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A COERCIVENESS INEQUALITY

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Let Ω be an open bounded set in R^n with smooth boundary ; by $H^1(\Omega)$ we denote the Hilbert space which is the completion of the smooth functions in Ω under the norm $\| u \|_1$ where

$$\| u \|_1^2 = \| u \|_0^2 + d_1(u).$$

Here $\| u \|_0$ is the usual L^2 norm over Ω and $d_1(u)$ is the Dirichlet integral given by

$$d_1(u) = \int_{\Omega} | \operatorname{grad} u |^2 \, dx .$$

It is well known that the elements of $H^1(\Omega)$ are equivalence classes of functions and that the study of these functions requires some elementary potential theory. We recall that the capacity associated with the space $H^1(\Omega)$ is the set function $\operatorname{cap}(A) = \inf \| u \|_1^2$ the infimum being taken over all smooth $u(x)$ which are $\geqq 1$ on A and that this function is an outer measure. The elements of $H^1(\Omega)$ are then determined as functions up to a set of capacity zero. If $u_n(x)$ is a minimizing sequence for the capacity of A, the u_n converge to a well defined element v_A of $H^1(\Omega)$ called the capacitary potential of A, and which may be taken equal to 1 on A. By a simple variational argument one finds that there corresponds to v_A a positive measure μ_A supported by the closure of A called the capacitary distribution such that $(u, v_A)_1 = \int u(x) \, d\mu_A$ for all u in $H^1(\Omega)$. Clearly $\| v_A \|_1^2 = \operatorname{cap}(A) = (v_A, v_A)_1 = \int v_A(x) \, d\mu_A(x) = \int 1 \, d\mu_A = | \mu_A |$. If $\mathcal{M}_A$ is the closure in $H^1(\Omega)$ of the smooth functions vanishing on a set A then this subspace is proper if and only if $\operatorname{cap}(A) > 0$.

G. Stampacchia has conjectured that the following coerciveness assertion holds: when $\operatorname{cap}(A)$ is positive, the quadratic norms $\| u \|_1$ and $\sqrt{d_1(u)}$ are

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equivalent norms on $\mathcal{M}_A$. It is our purpose here to establish this conjecture. Since it is obvious that $d_1(u) \leqq \| u \|_1^2$, what must be shown is the existence of a constant C (depending on A) such that for u in $\mathcal{M}_A$

$$\| u \|_0^2 \leqq C d_1(u). \tag{1}$$

Since we have supposed the boundary of Ω smooth, the Rellich theorem holds, i. e. the quadratic form $\| u \|_0^2$ is completely continuous relative to $\| u \|_1^2$. We may therefore write $\| u \|_0^2 = (Hu, u)_1$ where H is a positive operator which is completely continuous and of bound at most 1. It is easy to see that H has no null space, while H does have the eigenvalue 1 associated with the eigenfunction $u(x) =$ constant. That eigenvalue is simple, since $Hv = v$ implies $\| v \|_0 = \| v \|_1$ and therefore $d_1(v) = 0$, from which we infer that $v =$ constant, since its derivatives vanish almost everywhere.

Let P be the projection on the subspace $\mathcal{M}_A$; then the operator PHP is positive, completely continuous and has bound $\lambda = \| PHP \|$ at most 1. Since $PHPu = u$ implies $Hu = u$ and therefore $u =$ constant, and since the only constant function in $\mathcal{M}_A$ vanishes identically, we see that $\lambda < 1$. It follows that for u in $\mathcal{M}_A$

$$\| u \|_0^2 \leqq \lambda \| u \|_1^2 = \lambda \| u \|_0^2 + \lambda d_1(u)$$

and we obtain the desired inequality with $C = \dfrac{\lambda}{1 - \lambda}$.

It is possible to obtain an estimate for C in terms of the Lebesgue measure of Ω, the capacity of A and the number $\omega =$ smallest non-zero eigenvalue of the free membrane problem in Ω. For this purpose we write the eigenvalues of H in monotone decreasing order:

$$1 = \lambda_1 > \lambda_2 \geqq \lambda_3 \geqq \dots$$

and take $e_n(x)$ as the corresponding normalized eigenfunctions. Thus $e_1(x) = 1 / \sqrt{m}$ where m is the Lebesgue measure of Ω. For the second eigenfuction we have $\| e_2 \|_0^2 = \lambda_2 \| e_2 \|_1^2$, whence $\omega \| e_2 \|_0^2 = d_1(e_2)$ where $\omega = \lambda_2^{-1} - 1$, and this, by a classical argument, implies $- \Delta e_2(x) = \omega e_2(x)$ with the normal derivative of $e_2(x)$ vanishing on the boundary. Thus $e_2(x)$ is the eigenfuction of the free membrane problem for Ω and ω si the corresponding eigenvalue.

If v_A and μ_A are the capacitary potential and distribution associated with A we have

$$(e_1, v_A)_1 = \int e_1(x)\, d\mu_A = \frac{| \mu_A |}{\sqrt{m}} = \frac{\operatorname{cap}(A)}{\sqrt{m}}.$$

Our object is to estimate λ and hence C. We have $\lambda = \sup \frac{\| u \|_0^2}{\| u \|_1^2}$ the supremum being taken over all non-trivial u in $\mathcal{M}_A$.

Let $\mathcal{M}$ be the subspace consisting of all u in H^1 orthogonal to v_A. Since the capacity is positive, v_A is not 0 and $\mathcal{M}$ is proper; moreover $\mathcal{M}$ contains $\mathcal{M}_A$ since, for u in $\mathcal{M}_A$ we have $(u, v_A)_1 = \int u(x)\, d\mu_A(x) = 0$.

Let $\lambda^* = \sup \frac{\| u \|_0^2}{\| u \|_1^2}$ the supremum being taken over all non zero u in $\mathcal{M}$. We then have $\lambda^* \geq \lambda$, and since $\mathcal{M}$ contains no constant function other than $0, \lambda^* < 1$. If Q is the projection on $\mathcal{M}, \lambda^*$ is the largest eigenvalue of the positive, completely continuous operator QHQ. We estimate λ^* by the standard Aronszajn-Weinstein method. Let $R_\zeta = (H - \zeta I)^{-1}$ be the resolvent of H; λ^* is then the (unique) zero of the function $(R_\zeta v_A, v_A)_1$ in the interval $\lambda_2 < \zeta < 1$. For the sake of completeness, we give an elementary proof for this special case. If $QHQw = \xi w$, then $Hw = \xi w + cv_A$ where the coefficient c may be 0. If $c = 0, w$ is an eigenvector of H orthogonal to v_A, and therefore not e_1. The number ξ is then one of the $\lambda_n < 1$, hence $\xi \leqq \lambda_2$. If c is not 0 we have $(H - \xi I)\, w = cv_A$ whence $R_\xi v_A = c^{-1} w$, and therefore, since w in $\mathcal{M}$ is orthogonal to $v_A, (R_\xi v_A, v_A)_1 = 0$ and ξ is a zero of the function $(R_\zeta v_A, v_A)_1$. Thus the spectrum of QHQ is a subset of the zeros and poles of this function. Conversely, if $(R_\xi v_A, v_A)_1 = 0$ for some ξ, we write $w = R_\xi v_A$ which is $\mathcal{M}$ and obtain $(H - \xi I)\, w = v_A$ or $Hw = \xi w + v_A$, whence $QHQw = \xi w$ and therefore ξ is an eigenvalue of QHQ. We seek the largest eigenvalue of that operator, and note that the function $(R_\zeta v_A, v_A)_1$ is monotone increasing and assumes all real values in the interval $\lambda_2 < \zeta < 1$, and is negative to the right of 1; hence λ^* is the (unique) zero of the function in that interval.

We therefore write out the function explicitly:

$$(R_\zeta v_A, v_A)_1 = \sum_{n=1}^{\infty} \frac{|(v_A, e_n)_1|^2}{\lambda_n - \zeta}$$

and note that the root λ^* is surely to the left of the root of

$$\frac{\| v_A \|_1^2 - |(v_A, e_1)_1|^2}{\lambda_2 - \zeta} + \frac{|(v_A, e_1)_1|^2}{1 - \zeta}.$$

The root is easily computed, and we find

$$\lambda \leqq \lambda^* \leqq 1 - (1 - \lambda_2) \frac{|(v_A, e_1)_1|^2}{\| v_A \|_1^2}$$

and therefore

$$C \leqq (1 - \lambda_2)^{-1} \frac{\| v_A \|_1^2}{|(v_A, e_1)_1|^2} - 1 < (1 + 1/\omega) \frac{\text{meas}(\Omega)}{\text{cap}(A)}.$$

The foregoing estimate for the constant in Stampacchia's inequality has the disadvantage that it involves the capacity of A relative to the space $H^1(\Omega)$ and that this function is not known. However, as we shall presently show, this function is equivalent to the usual capacity for the corresponding Bessel potentials, a set function usually written $\gamma_2(A)$. There exists a constant M depending only on Ω such that

$$M\gamma_2(A) \leqq \text{cap}(A) \leqq \gamma_2(A)$$

for all subsets A of Ω, and therefore the constant occuring in inequality (1) involves a numerator which depends only on the domain Ω and a denominator $\gamma_2(A)$; it therefore is independent of any other property of A, for example, the distance of that set from the boundary.

The equivalence of the set functions $\gamma_2(A)$ and cap (A) is a consequence of the smoothness hypothesis made concerning the boundary of Ω; there exists a continuous linear transformation $u \longrightarrow \tilde{u}$ mapping $H^1(\Omega)$ into $P^1(R^n)$, the space of Bessel potentials on R^n such that $u(x) = \tilde{u}(x)$ for all x in Ω. The transformation is bounded; thus there exists a positive M such that $\| u \|_1^2 \geqq M \| \tilde{u} \|_1^2$. If v_A is the capacitary potential for A in the space $P^1(R^n)$ we have

$$\gamma_2(A) = \| v_A \|_1^2 \geq \| v^* \|_1^2 \geqq \text{cap}(A)$$

where v^* is the restriction of v_A to Ω considered as an element of $H^1(\Omega)$. Conversely, if v_A in $H^1(\Omega)$ is the capacitary potential of A,

$$\text{cap}(A) = \| v_A \|_1^2 \geqq M \| \tilde{v}_A \|_1^2 \geqq M \gamma_2(A).$$

It is natural to enquire to what extent inequality (1) is valid for the spaces $H^\alpha(\Omega)$ where the norm is defined by

$$\| u \|_\alpha^2 = \| u \|_0^2 + d_\alpha(u)$$

$$\text{with } d_\alpha(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} \, dx \, dy \qquad \text{when } 0 < \alpha < 1$$

and $d_\alpha(u) = \Sigma d_1(D_k u)$ when α is an integer, the summation being taken over all derivatives of order α, and finally when $\alpha > 1$ is not an integer, $d_\alpha(u) = \Sigma d_\beta(D_k u)$ where the summation is taken over all derivatives of order k, k being the largest integer $< \alpha$ and β defined by $\alpha = k + \beta$.

All of the arguments we have given carry over to the case $\alpha < 1$; Stampacchia's inequality is valid with a constant which depends only on the domain Ω and the reciprocal of the capacity $\gamma_{2\alpha}(A)$, this being the capacity for the corresponding space of Bessel potentials $P^\alpha(R^n)$.

The situation is essentially more complex when $\alpha > 1$; if we repeat our analysis we find that the operator H which represents the L^2 norm in the space $H^\alpha(\Omega)$ is positive, completely continuous and with bound 1, however, the eigenvalue 1 is no longer simple. The eigenspace corresponding to that eigenvalue consists of all polynomials of sufficiently low degree, and such a polynomial may vanish on a set of positive capacity. Thus the inequality does not hold, unless a further hypothesis is made, viz. that the set A is not contained in the set of zeros of a polynomial of degree $\leqq m =$ the largest integer strictly smaller than α. In this case inequality (1) is valid, but the constant depends in an essential way on the other data than simply the capacity $\gamma_{2\alpha}(A)$.

Let us remark that the surface on the unit sphere in R^n is a set of positive capacity, but is contained in the null set of the polynomail $1 - |x|^2$.

Throughout our discussion we have made use of the hypothesis that the boundary of Ω is smooth in order that the Rellich theorem guaranteeing the complete continuity of $\|u\|_0^2$ should hold. We have also used that hypothesis to have the extension theorem embedding $H^1(\Omega)$ into $P^1(R^n)$. The careful study of these questions given in [1] shows that the regularity hypotheses needed are very mild.

1. R. Adams, N. Aronszajn and K. T. Smith, « *Theory of Bessel Potentials II* » Annales Institut Fourier, to appear.

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