall be dealing with functions of x and y, both in R^n.

COROLLARY : If redefined on a set of measure zero, u is bounded from above in D_λ.

It is well known that elliptic operators of the form

$$L = a_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + a_i \frac{\partial}{\partial x_i} + a$$

enjoy special properties provided $a \equiv a(x) \leq 0$; for example Hopf's maximum principle is valid for positive solutions of $Lu \geq 0$. For our purposes it becomes necessary to ask, what is the analogous restriction on the coefficients b_{ij} etc, of L^*? The appropriate condition is

$$L[-1] \geq 0 \quad \text{weakly.}$$

It is easily seen that for smooth coefficients in L the last inequality is equivalent to $a \leq 0$.

THEOREM B : Under the « basic assumptions » made at the beginning of section 2, if

$$Lu \geq 0 \text{ weakly,} \qquad L[-1] \geq 0 \text{ weakly,}$$

and if, in addition, for a compact subdomain $D_\lambda \subset D$ we have

$$0 \leq M \equiv \operatorname*{ess\,sup}_{x \in D} u(x) = \operatorname*{ess\,sup}_{x \in D_\lambda} u(x), \tag{2.3}$$

then $u = M$ almost everywhere in D. (Note : the conclusion also holds if $L[1] \geq 0$ weakly and $M \leq 0$).

REMARKS : Theorem B is an improvement over our previous result [2] in several directions, the main one being the replacement of the requirement that u assume its essential supremum at a point of continuity by the weaker and more natural condition (2.3). We might also mention that A. Friedman [1] has extended the maximum principle in [2] to parabolic equations ; but here we restrict ourselves to the case of elliptic equation.

3. The Function G_h.

In this section we construct a function $G_h(x, y)$ which will eventually be used to construct the kernel $K_h(x, y)$ described in Theorem A. We begin with the Green's function $g(x, y)$ of the operator L_x with respect to the domain D and with singularity at $y \in D$; as construced, for example in [3].

$L_x^* g(x, y) = 0$ for $x, y \in D$, and $g(x, y) \to \infty$ like $|x - y|^{2-n}$ as $x \to y$. Furthermore, g is defined and continuous jointly in x and y for $x \in \bar{D}$, $y \in \bar{D}$, $x \neq y$; the first and second derivatives of g with respect to x are continuous for $x, y \in D$, $x \neq y$; $g(x, y) = 0$ for x on ∂D, i. e., the boundary of D. Also:

$$g(x, y) = 0(r^{2-n}) \quad (r = |x - y|)$$

$$g_{x_i} = 0(r^{1-n}), \qquad g_{x_i x_j} = 0(r^{-n}).$$

Let $p(t)$ be a C^∞ function of a single variable t defined for $t \leq t_0$ such that $p(0) = 1$, and

$$p(t) \geq p_0 > 0$$

for $t = |x-y| < 2\varrho_0$, $x, y \in D$.

$$L_x^* p(|x - y|) < 0$$

For example, we could take $p(t) = 1 - t^2$. Now let us denote by D_ϱ $(\varrho \geq 0)$ the set points of D whose distance from the complement of D does not exceed ϱ. For $h \geq 0$, $x \in D$ and $y \in D_\varrho$ we then define the function $G_h(x, y)$ as follows:

for $|x - y| \geq 2\varrho$: $\qquad G_h(x, y) = 0$

for $|x - y| < 2\varrho$

$$G_h(x, y) = \int_{-\infty}^{\infty} \Phi(\alpha - h) \ \operatorname{Max}[g - \alpha p, 0] \, d\alpha \tag{3.1}$$

$$= \int_{-\infty}^{\frac{g}{p}} \Phi(\alpha - h) \quad (g - \alpha p) \, d\alpha;$$

here $\Phi(\alpha)$ is a nonnegative C^∞ function defined for all real α, vanishing outside the interval $(0, 1)$ and positive in its interior such that

$$\int_{-\infty}^{\infty} \Phi(\alpha) \, d\alpha = 1.$$

From the definition of G_h it follows that if we take

$$h \geq h_\varrho = \frac{1}{p_0} \underset{\substack{\varrho \leq |x-y| \leq 2\varrho \\ x \in D,\, y \in D_{2\varrho}}}{\text{maximum}} g(x, y)$$

then $G_h(x, y) = 0$ holds even for $2\varrho \geq |x-y| \geq \varrho$. Namely, if $-\infty < \alpha \leq h_\varrho$ we have $(\alpha - h) = 0$; while if $\alpha > h_\varrho$, then $\alpha p > g$ and $\operatorname{Max}[g - \alpha p, 0] = 0$, which shows that the first integral defining G_h vanishes.

We shall also make use the function

$$(3.2) \qquad H_h(x, y) \equiv G_h(x, y) - g(x, y) = g \int_{\frac{g}{p}}^{\infty} \Phi(\alpha - h)\, d\alpha - p \int_{-\infty}^{\frac{g}{p}} \Phi(\alpha - h)\, \alpha d\alpha.$$

Please note that this function is continuous for $x \in \bar{D}$ and $y \in D_{2\varrho}$ and has two derivatives with respect to x continuous jointly in x, y.

Let us here call attention to the convention, already used, to abbreviate $g = g(x, y)$, $G_h = G_h(x, y)$ etc.

4. Properties of G_h.

In this section we derive a number of properties of the G_h. We state these properties, giving the shorter proofs with the statements, and leaving the more difficult of the proofs till later. In what follows we pick a positive number ϱ take x in D, y in $D_{2\varrho}$, and let $h \geq h_\varrho$.

$$(4.a) \qquad \frac{\partial G_h}{\partial h} \leq 0$$

Proof:

$$\frac{\partial G_h}{\partial h} = -\int_{-\infty}^{\frac{g}{p}} \Phi'(a - h)(g - \alpha p)\, d\alpha = -g \int_{-\infty}^{\frac{g}{p}} \Phi'(\alpha - h)\, d\alpha +$$

$$+ p \int_{-\infty}^{\frac{g}{p}} \alpha \cdot d\,\Phi(\alpha - h) = -g\,\Phi\left(\frac{g}{p} - h\right) + g\,\Phi\left(\frac{g}{p} - h\right) -$$

$$- p \int_{-\infty}^{\frac{g}{p}} \Phi(\alpha - h)\, d\alpha = -p \int_{-\infty}^{\frac{g}{p}} \Phi(\alpha - h)\, d\alpha \leq 0.$$

(4.b) $\dfrac{\partial G_h}{\partial h}$ has two continuous derivatives with respect to x for x in D (even for $x = y$). (The proof follows easily from the last equality).

(4.c) $L_x^* \, G_h(x, y) \geq 0$ and the strict inequality holds for $h_\varrho \leq h < \dfrac{g}{p}(x, y) < h + 1$. Furthermore, $L_x^* \, G_h(x, y)$ is continuous in y ($\in D_{2p}$) uniformly in x ($\in D$).

(4.d)
$$L_y \, H_h(x, y) \geq 0 \text{ (weakly).}$$

(4.e)
$$\text{If } b_{ij} \in C^{2+\alpha}(\overline{D}),\ b_i \in C^{1+\alpha}(\overline{D}),\ b \in C^\alpha(\overline{D}),$$

the $H_h(x, y)$ has two derivatives with respect to y continuos uniformly with respect to $x \in \overline{D}$ and $y \in D_{2\varrho}$ (see formula 3.2). In that case $L_y \, H_h(x, y) \geq 0$ in the strict sense.

(4.f) $\lim\limits_{h \to \infty} \displaystyle\int L_x^* \, G_h(x, y) \, dx = 1$, uniformly with respect to y for $y \in D_{2\varrho}$.

The next few properties will involve the functions u_h defined for y in $D_{2\varrho}$ as follows:

$$u_h(y) = \int u(x) \, L_x^* \, G_h(x, y), \, dx \tag{4.1}$$

(4.g) For any fixed y in $D_{2\varrho}$, $u_h(y)$ will be a non increasing function of h, $h \geq h_\varrho$.

PROOF: $\dfrac{\partial}{\partial h} u_h(y) = \dfrac{\partial}{\partial h} \displaystyle\int u(x) \, L_x^* \, G_h \, dx = \int u \, L_x^* \dfrac{\partial G_h}{\partial h} \, dx$. Now by

(4.a) $\dfrac{\partial G_h}{\partial h} \leq 0$, and $\dfrac{\partial G_h}{\partial h}$ has compact support in D (as function of x).

Hence, using the fact that $Lu \geq 0$ weakly, if follows that $\dfrac{\partial}{\partial h} u_h \leq 0$.

(4.h) $u_h(y)$ is continuous; if $b_{ij} \in C^{2+\alpha}$, $b_i \in C^{1+\alpha}$, $b \in C^\alpha$ then $u_h(y) \in C^{2+\alpha}$.

(4.i) $L_y \, u_h(y) \geq 0$ (weakly). If $b_i \in_j C^{2+\alpha}$, $b_i \in C^{1+\alpha}$, the « weakly » may be omitted.

Before giving the remaining proofs of the above properties let us use them to prove theorem A.

PROOF OF THEOREM A: Letting $K_h(x, y) = L_x^* G_h(x, y)$, we see that $u_h(y) = \int K_h(x, y)\, u(x)\, dx$. Using (4.*c*), (4.*f*), and the fact that the support of G_h (and therefore also of K_h) is contained in a sphere centered at y, whose radius $\to 0$ as $h \to \infty$, if follows that

$$\int_{D_{2p}} | u(y) - u_h(y) |\, dy \to 0 \text{ as } h \to \infty, \tag{4.2}$$

thus proving part *d*) of Theorem A. Next, we note that for almost all y in $D_{2\varrho}$ the quantity $u_h(y)$ remains bounded as $h \to \infty$. (If this were not the case, u_h, being nonincreasing, would have to approach $-\infty$ on a set of positive measure, which would contradict (4.2)). At these points, $u_h(y)$ (being monotonic) approaches a finite limit $= u^*(y)$. At the remaining points (constituting a set of measure zero) define $u^*(y) = -\infty$. Thus $u_h(y)$ approaches $u^*(y)$ pointwise in $D_{2\varrho}$, allowing the value $-\infty$. Since the sets $D_{2\varrho}$ exhaust D, this proves *e*). Assertions *a*, *b*, *c*, and *f* are immediate consequences of properties 4*h*, 4*i*, 4*g* and 4*h* (second part) respectively, while *g* is a consequence of the definition (4.1) of u_h. Thus the proof of theorem *A* is complete, modulo the proofs of the remaining properties.

PROOF OF THEOREM B: First we show that $u = M$ in an open subset of D.

Applying (4.*g*), but to the integral

$$\int (-1)\, L_x^* G_h(x, y)\, dx \tag{4.2}$$

instead of the integral appearing in (4.1), we see that the above integral must be monotonically non-increasing with h. Together with (4.*f*) this implies that the integral

$$J_h = \int L_x^* G_h(x, y)\, dx \to 1$$

in a nondecreasing manner as $h \to \infty$, hence, $0 < J_h \leq 1$.

Now let us suppose that

$$\operatorname*{ess\,sup}_{D} u = \operatorname*{ess.\,sup}_{D_\lambda} u = M \tag{4.3}$$

where $\lambda > 0$. Then the same equality must hold for u^*, and there must exist a point z in D_λ with the property that

$$\overline{\lim_{x \to z}}\ u^*(x) \geq M. \tag{4.4}$$

As we have seen, u^* is upper-semi continuous, hence

$$u^*(z) \geq \overline{\lim_{x \to z}}\ u^*(x). \tag{4.5}$$

Since

$$u_h(z) = \int u(x)\, L_x^*\, G_h(x, z)\, dx \tag{4.6}$$

is nonincreasing with h, we have

$$u^*(z) \leq \int u(x)\, L_x^*\, G_h(x, z)\, dx \tag{4.7}$$

for some value of h. Combining (4.4), (4.5), (4.7) with the fact that $0 < J_h \leq 1$, and assuming $M \geq 0$, we obtain

$$0 \leq \int \left(u(x) - \frac{M}{J_h} \right) L_x^*\, G_h\, dx \leq \int (u(x) - M)\, L_x^*\, G_h(x, z)\, dx. \tag{4.8}$$

From (4.c) and the fact that $M \geq u(x)$ almost everywhere in D, it follows then that $u(x) = M$ in an open subset of D.

Thus we have shown that under the hypothesis of theorem B $u = M$ almost everywhere in an open subset of D. The proof will be complete if we invoke the following lemma.

Lemma: Assume the hypothesis of theorem B. If $u = M$ almost everywhere in an open subset of D, the $u = M$ almost everywhere in D.

This lemma is essentially theorem 2 in [2]. The proof given there is not difficult and carries over with only superficial modifications to the above lemma. Hence we shall not give it here.

Proof of (4c) We denote the linear transformation depending on x corresponding to the matrix $b_{ij} = b_{ij}(x)$ by B, and the vectors $\text{grad}_x\, g$ and $\text{grad}_x\, p$ by M and N respectively. Since B is positive definite, it has a positive square root, and we can let $\widetilde{M} = \sqrt{B}\, M$ and $\widetilde{N} = \sqrt{B}\, N$. Considering the quantity

$$G_h(x, y) = \int_{-\infty}^{\frac{g}{p}} \Phi(\alpha - h)\,(g - \alpha p)\, d\alpha \equiv F(g, p, h),$$

and letting F_g, F_p, etc. denote the appropriate partial derivatives, a more or less straight forward calculation (for details see [2] p. 765) shows that

$$L_x^* G_h = L_x^* p \cdot F_p + \sum_{i=1}^{n} (F_{gg} \widetilde{M}_i^2 + 2F_{gp} \widetilde{M}_i \widetilde{N}_i + F_{pp} \widetilde{N}_i^2) \tag{4.9}$$

with

$$F_g = \int_{-\infty}^{\frac{g}{p}} \Phi(\alpha - h)\, d\alpha \geq 0 \qquad F_p = -\int_{-\infty}^{\frac{g}{p}} \Phi(\alpha - h)\, \alpha\, d\alpha \leq 0$$

$$F_{gg} = \frac{1}{p} \Phi\left(\frac{g}{p} - h\right), \qquad F_{pp} = \frac{g^2}{p^3} \Phi\left(\frac{g}{p} - h\right), \qquad F_{gp} = \frac{g}{p^2} \Phi\left(\frac{g}{p} - h\right).$$

Since $F_{gg} F_{pp} - F_{gp}^2 = 0$, the form $F_{gg} \xi^2 + 2F_{gp} \xi\eta + F_{pp} \eta^2$ is semi-definite. Noticing that F_{gg} and F_{pp} are ≥ 0 it follows that this form is nonnegative, hence $L_x^* G_h \geq 0$. The above formula for F_p shows that $F_p = F_p(g, p, h) < 0$, provided $h_p \leq h < \frac{g}{p} < h + 1$. Hence $L_x^* G_h(x, y) > 0$ for x, y satisfying the above inequalities. To prove the second part of the assertion in 4c), we simply notice that all terms in formula (4.9) for $L_x^* G_h(x, y)$ are continuous in x, y jointly. This follows from the fact that $g, p, \operatorname{grad}_x g$ and $\operatorname{grad}_x p$ depend continuously in x and y jointly for $x \in D$, $y \in D_{2p}$, $|x - y| \geq \text{const} > 0$.

PROOF OF 4d: First suppose that the coefficients of L^* are sufficiently smooth, say C^∞. Then the prof of 4.c also shows that $L_y H = L_y G_h(x, y) \geq 0$, using the fact that $g(x, y)$ is also the Green's function of the operator L_y, hence $L_y g(x, y) = 0$ $(x \neq y)$. Now for the general case, we wish to show that

$$\int H_h(x, y) \quad L_y^* v(y)\, dy \geq 0 \tag{4.10}$$

for all nonnegative smooth v with compact support in $D_{2\varrho}$. To that effect, we approximate L_y^* by operators $\widetilde{L}_y^*$ with sufficiently smooth coefficients, and replace H_h by the corresponding function $\widetilde{H}_h$ associated with $\widetilde{L}^*$. Then it is easily shown that as the coefficients of $\widetilde{L}^*$ approach those of L^* in the C^q norm, $\widetilde{H}_h \to H_h$ uniformly for $x \in \overline{D}$, $y \in D_{2\varrho}$, and hence that

$$\int \widetilde{H}_h \widetilde{L}_y^* v(y)\, dy \to \int H_h L_y^* v(y)\, dy,$$

thus proving (4.10).

PROOF 4f: $\int L_x^* G_h(x, y)\, dx \longrightarrow 1$

Formula (4.9) can be rewritten as follows:

$$L_x^* G_h = - L_x^* p \int_{-\infty}^{\frac{g}{p}} \Phi(\alpha - h)\, \alpha\, d\alpha \tag{4.11}$$

$$+ \frac{1}{p} \Phi(\widehat{g} - h) \Sigma b_{ij}(x)\, g_{x_i} g_{x_j} + 2 \frac{g}{p^2} \Phi(\widehat{g} - h) \Sigma b_{ij}(x)\, g_{x_i} p_{x_j}$$

$$+ \frac{g^3}{p^3} \Phi(\widehat{g} - h) \Sigma b_{ij}\, p_{x_i} p_{x_j},$$

where $\widehat{g} = \dfrac{g}{p}$.

On integrating with respect to x, and letting $h \longrightarrow \infty$, it is easily seen that the integrals of all terms on the right hand side will approach zero (uniformly for $y \in D_{2p}$) except the integral

$$\int \frac{1}{p} \Sigma b_{ij}\, g_{x_i} g_{x_j} \Phi(\widehat{g} - h)\, dx.$$

Since we can make a linear transformation at each point [with determinant bounded away from 0 and ∞) which takes the principal part of L_x^* at $x = y$ into the Laplacean, we may assume that

$$b_{ij}(y) = \delta_{ij}.$$

Since from the Hölder continuity of the coefficients it follows that

$$\int \Sigma (b_{ij}(x) - \delta_{ij}) \frac{1}{p} \Phi(\widehat{g} - h)\, g_{x_i} g_{x_j}\, dx \longrightarrow 0 \quad \text{as} \quad h \longrightarrow \infty;$$

and since $p(x - y) \longrightarrow p(0) = 1$ as $x \longrightarrow y$, it remains for us to show that

$$\int \Phi(\widehat{g} - h)(\text{grad } g)^2\, dx \longrightarrow 1.$$

Denoting the fundamental solution to the Laplacian by

$$g_0 = \frac{1}{w(n-2)} r^{2-n}$$

where w is the surface area of the unit sphere in R^n, and where $r = |x - y|$, the last limit relation is equivalent to

$$I \equiv \int \Phi\,(\widehat{g} - h)\,(\text{grad}\; g_0)^2\, dx \longrightarrow 1.$$

(This equivalence follows from: $\text{grad}\,(g - g_0) = 0\,(r^{1-n+\alpha})$).

Focusing our attention on the last integral, we rewrite it as

$$I = -\frac{1}{w} \iint \Phi\,(\widehat{g} - h) \left(\frac{dg_0}{dr}\right)_{\theta=\text{const}} dr\, d\theta = \frac{1}{w} \int \left[\int \Phi\,(g - h)\, dg_0 \right] d\theta.$$

Letting $\dfrac{dg_0}{d\widehat{g}} = \left(\dfrac{\partial g_0}{\partial r}\right) \cdot \left(\dfrac{\partial r}{\partial \widehat{g}}\right)$, ($\theta$ fixed),

$$\int_{\widehat{g}=0}^{\infty} \Phi\,(\widehat{g} - h)\, \frac{dg_0}{d\widehat{g}}\, d\widehat{g} = \int_{-h}^{\infty} \Phi\,(u) \left(\frac{dg_0}{d\widehat{g}_0}\right)_{g=u+h} du.$$

Now, as $h \longrightarrow \infty$, $\left(\dfrac{dg_0}{dg}\right)_{g=u+h} \longrightarrow 1$ uniformly for $0 \leq u \leq 1$, and all θ on the unit sphere. Hence

$$I \longrightarrow \frac{1}{w} \int_{\theta} \int_{-\infty}^{-\infty} \Phi\,(u)\, du\, d\theta = 1.$$

That this limit is uniform for $y \in D_{2p}$ is easily checked.

PROOF OF $4h$: (continuity of u_h; smoothness).

We again start with the fundamental formula

$$(4.12) \qquad L_x^*\, G_h\,(x, y) = -\, L_x^*\, p \int_{-\infty}^{\frac{g}{p}} \Phi\,(\alpha - h)\, \alpha\, d\alpha$$

$$+ \Sigma\, \frac{1}{p}\, b_{ij}(x)\, \Phi\left(\frac{g}{p} - h\right) \left(g_{x_i} g_{x_j} + 2\, \frac{g}{p}\, g_{x_i} p_{x_j} + \frac{g^2}{p^2}\, p_{x_i} p_{x_j}\right).$$

First we wish to show that $u_h(y)$ is continuous for $y \in D_{2p}$. We recall that g_{x_i} and g vary continuously in x and y jointly for x bounded away from y. Since $\Phi\left(\dfrac{g}{p} - h\right)$ vanishes in a neighborhood of $x = y$, we see that the second group of terms in the above right hand side is continuous in x, y jointly for $y \in D_{2p}$, $x \in D$; a similar contention is valid for the first term, since the integral

$$\int_{-\infty}^{\frac{g}{q}} \Phi(\alpha - h)\, \alpha \, d\alpha$$

is constant in a neighborhood of $x = y$ (for fixed h). From this it follows that $u_h(y)$ is continuous.

Next we wish to study the effect of applying L_y to the right hand side of (4.12). To facilitate this operation, we first assume that the coefficients of the operator L^* are C^∞ functions. We see that L_y applied to the last group of terms gives rise to terms of the form ΦI_1, where I_1 is a sum of products of g, q and their derivatives in x and y up to and including the second order for g and the third order for p; and where Φ is a function of (x, y) defined in $\overline{D} \times D_{2\varrho}$ with support contained in the set:

$$h < g < h + 1,$$

depending C^∞ on $p(x, y)$, $g(x, y)$ and their derivatives up to order two. The application of L_y to the first term will yield, apart from terms of the type already described, terms containing the factor

$$\int_{-\infty}^{\frac{g}{p}} \Phi(\alpha)\, \alpha \, d\alpha.$$

These terms will have support contained in the set $g/p > h$, and be constant in the set $g/p > h + 1$.

To estimate the second derivatives of the function $u_h(y)$ it will be necessary to investigate the smoothness of the Green's function $g(x, y)$ in the variables x, y jointly. To that effect we notice that $g(x, y)$ satisfies the two equations

$$L_x^* g(x, y) = 0 \qquad (x \neq y) \qquad L_y g(x, y) = 0 \qquad (x \neq y)$$

hence, also the equation

$$(L_x^* + L_y)\, g(x, y) = 0 \qquad x \neq y.$$

Since the operator $L_x^* + L_y$ is elliptic, it follows from the Schauder estimates that in every region R in xy space, of the type:

$$\{x, y : 0 < C_1 \leq g(x, y) \leq C_2 < \infty, \qquad y \in D_{C_3}$$

(C_1, C_2 sufficiently large) an estimate of the following type holds:

$$| g(x, y) |_{2+\alpha, R} \leq C$$

where C depends only on C_1, C_2, C_3, the ellipticity of L, and the Hölder continuity of the coefficients of $L_x^* + L_y$. The latter, in turn depends only on $K \equiv | b_{ij} |_{2+\alpha, \bar{D}} + | b_i |_{1+\alpha, \bar{D}} + | b |_{\alpha, \bar{D}}$. Thus we see that

$$| L_y L_x^* G_h |_{\alpha, D \times D_{2\varrho}} \leq \text{constant},$$

which by the Schauder estimates implies

$$| L_x^* G_h |_{2+\alpha, y \in D_{2\varrho}} \leq \text{constant, uniformly in } x. \text{ Since,}$$

$$u_h(y) = \int u(x) L_x^* G_h(x, y)\, dx,$$

we see that

$$| u_h(y) |_{2+\alpha, y \in D_{2\varrho}} \leq \text{constant},$$

the constant depending only on K, h, p, the ellipticity constant and D. So far we have been assuming that the coefficients are C^∞. However if only $b_{ij} \in C^{2+\alpha}$, $b_i \in C^{1+\alpha}$, $b \in C^\alpha$, then we approximate the coefficients by C^∞ functions, to get the operator $L^{(\nu)}$ and in the definition of u_h replace G_h by the appropriate function corresponding to $L^{(\nu)}$, however we leave $u(x)$ the same. We thus get a sequence function $u_h^{(\nu)} \longrightarrow u_h$ uniformly, with $| u_h^{(\nu)} |_{2+\alpha, D_{2\varrho}}$ uniformly bounded, which implies that $u_h \in C^{2+\alpha}(D_{2\varrho})$.

PROOF OF (4.*i*) $L_y u_h(y) \geq 0$ weakly.

Suppose first that L^* has C^∞ coefficients and v is C^2. Then, if v has compact support in $D_{2\varrho}$

$$\int L_y^* v(y) u_h(y)\, dy = \int v(y) L_y \left(\int u(x) L_x^* G_h\, dx \right) dy$$

$$= \int v(y) \left(\int u(x) L_y L_x^* H_h(x, y)\, dx \right) dy$$

$$\int L_y^* v(y) u_h(y)\, dy = \int u(x) L_x^* \left(\int H_h(x, y) L_y^* v(y)\, dy \right) dx. \tag{4.13}$$

Next let us assume that the coefficients of L^* are only Hölder continuous, and that $v \in C^2$. We wish to show that in this instance too the last identity holds To see this let us keep u, v fixed, but approximate the coefficients of L^* in the $|\ \ |_\alpha$ norm by a sequence of smooth functions, so as to obtain the operator $L^{*(\nu)}$. There will be a corresponding Green's function $g^{(\nu)}$, and correspondingly, functions $G_h^{(\nu)}, H_h^{(\nu)}, u_h^{(\nu)}(y)$. Then an equation $(4.13)_\nu$ will hold, i. e. (4.13) with the obvious modifications. It is easy to check that

$$L_x^{*(\nu)} G_h^{(\nu)} \longrightarrow L_x^* G_h \text{ uniformly, that } u_h^{(\nu)} \longrightarrow u_h$$

uniformly and hence that the left hand side of $(4.13)_\nu \longrightarrow$ l. h. s. of (4.13), i. e.,

$$\int L_y^{*(\nu)} v(y) \cdot u_h^{(\nu)}(y)\, dy \longrightarrow \int L_y^* v(y)\, u_h(y)\, dy.$$

We turn to the right hand side of $(4.13)_\nu$:

From the form of the expression

$$(4.14) \qquad H_h^{(\nu)}(x, y) \equiv G_h^{(\nu)}(x, y) - g^{(\nu)}(x, y)$$

$$= g^{(\nu)} \int_{\frac{g^{(\nu)}}{p}}^{\infty} \Phi(\alpha - h)\, d\alpha - p \int_{-\infty}^{\frac{g^{(\nu)}}{p}} \Phi(\alpha - h)\, \alpha\, d\alpha,$$

it is easily seen that this expression approaches its limit $H_h(x, y)$ as $\nu \longrightarrow \infty$, in the $|\ \ |_{2+\alpha, D}$ norm (with respect to x) uniformly with respect to $y \in D_{2\varrho}$. Therefore the right hand side of $(4.13)_\nu$ approaches the right hand side of (4.13). The validity of (4.13) is thus established for the case of merely Hölder continuous coefficient of L^*.

Now if $v \geq 0$, the function of x

$$\int H_h L_y^* v(y)\, dy$$

has compact support in D, is $C^2(D)$ and is nonnegative by (4.d).

Invoking the assumption $Lu \geq 0$ (weakly), then enables us to conclude that the left hand side of (4.13) is nonnegative.

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