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NONLOCAL ELLIPTIC BOUNDARY
VALUE PROBLEMS (*)

MARTIN SCHECHTER

1. Introduction.

In [13, 14] Beals considers boundary value problems of the form

\begin{align}
Au &= f \quad \text{in } \Omega \\
B_j u &= \sum_{k=1}^{r} M_{jk} C_k u \quad \text{on } \partial\Omega, \quad 1 \leq j \leq r,
\end{align}

where $A$ is an elliptic operator of order $m = 2r$ in a domain $\Omega \subset \mathbb{R}^n$, $\{B_j\}_{j=1}^{r}$, $\{C_j\}_{j=1}^{r}$ are sets of differential boundary operators and the $M_{jk}$ are arbitrary linear operators bounded in a certain sense. He considers the problem for those $u \in L^2(\Omega)$ for which $Au \in L^2(\Omega)$ and all derivatives $< m$ are in $L^2(\partial\Omega)$ and such that (1.2) holds. Under suitable hypotheses he proves that the operator $A(M)$ thus defined is closed (and a Fredholm operator for $\Omega$ bounded) and that its adjoint is of the form

\begin{align}
A' v &= g \quad \text{in } \Omega, \\
B'_j v &= \sum_{k=1}^{r} M'_{kj} C'_k v \quad \text{on } \partial\Omega, \quad 1 \leq j \leq r,
\end{align}

where $A'$ is the formal adjoint of $A$ and the $B'_j$, $C'_k$, $M'_{kj}$ are related to the $B_j$, $C_k$, $M_{jk}$ by integration by parts. Such problems are called « nonlocal », since the $M_{jk}$ need not be differential operators.

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For second order, self-adjoint operators, previous work along these lines was done by Calkin [15], Bade and Freeman [16] and Freeman [17]. When the $M_{jk}$ are singular integral operators, problem (1,1,2) was studied by Dynin [18] and Agronovich and Dynin [19]. Abstract boundary value problems were considered by Visik [20], Hörmander [21], Browder [22-25], Peetre [26], Phillips [27], and Schechter [2,28].

In this paper we extend some of the results of Beals (but restrict ourselves to bounded domains). We consider the problem (1.1,2) in a slightly more general framework, namely the $H^{t,p}(\Omega)$ spaces for $t$ real and $1 < p < \infty$. The boundary values of the functions and their derivatives of order $< m$ are taken in the $W^{s,p}(\partial\Omega)$ spaces, with $t \leq s + 1/p \leq t + m$. This allows a bit more latitude in applications. It also allows one to relax the assumptions on the $M_{jk}$.

We also consider the more general type of problem where (1.2) is replaced by

$$B_j u = \sum_{k=1}^{m} L_{jk} \frac{\partial^{k-1} u}{\partial n^{k-1}} \quad \text{on} \quad \partial\Omega, \quad 1 \leq j \leq r,$$

where $\partial^j u/\partial n^j$ denotes the normal derivative of order $l$ on $\partial\Omega$. We investigate the regularity of solutions of (1.1) (1.5) under varying assumptions on the $L_{jk}$ and obtain a priori estimates. Our methods make use of $L^p$ estimates of [1, 4, 6].

As an example, let $A$ be a second order operator. Consider the closure in $L^p(\Omega)$ of $A$ acting on those $u \in C^\infty(\Omega)$ satisfying boundary conditions of the form

$$\frac{\partial u}{\partial n} = R_1 u + R_2 \frac{\partial u}{\partial n} \quad \text{on} \quad \partial\Omega,$$

where the $R_i$ are arbitrary linear operators. Sufficient conditions are given for the operator $A(R)$ thus defined to be Fredholm (and therefore closed) and for the solutions of $A(R) u = 0$ to be smooth. When $R_2 = 0$, we give sufficient conditions for the adjoint of $A(R)$ to be the closure in $L^{p'}(\Omega)$, $p' = p/(1 - p)$, of the formal adjoint $A'$ of $A$ restricted to those $v \in C^\infty(\overline{\Omega})$ such that

$$\frac{\partial v}{\partial n} = R_1^* v$$

where $R_1^*$ is the adjoint of $R_1$ and $\partial v/\partial n$ is a first order derivative obtained by integration by parts. Similar results hold if (1.6) is replaced by

$$u = S_1 u + S_2 \frac{\partial u}{\partial n} \quad \text{on} \quad \partial\Omega.$$
2. Main results.

Let \( \Omega \) be a bounded domain in Euclidean \( n \)-dimensional space \( E^n \) with boundary \( \partial \Omega \) of class \( C^\infty \). Throughout the paper we shall assume that \( A \) is a properly elliptic operator of order \( m = 2r \) with coefficients in \( C^\infty (\bar{\Omega}) \), where \( \bar{\Omega} \) denotes the closure of \( \Omega \) (for definitions for all terms we refer to [1]). \( \{B_j\}_{j=1}^r \) will denote a normal set of differential operators of orders \( < m \) with coefficients in \( C^\infty (\partial \Omega) \). This means that the orders of the \( B_j \) are distinct and that \( \partial \Omega \) is nowhere characteristic for any of them. We shall also assume that \( \{B_j\}_{j=1}^r \) covers \( A \).

It is convenient to discuss boundary value problems for \( A \) within the framework of the \( H^{s,p} (\Omega) \) spaces, \( s, p \) real, \( 1 < p < \infty \). We give brief definitions here; for further details we refer to [2]. The space \( H^{s,p} (E^n) \) is the completion of \( C_0^\infty (E^n) \) with respect to the norm given by

\[
\| u \|_{E^n} = \left[ \int | \mathcal{F}^{-1} (1 + | \xi |^2)^{s/2} \mathcal{F} u |^p \ dx \right]^{1/p}
\]

where \( \mathcal{F} \) denotes the Fourier transform and \( C_0^\infty (E^n) \) is the set of infinitely differentiable functions on \( E^n \) with compact supports. For \( s \geq 0 \) we let \( H^{s,p} (\Omega) \) denote the restrictions to \( \Omega \) of functions in \( H^{s,p} (E^n) \) with the norm

\[
\| u \|_{s,p} = \| gL u \| \| v \|_{s,p}, \quad v = u \text{ on } \Omega.
\]

For \( s < 0 \), \( u \in C^\infty (\bar{\Omega}) \) we set

\[
\| u \|_{s,p} = \int_{\Omega} \frac{|(u,v)|}{\| v \|_{s,p}},
\]

where \( (u,v) \) is the scalar product in \( L^2 (\Omega) \) and \( p' = p/(p - 1) \). We then complete \( C^\infty (\bar{\Omega}) \) with respect to this norm to obtain \( H^{s,p} (\Omega) \).
For \( s > 0 \) we let \( W^{s,p}(\partial \Omega) \) denote the restrictions to \( \partial \Omega \) of functions in \( H^{s+1/p, p}(\Omega) \) with norm
\[
\langle g \rangle_{s,p} = g|b|_v \| v \|_{s+1/p, p}, \quad v = g \text{ on } \partial \Omega.
\]

It follows from the results of [3] that they are Banach spaces (cf. [2,4]). Moreover for \( g \in C^\infty(\partial \Omega) \) we set
\[
\langle g \rangle_{0,p} = \lim_{0 < s \to 0} \langle g \rangle_{s,p}.
\]

This limit exists and gives a norm (cf. [4]). For \( s < 0 \) we set
\[
\langle g \rangle_{s,p} = \inf_{h \in C^\infty(\partial \Omega)} \frac{\langle g, h \rangle}{\langle h \rangle_{s-1, p}},
\]
where \( \langle g, h \rangle \) denotes the \( L^2(\partial \Omega) \) scalar product. For \( s \leq 0 \) we let \( W^{s,p}(\partial \Omega) \) denote the completions of \( C^\infty(\partial \Omega) \) with respect to these norms.

For \( u \in C^\infty(\overline{\Omega}) \) and \( s \) real we introduce the norm
\[
\| u \|_{s,p} = \| u \|_{s,p} + \| Au \|_{s-m, p}
\]
and denote the completion of \( C^\infty(\overline{\Omega}) \) with respect to this norm by \( H^{s,p}_A(\Omega) \).

Let \( \gamma_l \) denote the normal derivative of order \( l \) on \( \partial \Omega \). We shall show that for \( l < m \), \( \gamma_l \) can be defined for elements of \( H^{s,p}_A(\Omega) \).

**Lemma 2.1.** For each \( s \) there is a constant \( K \) such that
\[
\sum_{l=0}^{m-1} \langle \gamma_l u \rangle_{s-1, p} \leq K \| u \|_{s,p}
\]
for all \( u \in C^\infty(\overline{\Omega}) \).

From the lemma we see that the mapping \( \gamma = \{\gamma_0, \ldots, \gamma_{m-1}\} \) can be extended by continuity to a bounded mapping from \( H^{s,p}_A(\Omega) \) to the space
\[
E_{s, p} = \bigoplus_{l=0}^{m-1} W^{s-1, p}(\partial \Omega).
\]
Similarly the mapping \( B = \{B_1, \ldots, B_r\} \) can be extended to a bounded mapping from \( H^{s,p}_A(\Omega) \) to
\[
F_{s, p} = \bigoplus_{j=1}^{l} W^{s-m, p}(\partial \Omega),
\]
where $m_j$ is the order of $B_j$. We also extend $A$ to be a mapping from $H^s_{A_p}(\Omega)$ to $H^{s-m}_{A_p}(\Omega)$.

Suppose we are given an $rxm$ matrix $L = (L_{ij})$ of operators such that $L_{ij}$ is a linear operator from $W^{s-i,j}_{1,q}(\partial \Omega)$ to $W^{s-j}_{1,q}(\partial \Omega)$. Then the (matrix) operator $L$ is a linear map from $E_{s,p}$ to $F_{s,p}$. Let $t$ be a number satisfying $s - m \leq t \leq s$. We define the operator $A_{s,p}(L)$ as the restriction of $A$ to those $u \in H^s_{A_p}(\Omega)$ such that $Au \in H^{t}_{A_p}(\Omega)$ and

$$Bu = L \gamma u.$$ 

We consider $A_{s,p}(L)$ as an operator in $H^1(\Omega)$.

Let $T$ be a linear operator on a Banach space $X$. It is called a semi-Fredholm operator if

1) the domain $D(T)$ of $T$ is dense in $X$
2) $T$ is closed
3) the null space $N(T)$ of $T$ has finite dimension
4) the range $R(T)$ of $T$ is closed in $X$.

It is called a Fredholm operator if, in addition,
5) the codimension of $R(T)$ in $X$ is finite.

**Theorem 2.1.** If there are constants $c < 1$, $c$, such that

$$\|L \gamma u\|_{E_{s,p}} \leq \epsilon \|Bu\|_{E_{s,p}} + c \|Au\|_{s,p} + \|u\|_{s-m,p},$$

holds for all $u \in C^\infty(\overline{\Omega})$, then $A_{s,p}(L)$ is a semi-Fredholm operator.

The proof of Theorem 2.1 can be made to depend on

**Theorem 2.2.** If (2.2) holds, then

$$\|u\|_{s,p} \leq C \|Au\|_{s,p} + \|u\|_{s-m,p} + \|(B - L \gamma) u\|_{E_{s,p}}$$

for all $u \in C^\infty(\overline{\Omega})$.

Another criterion is given by

**Theorem 2.3.** If $L$ is a compact operator from $E_{s,p}$ to $F_{s,p}$, then (2.3) holds and hence $A_{s,p}(L)$ is a semi-Fredholm operator. In particular, this is true if $L$ maps $E_{s-r,p}$ boundedly into $F_{s,p}$ or $E_{s,p}$ boundedly into $F_{s+r,p}$ for some $\epsilon > 0$.

**Theorem 2.4.** Assume that $L$ maps $E_{s,p}$ into $F_{s,q}$, $1 < q < \infty$. If $u \in H^s_{A_p}(\Omega)$, $Au \in H^{s-m,q}(\Omega)$ and $Bu = L \gamma u$ then $u \in H^{s,q}_{A_p}(\Omega)$. 
THEOREM 2.5. If there is an \( \varepsilon > 0 \) such that \( L \) maps \( E_{\sigma, p} \) into \( F_{\nu+\varepsilon, p} \) for \( \sigma \leq \varepsilon \leq t+m-\varepsilon \), then \( A_{\sigma, p}(L) = A_{t+m, p}(L) \).

The dual space of \( F_{\nu, p} \) is

\[
F_{\nu, p}^* = \prod_{j=1}^{r} W^{m_{j-1}+1/p', p'} (\partial \Omega)
\]

(cf. [4,6]). A similar formula gives \( E_{\nu, p}^* \).

THEOREM 2.6. Suppose \( \sigma > \nu \) and that for some \( \varepsilon > 0 \) \( L \) maps \( E_{\nu, p} \) boundedly into \( F_{\sigma+\varepsilon, p} \) for \( \sigma \leq \nu \leq \sigma - \varepsilon \). Assume further that \( f \in H^{m-\nu, p'}(\Omega), \ G = g_1, \ldots, g_r \in F_{\sigma, p}^* \) and

\[
(2.4) \quad |(f, Au) + \langle G, Bu - L \gamma u \rangle| \leq c_0 \|u\|_{s, p}
\]

for all \( u \in C^\infty(\overline{\Omega}) \), where \( \langle \cdot, \cdot \rangle \) denotes duality between \( F_{\nu, p} \) and \( F_{\sigma, p}^* \). Then \( f \in H^{m-\nu, p'}(\Omega), G \in F_{\sigma, p}^* \) and

\[
(2.5) \quad \|f\|_{m-\nu, p'} + \|G\|_{F_{\sigma, p}^*} \leq \text{const.} (c_0 + \|f\|_{m-\nu, p'} + \|G\|_{F_{\sigma, p}^*})
\]

COROLLARY 2.1. If \( L \) maps \( E_{\sigma, p} \) boundedly into \( F_{\sigma+\varepsilon, p} \) for some \( \varepsilon > 0 \) and each real \( \sigma \) and

\[
(2.6) \quad (f, Au) + \langle G, Bu - L \gamma u \rangle = 0
\]

for all \( u \in C^\infty(\overline{\Omega}) \), then \( f \in C^\infty(\overline{\Omega}) \) and each \( g_j \in C^\infty(\partial \Omega), 1 \leq j \leq r \). Moreover the set of all such \( f, G \) is finite dimensional.

For Theorems 2.7-2.10 we assume that there is an \( \varepsilon > 0 \) such that \( L \) is a bounded mapping from \( E_{\sigma, p} \) to \( F_{\sigma+\varepsilon, p} \) for each real \( \sigma \).

Let \( V(L) \) denote the set of those \( u \in C^\infty(\overline{\Omega}) \) satisfying (2.1) and \( V(L)' \) the set of those \( v \in C^\infty(\overline{\Omega}) \) satisfying

\[
(Au, v) = (u, A' v)
\]

for all \( u \in V(L) \), where \( A' \) is the formal adjoint of \( A \). By \( N(A(L)) \) [resp. \( N(A'(L)) \)] we shall denote the set of those \( u \in V(L) \) [resp. \( v \in V(L)' \)] which satisfy \( Au = 0 \) [resp. \( A' v = 0 \)]. We have

THEOREM 2.7. For each real \( \sigma \)

\[
N(A_{\sigma, p}(L)) = N(A(L)).
\]
Let $\tilde{N}$ denote the set of those $h \in C^\infty(\Omega)$ for which there is a $G \in C^\infty(\partial\Omega)$ such that
\begin{equation}
(h, Au) + \langle G, Bu - Lyu \rangle = 0
\end{equation}
for all $u \in C^\infty(\tilde{\Omega})$. By Corollary 2.1, $\tilde{N}$ is finite dimensional. Clearly $\tilde{N} \subseteq N(A(L'))$. We shall prove

**Theorem 2.8.** $\tilde{N} = N(A(L'))$. Hence the latter is finite dimensional.

In proving Theorem 2.8 we shall make use of

**Theorem 2.9.** $R(A_{s,p}(L))$ consists of those $f \in H^{s,p}(\Omega)$ which are orthogonal to $\tilde{N}$, i.e., which satisfy $(f,h) = 0$ for all $h \in \tilde{N}$.

**Theorem 2.10.** If $v \in H^{s,p}(\Omega)$ for some $\sigma$ and
\begin{equation}
(v, Au) = 0
\end{equation}
for all $u \in V(L)$, then $v \in \tilde{N}$.

**Corollary 2.2.** If and
\begin{equation}
(\psi, Au) = (f, u)
\end{equation}
for all $u \in V(L)$, then $v \in H^{s,m,p}(\Omega)$.

**Corollary 2.3.** $A_{s,p}(L)$ is a Fredholm operator.

Let $v_1, \ldots, v_r$ be the complementary set of the $m_1$ among the integers $0, \ldots, m - 1$. Let $\{C_k\}_{k=1}^r$ be any normal set of differential boundary operators with coefficients in $C^\infty(\partial\Omega)$ and such that the order of $C_k$ is $\nu_k$. Then there are normal sets $\{B^r_{ij}\}_{i,j=1}^r$ and $\{C_k\}_{k=1}^r$ such that
\begin{equation}
(Au, v) - (u, A' v) + \sum_{j=1}^r \langle B_{ij} u, C_{ij} v \rangle - \sum_{k=1}^r \langle C_k u, B^r_{kj} v \rangle = 0
\end{equation}
holds for $u, v \in C^\infty(\partial\Omega)$ (cf. [5,1]). The order of $B^r_{ij}$ in $m - \nu_j - 1$, while that of $C_{ij}$ is $m - m_j - 1$. We set $B' = \{B'_{1}, \ldots, B'_{r}\}$, $C = \{C_1, \ldots, C_r\}$, $C' = \{C'_{1}, \ldots, C'_{r}\}$.

Let $M = (M_{jk})$ be an $r \times r$ matrix of linear operators such that $M_{jk}$ maps $W^{s-k-1/p, p}(\partial\Omega)$ into $W^{s-m_j-1/p, p}(\partial\Omega)$. 

Thus $M$ maps $J_{s,p}$ into $F_{s,p}$, where

$$J_{s,p} = \prod_{k=1}^{r} W^{s-k-1/p',p}(\partial \Omega).$$

By expressing each $C_i$ in terms of normal and tangential derivatives on $\partial \Omega$, we obtain a unique operator $L_1$ from $E_{s,p}$ to $F_{s,p}$ such that

$$L_1 \gamma = MC.$$

We have

**Proposition 2.1.** If $s = t + m$ and (2.3) holds, then $A_{s,p}(L_1)$ is the closure of $A$ in $H^{s,p}(\Omega)$ defined for those $u \in C^\infty(\Omega)$ satisfying $(B - MC) u \equiv 0$ on $\partial \Omega$.

**Remark.** By Theorem 2.3, the inequality (2.3) holds when $M$ is a compact operator from $J_{s,p}$ to $F_{s,p}$.

We now assume that there is a number $\tau$ such that $t \leq \tau \leq s$ and such that $M$ is a bounded operator from $J_{s,p}$ to $F_{s,p}$. If $\tau < s$, it follows that $M$ is compact from $J_{s,p}$ to $F_{s,p}$. If $\tau = s$, we assume this. We set

$$L_1' = M^* C',$$

where $M^*$ is the adjoint of $M$. It follows that $L_1'$ is a bounded operator from $E_{s,p}$ to $F_{s,p}$ while $L_1'$ is bounded from $F_{s,p}$ to $E_{s,p}$. For $-t \leq \sigma \leq \leq m - t$, $1 < q < \infty$, we let $A_{\sigma,q}'(L_1)$ denote the restriction of $A'$ to those $v \in H^{\sigma,q}_s(\Omega)$ for which $A' v \in H^{-t,q}(\Omega)$ and

(2.11) $$B' v = M^* C' v.$$

We have

**Theorem. 2.11.** Under the above hypotheses,

(2.12) $$(A_{s,p}(L_1))^* = A'_{m-t,p'}(L_1')$$

and

(2.13) $$(A'_{m-t,p'}(L_1))^* = A_{s,p}(L_1).$$

The case considered by Beals [13, 14] is $t = 0$, $p = 2$, $s = m - 1/p'$, $\tau = 1/p$ (we have been able to avoid the consideration of the operator $S$ of his papers). By known interpolation theorems (cf. [11, 6]) the assumption that $L$ maps $E_{s,p}$ boundedly into $F_{s+p',p}$ for all $\sigma$ need only be verified for sequences $o_k \to \infty$, $o_k'' \to -\infty$. 
3. Background Material.

We now list those known results which will be used in our proofs.

**THEOREM 3.1.** For each number $q$ there is a constant $C$ such that

$$
\| u \|_{q,p} \leq C \left( \| Au \|_{q-m,p} + \| u \|_{q-m,p} + \| Bu \|_{p,p} \right)
$$

for all $u \in C^\infty(\overline{\Omega})$.

Inequality (3.1) is a weaker form of Theorem 2.1 of [6].

**THEOREM 3.2.** For $\sigma > q$, the unit sphere in $H^{q,p}(\Omega)$ is conditionally compact in $H^{\sigma,p}(\Omega)$. The same is true for the spaces $W^{\sigma,p}(\Omega)$.

For $q = 0$ and $\sigma$ positive and large, Theorem 3.2 follows easily from Sobolev's Lemma. For the other cases one applies an abstract interpolation result of Lions Peetre [7, Theorem 2.3, p. 38].

**THEOREM 3.3.** If $f \in H^{m-q,p'}(\Omega)$, $G \in F^*_p(p)$ and

$$
\langle f, Au \rangle + \langle G, Bu \rangle \leq c_1 \| u \|_{q,p}
$$

for all $u \in C^\infty(\overline{\Omega})$, then $f \in H^{m-q,p'}(\Omega)$, $G \in F^*_p(p)$ and

$$
\| f \|_{m-q,p'} + \| G \|_{F^*_p(p)} \leq \text{const.} \left( c_1 + \| f \|_{m-q,p} \right)
$$

Theorem 3.3. follows from Theorem 2.1 of [4]. (The term $\| Au \|_{q-m,p}$ was missing from the right hand side of the inequality corresponding to (3.2). However, one checks easily from the proof given there that it could have been included.)

**THEOREM 3.4.** For each set $\Phi = \{ \Phi_1, \ldots, \Phi_r \}$ of functions in $C^\infty(\partial \Omega)$ there is a $u \in C^\infty(\overline{\Omega})$ such that

$$
Bu = \Phi
$$

and for each real $q$

$$
K^{-1} \| Au \|_{q+m,p} \leq \| u \|_{q,p} \leq K \| \Phi \|_{F^*_p(p)},
$$

where the constant $K$ does not depend on $\Phi$ or $u$. 
**Proof.** Consider the boundary problem

\[(3.6) \quad (A' A + 1) u = 0; \]

\[(3.7) \quad B u = \Phi, \quad B' A u = 0.\]

By (2.10)

\[(3.8) \quad \langle A w, A v \rangle - \langle v, A' A v \rangle = \langle B w, C' A v \rangle - \langle C w, B' A v \rangle\]

for all \(w, v \in C^\infty (\overline{\Omega})\). From this one easily checks that the problem (3.6, 7) is self-adjoint. Moreover, it is a well posed elliptic boundary value problem. (Here we make use of the fact that \(B'\) covers \(A'\) [8, 1]). In addition, when \(\Phi = 0\) we have by (3.8).

\[
\| A u \|^2 + \| u \|^2 = (u, (A' A + 1) u) = 0,
\]

showing that \(u = 0\). Applying the theory of such problems, we see that for each \(\Phi \in C^\infty (\partial \Omega)\) there is a unique solution \(u \in C^\infty (\overline{\Omega})\) of (3.6, 7) (cf. [1]). We can also apply Theorem 3.1 to this problem, taking into consideration the fact that the term \(\| u \|_{l - m, p}\) may be dropped in (3.1) when there is uniqueness. Thus we have for each \(\varphi\)

\[
\| u \|_{\varphi, p} \leq K \| \Phi \|_{\varphi, p},
\]

where the constant \(K\) does not depend on \(\Phi\) or \(u\). We now note that \(A u\) is a solution of

\[
A' w = -u, \quad B' w = 0.
\]

Applying Theorem 3.1 to this problem we obtain

\[(3.9) \quad \| A u \|_{m+\varphi, p} \leq C (\| u \|_{\varphi, p} + \| A u \|_{\varphi, p})\]

for each \(\varphi\). We claim that this implies

\[(3.10) \quad \| A u \|_{m+\varphi, p} \leq K \| u \|_{\varphi, p}.\]

For otherwise there would be a sequence \([u_k]\) of functions \(u_k \in C^\infty (\overline{\Omega})\) satisfying \((A' A + 1) u_k = 0, B' A u_k = 0\) such that

\[
\| A u_k \|_{m+\varphi, p} = 1, \quad \| u_k \|_{\varphi, p} \to 0.
\]
By Theorem 3.2 there is a subsequence (also denoted by \([u_k]\)) for which 
\(Au_k\) converges in \(H^{\sigma,p}(\Omega)\). By (3.9) \(Au_k\) converges in \(H^{m+\sigma,p}(\Omega)\). On one hand the limit must be zero, since for \(w \in C^\infty_0(\Omega)\)
\[
(Au_k, w) = (u_k, A'w) \to 0
\]
while on the other, the limit must have norm 1. This gives a contradiction and (3.10) holds. This completes the proof.

**Theorem 3.5.** For each real \(\sigma\)
\[
\|u\|_{C^\infty(\Omega)} \leq C \left( \|Au\|_{C^{m-\sigma,p}} + \|u\|_{C^{m-\sigma,p}} + \|Bu\|_{L^p(\omega)} \right)
\]
holds for all \(u \in C^\infty(\Omega)\).

This is just Theorem 2.3 of [6].

**Theorem 3.6.** If \(f \in H^{\sigma,p}(\Omega)\) and
\[
(f, A' \psi) \leq C \|\psi\|_{C^{m-\sigma,p}}
\]
for all \(\psi \in C^\infty_0(\Omega)\) (the set of infinitely differentiable functions with compact supports in \(\Omega\)), then \(f \in H^{\sigma,p}_r(\Omega)\).

**Proof.** We follow the reasoning of [10, p. 14]. We consider \(A\) as an operator in \(H^{\sigma,p}(\Omega)\) with domain \(H^{\sigma,p}_r(\Omega)\). Let \(A_w\) be the extension of \(A\) to those \(f \in H^{\sigma,p}(\Omega)\) satisfying (3.11). For such an \(f\) there is an \(h \in H^{\sigma-m,p}\) such that
\[
(f, A' \psi) = (h, \psi)
\]
for all \(\psi \in C^\infty_0(\Omega)\). We then define \(A_w f = h\). Clearly \(A \subseteq A_w\). We now show that \(A^* \subseteq A_w^*\). This will mean that \(A = A_w\) and the theorem will follow.

Suppose \(v \in D(A^*)\). Then
\[
(v, Au) = (w, u)
\]
for some \(w \in H^{-\sigma,p}(\Omega)\) and all \(u \in C^\infty(\Omega)\). In particular, this holds for all \(u\) satisfying zero Dirichlet or Neumann data on \(\partial\Omega\). From this it follows that \(v \in H^{\sigma,m-\sigma,p}(\Omega)\), the closure of \(C^\infty(\Omega)\) in \((H^{m-\sigma,p}(\Omega))\) (cf., e.g., [2]). Hence there is a sequence \([v_k]\) \(C^\infty_0(\Omega)\) converging to \(v\) in \(H^{m-\sigma,p}(\Omega)\). If \(f \in D(A_w)\), then
\[
(f, A' v_k) = (A_w f, v_k).
\]
But
\[ \| A'(v_k - v_0) \|_{m, p'} \leq C \| v_k - v_0 \|_{m, p'} \]
and hence \( A' v_k \to g \in H^{-v, p'} (\Omega) \). Thus
\[ (f, g) = (A_w f, v) \]
for all \( f \in D(A_w) \). Thus \( v \in D(A_{w}^*) \). This completes the proof.

**COROLLARY 3.1.** If \( f \in H^{m-v, p'} (\Omega), g \in F^*_{e, p} \) and
\[
(\mathbf{3.12}) \quad | (f, Au) + \langle G, Bu \rangle | \leq c_2 (\| u \|_{r, p} + \| Cu \|_{r, p})
\]
for all \( u \in C^\infty(\overline{\Omega}) \), then \( f \in H^{m-v, p'} (\Omega), G = C' f \in F^*_{e, p} \) and
\[
\| f \|_{m-v, p'} \leq C (c_2 + \| f \|_{m-v, p})
\]

**PROOF.** The only thing which does not follow immediately from Theorems 3.3 and 3.6 is the fact that \( G = C' f \). By (3.12) there are \( h \in H^{-v, p'} (\Omega) \) and \( \Phi \in J^*_{e, p} \) such that
\[
(f, Au) + \langle G, Bu \rangle = (h, u) + \langle \Phi, Cu \rangle
\]
for all \( u \in C^\infty(\overline{\Omega}) \). By (2.10) this becomes
\[
(A' f - h, u) + \langle G - C' f, Bu \rangle + \langle B' f - \Phi, Cu \rangle = 0.
\]
Since this is true for all \( u \in C^\infty(\overline{\Omega}) \), it follows that \( G = C' f \). This completes the proof.

**THEOREM 3.7.** Let \( \{Q_j\}_{j=1}^k, k \leq m \), be a normal set of boundary operators of orders \( \mu_j \leq m \). Then for each set \( \{\Phi_j\}_{j=1}^k \) of functions in \( C^\infty(\partial \Omega) \) there is \( u \in C^\infty(\overline{\Omega}) \) such that
\[
Q_j u = \Phi_j \text{ on } \partial \Omega, \quad 1 \leq j \leq k,
\]
and for each \( \varrho \)
\[
(\mathbf{3.13}) \quad \| u \|_{\varrho, p} \leq C \sum_{j=1}^k \langle \Phi_j \rangle_{\varrho - \mu_j - 1, p},
\]
where the constant \( C \) does not depend on \( u \) or the \( \Phi_j \).

**PROOF.** By adding appropriate operators to the \( Q_j \) and taking the corresponding \( \Phi_j \) to be zero, we may assume that \( k = m \). Consider the boun-
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dary value problem

\((A - 1)^m u = 0\) in \(\Omega\),

\[ Q_j u = \Phi_j \text{ on } \partial\Omega, \quad 1 \leq j \leq m, \]

where \(A\) is the Laplacian. This problem is equivalent to the Dirichlet problem, and hence we know that there always exists a unique solution. Applying (3.1) to this problem, we obtain

\[
\| u \|_{\theta, p} \leq C \sum_{j=1}^{m} \langle \Phi_j \rangle_{\theta \mu - 1/p, p}. \]

For \(\theta \geq m\) we have

\[
\| Au \|_{\theta - m, p} \leq C \| u \|_{\theta, p}. \]

For \(\theta \leq 0\) we have

\[
\| Au \|_{\theta - m, p} = 1. \text{ u. b. } \frac{|(Au, v)|}{\| v \|_{\theta - m, p}} \leq 1. \text{ u. b. } \| v \|_{\theta - m, p}^{-1} |(u, A' v) + \sum Q_j u, Q_{m-j+1} v| \leq C \| u \|_{\theta, p} + \sum Q_j u, \langle \Phi_j \rangle_{\theta \mu - 1/p, p}, \]

where \(Q_{m-j+1}\) is an appropriate boundary operator of order \(m - \mu_j - 1\).

We now apply an abstract interpolation theorem due to Calderón [11, 10.1] to the spaces considered (cf. [6, Theorem 3.1]) to conclude that (3.13) holds for all real \(\theta\).

4. Proofs.

**Proof of Lemma 2.1.** By (2.10) there is a normal set \(\{N_j\}_{j=1}^{m}\) of boundary operators such that

\[
(Au, v) - (w, A' v) = \sum_{j=0}^{m-1} \gamma_j w, N_{m-j} v \]

for \(w, v \in C^\infty(\overline{\Omega})\), where the order of \(N_j\) is \(j - 1\).

By Theorem 3.7, for each set \(\Phi_1, ..., \Phi_m\) of functions in \(C^\infty(\partial\Omega)\) there is a function \(v \in C^\infty(\overline{\Omega})\) such that

\[
N_j v = \Phi_j \text{ on } \partial\Omega, \quad 1 \leq j \leq m, \]

for \(\theta \geq m\) we have

\[
\| Au \|_{\theta - m, p} \leq C \| u \|_{\theta, p}. \]

For \(\theta \leq 0\) we have

\[
\| Au \|_{\theta - m, p} = 1. \text{ u. b. } \frac{|(Au, v)|}{\| v \|_{\theta - m, p}} \leq 1. \text{ u. b. } \| v \|_{\theta - m, p}^{-1} |(u, A' v) + \sum Q_j u, Q_{m-j+1} v| \leq C \| u \|_{\theta, p} + \sum Q_j u, \langle \Phi_j \rangle_{\theta \mu - 1/p, p}, \]

where \(Q_{m-j+1}\) is an appropriate boundary operator of order \(m - \mu_j - 1\).

We now apply an abstract interpolation theorem due to Calderón [11, 10.1] to the spaces considered (cf. [6, Theorem 3.1]) to conclude that (3.13) holds for all real \(\theta\).
while for each \( q \), \( 1 < q < \infty \),

\[
\| v \|_{s, q} + \| A' v \|_{s-m, q} \leq C \sum_{j=1}^{m} \langle \phi_j \rangle_{s-j+1}^{q},
\]

where the constant \( C \) does not depend on \( v \) or the \( \phi_j \). Now

\[
(4.3) \quad \| v \|_{s, q} + \| A' v \|_{s-m, q} \leq C \sum_{j=1}^{m} \langle \phi_j \rangle_{s-j+1}^{q},
\]

Setting \( e = m - s, q = p' \) in (4.3) and applying (4.1) we have

\[
(4.4) \quad |(Au, v)| \leq \| Au \|_{s-m, p} \| v \|_{s, p'},
\]

\[
(4.5) \quad |(u A' v)| \leq \| u \|_{s, p} \| A' v \|_{s, p'}.
\]

Setting \( q = m - s, q = p' \) in (4.3) and applying (4.1) we have

\[
(4.6) \quad \sum \langle \gamma_j u, \phi_{m-j} \rangle \leq C' \| u \|_{s, p} \sum \langle \phi_{m-j} \rangle_{s-j+1}^{p, p'}.
\]

Taking all of the \( \phi_j \) but one to be zero in (4.6), we obtain estimates for each \( \gamma_j u \), namely

\[
(4.7) \quad \sum \langle \gamma_j u \rangle_{s-j+1} \leq C' \| u \|_{s, p}.
\]

This completes the proof.

**Proof of Theorem 2.2.** By (2.2)

\[
\| Bu \|_{s, p} \leq \| (B - L\gamma) u \|_{s, p} + \| L\gamma u \|_{s, p} + \| \gamma \|_{s, p} \leq \| (B - L\gamma) u \|_{s, p} + \| Bu \|_{s, p} + C \| u \|_{s, p},
\]

Thus

\[
(4.8) \quad \| Bu \|_{s, p} \leq C \| u \|_{s, p} + \| \gamma \|_{s, p} \leq \| u \|_{s-m, p}.
\]

Combining this with (3.1) we obtain (2.3).

**Proof of Theorem 2.1.** Since smooth functions with compact support in \( \Omega \) are in \( D(A_{s, p}(L)) \) and they are dense in \( H^s(L) \), 1 holds. By completion, (2.3) holds for functions in \( H^s(L) \). This gives immediately that \( A_{s, p} \) is closed. Moreover

\[
\| u \|_{s, p} \leq C \| u \|_{s-m, p}
\]

holds for all \( u \in N(A_{s, p}(L)) \). A standard argument using Theorem 3.2 shows that this set must be finite dimensional. Another application of Theorem
3.2 shows that
\[ \| u \|_{s,p} \leq C \| Au \|_{s,p} \]
holds for all \( u \in D(A_{s,p}(L))/N(A_{s,p}(L)) \).
This gives immediately that the range of \( A_{s,p} \) is closed.

**Proof of Theorem 2.3.** We show that (2.3) holds. If it did not, there would be a sequence \( \{ u_k \} \) of functions in \( C^\infty(\overline{Q}) \) such that
\[ \| u_k \|_{s,p} = 1 \]
while
\[ \| Au_k \|_{s,p} + \| u_k \|_{s-m,p} + \| (B - L)u_k \|_{s,p} \rightarrow 0. \]
By Lemma 2.1
\[ \| \gamma u_k \|_{s,p} \leq \text{const.} \]
and hence there is a subsequence (also denoted by \( \{ u_k \} \)) for which \( Ly_{u_k} \) converges in \( F_{s,p} \). Thus \( Bu_k \) converges in the same space. If we now make use of (3.1) we see that \( u_k \) converges in \( H^{s,p}_A(\Omega) \). Since it converges in \( H^{s-m,p}(\Omega) \) to zero, it must converge to the same limit in \( H^{s,p}_A(\Omega) \). But this is impossible, since the \( H^{s,p}_A(\Omega) \) norm of the limit must be unity. The last part of the theorem follows from Theorem 3.2.

**Proof of Theorem 2.4.** Since \( \gamma u \in E_{s,p}, Ly \in F_{q,q} \). By Theorem 3.7 there is a \( \nu \in H^{s,q}_A(\Omega) \) such that \( Bv = Ly u \). Set \( w = u - \nu \). Then \( Bw = 0 \) while \( Aw \in F_{q,s-m} \). Thus by (2.10)
\[ \langle (w, A'g) + \langle Cw, B'g \rangle \rangle = \langle (Aw, g) \rangle \leq C \| g \|_{s-m,q} \]
for all \( g \in C^\infty(\overline{Q}) \). Thus \( \nu \in H^{s,q}_A(\Omega) \) by Corollary 3.1 (applied to \( A', B' \), where we use the fact [8,1] that \( B' \) covers \( A' \)). Thus \( \nu = w + \nu \in H^{s,q}_A(\Omega) \).

**Proof of Theorem 2.5.** By Theorem 2.4 \( u \in H^{s,q}_A(\Omega) \), where \( q = \min(t + s + e) \). If \( q = t + m \), the theorem is proved. Otherwise we replace \( s \) by \( s - m \) and repeat the process as many times as needed to reach \( t + m \).

**Proof of Theorem 2.6.** If (2.4) holds
\[ \langle (f, Au) + G, Bu \rangle \leq c_0 \| u \|_{s,p} + C \| G \|_{s,p} \| u \|_{s-m,p} \]
\[ \leq (c_0 + C \| G \|_{s,p}) \| u \|_{s,p} \]
where \( \tau = \max (s, q - \varepsilon) \). Thus \( f \in H^{m-r, p}(\Omega) \), \( G \in F^q_{r,p} \) and

\[
\| f \|_{m-r, p'} + \| G \|_{F^q_{r,p}} \leq C \left( c_0 + \| G \|_{F^q_{r,p}} + \| f \|_{m-q, p'} \right).
\]

If \( \tau = s \), we are finished. Otherwise we continue the process until we obtain the desired result.

**Proof of Corollary 2.1.** Taking \( c_0 = 0 \) in (2.4), we have by Theorem 2.6 that \( f \in H^{m-s, p'}(\Omega) \) and \( G \in F^q_{r,p} \) for each real \( s \). By Sobolev's lemma \( f \in C^\infty(\Omega) \) and each \( g_j \in C^\infty(\partial \Omega) \). Moreover for any \( s, q \), we have by (2.5)

\[
\| f \|_{m-s, p'} + \| G \|_{F^q_{r,p}} \leq C \left( \| f \|_{m-q, p'} + \| G \|_{F^q_{r,p}} \right)
\]

where the constant \( C \) does not depend on \( f, G \). An application of Theorem 3.2 shows that the set of such \( f, G \) is finite dimensional.

**Theorem 2.7** follows immediately from Theorem 2.5 and Sobolev's Lemma.

**Proof of Theorem 2.9.** By Theorem 2.5 we may assume that \( s = t + m \). Suppose \( f \in H^{t, p}(\Omega) \) is orthogonal to \( \tilde{N} \). Then \( \langle f, h \rangle + \langle 0, G \rangle = 0 \) for all \( h \in \tilde{N} \), where \( G \) is any vector corresponding to \( h \). This shows that there is a sequence \( \{ u_k \} \) of functions in \( C^\infty(\Omega) \) such that \( Au_k \to f \) in \( H^{t, p}(\Omega) \) and \( \langle B - L \gamma \rangle u_k \to 0 \) in \( E_{s,p} \). Moreover, we may take the \( u_k \) to be orthogonal to \( N(A(L)) \). Thus by Theorem 2.2, \( u_k \) converges in \( H^{t, p}(\Omega) \) to some element \( u \). Thus \( Au = f \) and \( Bu = L \gamma u \). Hence \( f \in R(A_{s,p}(L)) \). Conversely, if \( f \in R(A_{s,p}(L)) \), such a sequence exists. If \( h \in \tilde{N} \) and \( G \) is any corresponding vector, then

\[
\langle h, Au_k \rangle + \langle G, Bu_k - L \gamma u_k \rangle = 0
\]

for each \( k \). Taking the limit, we have

\[
\langle h, f \rangle = 0.
\]

Since \( h \) was any element of \( \tilde{N} \), \( f \) is orthogonal to \( \tilde{N} \).
PROOF OF THEOREM 2.10. Write \( v = v' + v'' \), where \( v'' \in \tilde{N} \) and \( v' \) is orthogonal to \( \tilde{N} \) (cf. [9]). Let \( w \) be any function in \( C^\infty(\overline{\Omega}) \) and write \( w = w' + w'' \), where \( w'' \in \tilde{N} \) while \( w' \) is orthogonal to it. By Theorem 2.9 there is a \( u \in V(L) \) such that \( Au = w' \). Now \( (v', w'') = 0 \), while \( (v', w') = (v', Au) = (v, Au) - (v'', Au) = 0 \).

Hence \( (v', w) = 0 \). Since this is true for all \( w \in C^\infty(\overline{\Omega}) \), \( v' = 0 \). Thus \( v = v'' \in \tilde{N} \).

PROOF OF COROLLARY 2.2. If \( u \in V(L) \) is orthogonal to \( N(A(L)) \), then by Theorem 2.2

\[
| (f, u) | \leqslant \| f \|_{m \to p} \| u \|_{m \to s, p} \leqslant C \| Au \|_{s, p}.
\]

Hence there is a \( v_0 \in H^{s, p'}(\Omega) \) orthogonal to \( N(A(L)) \) and such that

\[
(4.10) \quad (f, u) = (v_0, Au)
\]

for all \( u \in V(L) \) orthogonal to \( N(A(L)) \). By (2.9) we see that \( f \) itself is orthogonal to \( N(A(L)) \).

Thus (4.10) holds for all \( u \in V(L) \). Subtracting (2.9) from (4.10), we have

\[
(v - v_0, Au) = 0
\]

for all \( u \in V(L) \). Thus \( v - v_0 \in \tilde{N} \subseteq C^\infty(\overline{\Omega}) \). Hence \( v \in H^{s, p'}(\Omega) \).

PROOF OF THEOREM 2.8. Clearly \( \tilde{N} \subseteq N(A(L)') \).

Conversely, if \( v \in N(A(L)') \), then \( (v, Au) = (A' v, u) = 0 \) for all \( u \in V(L) \).

But then by Theorem 2.10, we have \( v \in \tilde{N} \).

COROLLARY 2.3 follows from Theorems 2.3 and 2.9 and Corollary 2.1.

PROOF OF PROPOSITION 2.1. If \( u \) is in the domain of the closure of \( A \) as described, then there is a sequence \( \{u_k\} \) of functions in \( C^\infty(\overline{\Omega}) \) such that \( u_k \to u \), \( Au_k \to Au \) in \( H^{s, p}(\Omega) \) while \( (B - MC) u_k = 0 \). By (2.3) we see that \( u \in H^{s, p}_A(\Omega) \) and \( (B - MC) u = 0 \).

Conversely if \( u \in D(A_{s, p}(L_1)) \), there is a sequence \( \{u_k\} \) of functions in \( C^\infty(\overline{\Omega}) \) such that \( u_k \to u \) in \( H^{s, p}_A(\Omega) \) while \( (B - MC) u_k \to 0 \) in \( F_{s, p} \). By

Theorem 3.7 there is a linear mapping \( W \) from \( F_{s, p} \) to \( H_{A}^{s, p}(\Omega) \) such that
\[
(4.11) \quad BW \Phi = \Phi, \quad CW \Phi = 0,
\]
\[
(4.12) \quad \| W \Phi \|_{s, p} \leq C \| \Phi \|_{s, p},
\]
for all \( \Phi \in F_{s, p} \). Set \( v_k = W (B - MC) u_k, w_k = u_k - v_k \).
Then \( (B - MC) w_k = 0 \) while by (4.12)
\[
\| w_k - u \|_{s, p} \leq \| u_k - u \|_{s, p} + \| (B - MC) u_k \|_{s, p} \to 0.
\]
Hence \( u \) is in the domain of the closure of \( A \) as described.

**Proof of Theorem 2.11.** If \( u \in D(A_{m, p} (L_i)) \) and \( v \in D(A_{m-r, p'} (L_i)) \), then by (2.10)
\[
(u, A' v) - (Au, v) = \langle Bu, C' v \rangle - \langle Cu, B' v \rangle
\]
\[
- \langle MCu, C' v \rangle - \langle Cu, M^* C' v \rangle = 0.
\]
Thus \( A_{m-r, p'} (L_i) \subseteq (A_{s, p} (L_i))^* \). Next suppose \( v, f \in H^{-l, p'}(\Omega) \) satisfy
\[
(4.13) \quad (u, f) = (Au, v)
\]
for all \( u \in D(A_{s, p} (L_i)) \). By Theorem 3.4 there is a mapping \( U \) from \( F_{s, p} \) to \( H_{A}^{s, p}(\Omega) \) such that
\[
BU \Phi = \Phi, \quad AU \Phi \in H^{s, p}(\Omega)
\]
\[
(4.14) \quad \| U \Phi \|_{s, p} \leq \| AU \Phi \|_{s, p} \leq K \| \Phi \|_{s, p},
\]
for all \( \Phi \in F_{s, p} \). Consider the operator \( (B - MC) U = I - MCU \) or \( F_{s, p} \).
Since \( MCU \) is compact, this operator is Fredholm. It thus has a bounded inverse from its range \( K_{s, p} \) onto a complement of its null space. Let \( L_{s, p} \)
be some finite dimensional complement of \( K_{s, p} \). The set \( S \) of those \( \Phi \in L_{s, p} \)
for which there is a \( u \in H_{A}^{s, p}(\Omega) \) such that
\[
Au \in H^{s, p}(\Omega), \quad (B - MC) u = \Phi
\]
is thus finite dimensional. Thus there is a mapping \( U_1 \) from \( S \) to \( H^{s,p}(\Omega) \) such that

\[
AU_1 \Phi \in H^{s,p}(\Omega), \quad (B - MC) U_1 \Phi = \Phi
\]

(4.15)

\[
\| U_1 \Phi \|_{s,p} + \| AU_1 \Phi \|_{s,p} \leq K \| \Phi \|_{r,p}
\]

for all \( \Phi \in S \). Now let \( u \) be any function in \( C^\infty(\Omega) \) and set \( \Phi = (B - MC) u \).

We decompose \( \Phi \) in the form \( \Phi = \Phi' + \Phi'' \), where \( \Phi' \in K_{s,p} \) and \( \Phi'' \in S \). Set \( u_0 = u' + u'' \), where

\[
u' = U(I - MCU)^{-1} \Phi', \quad u'' = U_1 \Phi''.\]

Then \( (B - MC) u_0 = \Phi \) and hence \( u - u_0 \) is in \( D(A_{s,p}(L_1)) \). Thus

(4.16)

\[
(u, f) - (Au, v) = (u_0, f) - (Au_0, v),
\]

showing that the expression on the right depends only on \( \Phi \). Denoting it by \( F \Phi \) we see by (4.14) and (4.15) that it is a bounded linear functional defined on a subspace of \( F_{s,p} \) (actually, this subspace is the whole of \( F_{s,p} \), but we need not know this fact here). Thus, by the Hahn-Banach theorem, there is a \( G \in F^{*,p} \) such that

\[
F \Phi = \langle \Phi, G \rangle
\]

for all \( \Phi \) in the domain of definition of \( F \). Thus

(4.17)

\[
(u, f) - (Au, v) = \langle (B - MC) u, G \rangle
\]

for all \( u \in C^\infty(\Omega) \). In particular,

\[
| \langle Au, v \rangle + \langle Bu, G \rangle | \leq c (\| u \|_{r,p} + \| Cu \|_{r,p})
\]

for all such \( u \). This allows us to apply Corollary 3.1 to obtain that \( v \in H^{m-r,p'}(\Omega) \) and \( G = C' v \in F^{*,p'} \). From (4.13) it is clear that \( f = A' \).

Thus \( (A_{s,p}(L_1))^* \subseteq A_m^{*-r,p'}(L_1) \), and (2.12) is proved. Since \( L_1 \) is compact from \( E_{s,p} \) to \( F_{s,p} \), it follows from Theorem 2.3 that \( A_{s,p}(L_1) \) is closed. Since \( H^{r,p}(\Omega) \) is reflexive, (2.13) follows from the fact that

\[
(A_{s,p}(L_1))^{**} = A_{s,p}(L_1).
\]

After this paper was completed, R. S. Freeman sent us a copy of his work \[29\] which treats similar problems. He considers the \( L^2 \) theory for bounded or unbounded domains. Although not explicitly stated in his paper, his methods also apply to boundary conditions of the form (1.2) as considered here.
BIBLIOGRAPHY


