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# A THEOREM ON GENERALIZED ABSOLUTE RIESZ SUMMABILITY

by S. M. MAZHAR

1.1. Let  $\sum a_n$  be a given infinite series and  $\{\lambda_n\}$  be an increasing sequence of positive numbers tending to infinity with  $n$ . We write

$$\begin{aligned} A_\lambda(\omega) &= A_\lambda^0(\omega) = \sum_{\lambda_n < \omega} a_n, & \omega > \lambda_1, \\ &= 0 & \omega \leq \lambda_1, \end{aligned}$$

$$\begin{aligned} A_\lambda^\alpha(\omega) &= \sum_{\lambda_n < \omega} (\omega - \lambda_n)^\alpha a_n, & \alpha > 0, \\ &= \int_{\lambda_1}^{\omega} (\omega - t)^\alpha dA_\lambda(t) \end{aligned}$$

and

$$C_\lambda^\alpha(\omega) = A_\lambda^\alpha(\omega) / \omega^\alpha.$$

A series  $\sum a_n$  is said to be summable by Riesz means of « type »  $\lambda$  and « order »  $\alpha$  or, simply, summable  $(R, \lambda, \alpha)$ ,  $\alpha \geq 0$  to the sum  $s$  if

$$\lim_{\omega \rightarrow \infty} C_\lambda^\alpha(\omega) = s,$$

where  $s$  is any finite number [8].

The series  $\sum a_n$  is said to be summable  $|R, \lambda, \alpha|$ ,  $\alpha \geq 0$ , if the function  $C_\lambda^\alpha(\omega) \in BV(h, \infty)$ , that is to say, if

$$\int_h^\infty \left| \frac{d}{d\omega} C_\lambda^\alpha(\omega) \right| d\omega < \infty,$$

where  $h$  is a finite positive number [6, 7].

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Similarly the series  $\Sigma a_n$  is said to be summable  $|R, \lambda, \alpha|_k$ ,  $\alpha > 0$ ,  $k \geq 1$ ,  $\alpha k' > 1$ ,  $\frac{1}{k} + \frac{1}{k'} = 1$ , if the integral

$$\int_h^\infty \omega^{k-1} \left| \frac{d}{d\omega} C_\lambda^\alpha(\omega) \right|^k d\omega$$

is convergent [4].<sup>(1)</sup>

1.2. In 1915 Hardy and Riesz [2] proved the following interesting theorem concerning the Riesz summability of an infinite series.

**THEOREM A.** If  $\lambda_1 > 0$  and  $\Sigma a_n$  is summable  $(R, \lambda, \alpha)$ , then the series  $\Sigma a_n \lambda_n^{-\alpha}$  is summable  $(R, l, \alpha)$ , where  $l_n = e^{\lambda_n}$ .

Analogous problem was considered by Tatchell [9] for absolute Riesz summability. He proved the following theorem :

**THEOREM B.** If  $\alpha \geq 0$  and  $\Sigma a_n$  is summable  $|R, \lambda, \alpha|$ , then  $\Sigma a_n \lambda_n^{-\alpha}$  is summable  $|R, l, \alpha|$ , where  $l_n = e^{\lambda_n}$ .<sup>(2)</sup>

The object of the present note is to establish the corresponding result for the generalized absolute Riesz summability, namely summability  $|R, \lambda, \alpha|_k$  for integral values of  $\alpha$ . In a subsequent note it is proposed to discuss the non-integral case.

2.1. In what follows we shall prove the following theorem :

**THEOREM.** If  $\alpha$  is a positive integer and  $\Sigma a_n$  is summable  $|R, \lambda, \alpha|_k$ , then  $\Sigma a_n \lambda_n^{-\alpha+1/k'}$  is summable  $|R, l, \alpha|_k$ , where  $l_n = e^{\lambda_n}$ ,  $k \geq 1$ ,  $\lambda_1 > 0$  and  $\frac{1}{k} + \frac{1}{k'} = 1$ .

It is evident that for  $k = 1$  our theorem includes the above theorem of Tatchell for integral values of  $\alpha$ .

2.2. We require the following lemmas for the proof of this theorem :

**LEMMA 1** [3]. If  $\alpha > 0$  and  $B_\alpha(\omega)$  is the Rieszian sum of type  $\lambda$  and order  $\alpha$  of the series  $\Sigma a_n \lambda_n$ , then

$$\omega^{\alpha+1} \frac{d}{d\omega} C_\lambda^\alpha(\omega) = \alpha B_{\alpha-1}(\omega) = \frac{d}{d\omega} B_\alpha(\omega).$$

<sup>(1)</sup> See also Borwein [1] who defined the summability  $|R, n, \alpha|_k$ .

<sup>(2)</sup> This theorem for the case  $\alpha = 1$  is due to Mohanty [5].

LEMMA 2 [2]. *If  $l$  is a positive integer, then*

$$A_\lambda(t) = \frac{1}{l!} \left(\frac{d}{dt}\right)^l A_\lambda^l(t).$$

3.1. PROOF OF THE THEOREM. Under the hypothesis of the theorem we have by Lemma 1

$$(3.1.1) \quad \int_{\lambda_1}^{\infty} \omega^{-(1+\alpha k)} |B_{\alpha-1}(\omega)|^k d\omega < \infty$$

and we have to establish the convergence of the integral

$$(3.1.2) \quad \int_{\lambda_1}^{\infty} \omega^{-(1+\alpha k)} |E_{\alpha-1}(\omega)|^k d\omega,$$

where  $E_{\alpha-1}(\omega)$  is the Rieszian sum of order  $(\alpha - 1)$  and of type  $l$ , of the series  $\sum a_n \lambda_n^{-\alpha+1/k'} e^{\lambda_n}$ .

By writing  $\omega = e^x$  in the above integral (3.1.2) we find that the required condition can also be written in the form

$$(3.1.3) \quad \int_{\lambda_1}^{\infty} e^{-\alpha x k} |E_{\alpha-1}(e^x)|^k dx < \infty.$$

We have

$$\begin{aligned} E_{\alpha-1}(e^x) &= \int_{\lambda_1}^{e^x} (e^x - u)^{\alpha-1} dE(u) \\ &= \int_{\lambda_1}^x (e^x - e^t)^{\alpha-1} dE(e^t) \\ &= \int_{\lambda_1}^x (e^x - e^t)^{\alpha-1} e^t t^{-\alpha-1/k} dB(t) \\ &= [(e^x - e^t)^{\alpha-1} e^t t^{-\alpha-1/k} B(t)]_{\lambda_1}^x \\ &\quad - \int_{\lambda_1}^x B(t) \frac{d}{dt} \{(e^x - e^t)^{\alpha-1} e^t t^{-\alpha-1/k}\} dt. \end{aligned}$$

Applying Lemma 2 and integrating  $(\alpha - 1)$  times we have

$$\begin{aligned} E_{\alpha-1}(e^x) &= [(e^x - e^t)^{\alpha-1} e^t \cdot t^{-\alpha-1/k} B(t)]_{\lambda_1}^x + \\ &+ C^{(3)} \left[ \sum_{i=1}^{\alpha-1} (-1)^i \left( \frac{d}{dt} \right)^{\alpha-i-1} B_{\alpha-1}(t) \left( \frac{d}{dt} \right)^i \{(e^x - e^t)^{\alpha-1} e^t t^{-\alpha-1/k}\} \right]_{\lambda_1}^x \\ &+ C \int_{\lambda_1}^x B_{\alpha-1}(t) \left( \frac{d}{dt} \right)^\alpha \{(e^x - e^t)^{\alpha-1} e^t t^{-\alpha-1/k}\} dt \\ &= C B_{\alpha-1}(x) e^{\alpha x} x^{-\alpha-1/k} + C \int_{\lambda_1}^x B_{\alpha-1}(t) \left( \frac{d}{dt} \right)^\alpha \{(e^x - e^t)^{\alpha-1} e^t t^{-\alpha-1/k}\} dt \\ &= L_1 + L_2, \text{ say.} \end{aligned}$$

Since

$$\int_{\lambda_1}^{\infty} e^{-xak} |L_1|^k dx \leq C \int_{\lambda_1}^{\infty} x^{-(1+ak)} |B_{\alpha-1}(x)|^k dx < \infty,$$

it is, therefore, by virtue of Minkowski's inequality sufficient to prove that

$$\int_{\lambda_1}^{\infty} e^{-xak} |L_2|^k dx < \infty.$$

Now

$$\begin{aligned} L_2 &= O \left\{ \int_{\lambda_1}^x |B_{\alpha-1}(t)| t^{-\alpha-1/k} e^{at} dt \right\} + O \left\{ \int_{\lambda_1}^x |B_{\alpha-1}(t)| \sum_{i=1}^{\alpha-1} e^{ix} e^{(\alpha-i)t} t^{-\alpha-1/k} dt \right\} \\ &= L_{21} + L_{22}. \end{aligned}$$

Applying Hölder's inequality, we observe that

$$\int_{\lambda_1}^{\infty} e^{-xak} |L_{21}|^k dx = O \left\{ \int_{\lambda_1}^{\infty} e^{-xak} \int_{\lambda_1}^x |B_{\alpha-1}(t)|^k t^{-(1+ak)} e^{at} e^{\alpha x(k-1)} dt dx \right\}$$

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(3) Where  $C$  denotes a constant not necessarily the same at each occurrence.

$$\begin{aligned}
 &= O \left\{ \int_{\lambda_1}^{\infty} t^{-(1+ak)} |B_{\alpha-1}(t)|^k dt \right\} \\
 &= O(1).
 \end{aligned}$$

Also, in order to show that

$$\int_{\lambda_1}^{\infty} e^{-xak} |L_{22}|^k dx < \infty$$

it is sufficient to prove the convergence of the integral

$$\int_{\lambda_1}^{\infty} e^{-xak} \left\{ \int_{\lambda_1}^x |B_{\alpha-1}(t)| e^{ix} e^{(a-i)t} t^{-a-1/k} dt \right\}^k dx$$

for  $1 \leq i \leq \alpha - 1$ .

Using Hölder's inequality we find that the above integral is

$$\begin{aligned}
 &< \int_{\lambda_1}^{\infty} e^{-xak+ixk} \int_{\lambda_1}^x |B_{\alpha-1}(t)|^k t^{-(1+ak)} e^{(a-i)t} dt e^{(a-i)x(k-1)} dx \\
 &= C \int_{\lambda_1}^{\infty} t^{-(1+ak)} |B_{\alpha-1}(t)|^k e^{(a-i)t} \int_t^{\infty} e^{x(i-\alpha)} dx dt \\
 &= C \int_{\lambda_1}^{\infty} t^{-(1+ak)} |B_{\alpha-1}(t)|^k dt < \infty,
 \end{aligned}$$

by hypothesis.

This completes the proof of the theorem.

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