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PRIMAL CLUSTERS AND LOCAL BINARY ALGEBRAS (*)

ADIL YAQUB

The theory of a primal (= strictly functionally complete) algebra subsumes and substantially generalizes the classical Boolean theory as well as that of p -rings and Post algebras. Here a primal algebra is a finite algebra in which each map is expressible in terms of the primitive operations of the algebra. The concept of independence is essentially a generalization to universal algebras of the Chinese remainder theorem in number theory. A primal cluster is a class $\{U_i\}$ of universal algebras U_i of the same species in which each U_i is primal and such that every finite subset of $\{U_i\}$ is independent.

Our present object is to show that the class $\{(B_n, \times)\}$ of all «local» binary algebras of distinct orders, endowed with a suitably chosen permutation $\hat{\cdot}$ of B_i , forms a primal cluster $\{(B_n, \times, \hat{\cdot})\}$ (of species $(2, 1)$). Here a local binary algebra is a finite associative binary algebra (B, \times) such that every element in B is either nilpotent or has an inverse (in B). In Theorem 5, which is our main result, we show that a much more comprehensive class K of algebras of rather *diverse* nature nevertheless forms a primal cluster. Here K is the union of all algebras in $\{(B_n, \times, \hat{\cdot})\}$ and $\{(P_m, \times, \hat{\cdot})\}$, where $(P_m, \times, \hat{\cdot})$ is the basic Post algebra of order m (species $(2, 1)$). This theorem subsumes Foster's results on prime fields, « n -fields», groups with null, basic Post algebras, and, in addition, yields new results especially in regard to local rings (viewed as binary algebras). Our methods for obtaining the permutation $\hat{\cdot}$, as well as for establishing independence, are *constructive*. Indeed, the permutation $\hat{\cdot}$ turns out to be of rather general (but not entirely arbitrary) nature.

1. Fundamental concepts. We begin this section by recalling the following results of [3]. Let $U = (U, \theta_1, \theta_2, \dots)$ be an algebra and let

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$S = (n_1, n_2, \dots)$ be its species, where θ_i is a primitive operation of finite rank n_i . An S -expression is a primitive composition of indeterminate symbols ζ_1, ζ_2, \dots via the primitive operations θ_i . A map $f(\zeta_1, \dots, \zeta_n)$ from $U \times \dots \times U$ to U is *expressible* if there exists an S -expression $\varphi(\zeta_1, \dots, \zeta_n)$ such that $f = \varphi$ for all ζ_1, \dots, ζ_n in U . An algebra U is *primal*, or *strictly functionally complete* if it is finite, with at least two elements, and if every map $f(\zeta_1, \dots, \zeta_n)$ in U is expressible. Now, let $\{U_i\} = \{U_1, \dots, U_t\}$ be a finite set of algebras of species S . We say that $\{U_i\}$ satisfies the *Chinese remainder condition*, or is *independent* if, corresponding to each set of S -expressions $\varphi_1, \dots, \varphi_t$, there exists a single expression ψ such that $\psi = \varphi_i$ is an identity of U_i ($i = 1, \dots, t$). A *primal cluster* of species S is a class $\tilde{U} = \{\dots, U_i, \dots\}$ of primal algebras U_i of species S any finite subset of which is independent. We now have the following (compare Definition 2 below with Definition 3 and Theorem 6).

Definition 1. A *binary algebra* is an algebra (B, \times) , possessing two distinguished elements $0, 1$ ($0 \neq 1$) such that

$$0 \times \zeta = \zeta \times 0 = 0, \quad 1 \times \zeta = \zeta \times 1 = \zeta \quad (\zeta \in B).$$

Definition 2. A *local binary algebra* is a finite binary algebra (B, \times) which is associative and such that ζ in B implies ζ is nilpotent or ζ has an inverse (in B).

Examples of local binary algebras are wide-spread. Thus, the multiplicative structure of any (finite) Galois field, $(GF(p^k), \times)$, and more generally any (not necessarily cyclic) finite group with null $(G \cup \{0\}, \times)$ is a local binary algebra. Other examples include the multiplicative structure of each of the following rings: the integers $(\text{mod } p^k, p \text{ prime})$, the hypercomplex ring $GF(p^k)[\eta_1, \dots, \eta_t]$ obtained by adjoining any finite number η_1, \dots, η_t of commuting nilpotent elements to any Galois field, and more generally, any local finite commutative ring with identity (see Definition 3).

2. The main theorems. Let (B, \times) be any local binary algebra. Then, for some $r \geq 1$ and some $s \geq 0$, we may denote:

$$(1) \quad B = \{0, 1, \zeta_2, \zeta_3, \dots, \zeta_r, \eta_1, \eta_2, \dots, \eta_s\}, \quad \zeta_i \text{ unit}, \quad \eta_i \text{ nilpotent}.$$

For every element α in B , define the *characteristic function* $\delta_\alpha(\zeta)$ as follows:

$$(2) \quad \delta_\alpha(\zeta) = 1 \quad \text{if } \zeta = \alpha, \quad \delta_\alpha(\zeta) = 0 \quad \text{if } \zeta \neq \alpha \quad (\zeta \in B).$$

Now, define a permutation $\widehat{\cdot}$ of B by the ordering (1) above, i. e.,

$$(3) \quad \widehat{\cdot} : \text{def} : 0^{\widehat{\cdot}} = 1, 1^{\widehat{\cdot}} = \zeta_2, \zeta_2^{\widehat{\cdot}} = \zeta_3, \dots, \zeta_r^{\widehat{\cdot}} = \eta_1, \dots, \eta_s^{\widehat{\cdot}} = 0.$$

Clearly, $\widehat{\cdot}$ is a cyclic $0 \rightarrow 1$ permutation of B . Following [1], we define :

$$(4) \quad a \times_{\widehat{\cdot}} b = \text{def} = (a^{\widehat{\cdot}} \times b^{\widehat{\cdot}})^{\sim}, \quad \zeta^{\sim} = \text{def} = \text{inverse of } \zeta^{\widehat{\cdot}}.$$

It is readily verified that

$$(5) \quad a \times_{\widehat{\cdot}} 0 = 0 \times_{\widehat{\cdot}} a = a.$$

Moreover, for any function f on B , we have (compare with [1])

$$(6) \quad f(x_1, \dots, x_k) = \sum_{\alpha_1, \dots, \alpha_k}^{\widehat{\cdot}} f(\alpha_1, \dots, \alpha_k) \delta_{\alpha_1}(x_1) \delta_{\alpha_2}(x_2) \dots \delta_{\alpha_k}(x_k).$$

In (6), $\alpha_1, \dots, \alpha_k$ range independently over all the elements of B . Furthermore, the notation $\sum_{\beta_i \in B}^{\widehat{\cdot}} \beta_i$ means $\beta_1 \times_{\widehat{\cdot}} \beta_2 \times_{\widehat{\cdot}} \beta_3 \times_{\widehat{\cdot}} \dots$ where $\beta_1, \beta_2, \beta_3, \dots$ are all the elements of B . The verification of (6) is immediate upon using (2) and (5).

We also need the following easily proved

LEMMA 1. *Let (B, \times) be any local binary algebra, and let η be any nilpotent element in B . Then ηx and $x\eta$ are nilpotent for all x in B .*

PROOF. Suppose η is a nilpotent element of B and x is in B . Suppose $\eta x = y = \text{unit}$. We show that this leads to a contradiction. Indeed, if y were a unit, then $\eta = yx^{-1} = \text{unit}$, contradiction. Moreover, if x were not a unit, then, since B is local, x would be nilpotent. Now, let k be the least positive integer such that $x^k = 0$. Clearly, $k > 1$. Then $\eta x^k = yx^{k-1}, 0 = yx^{k-1}, y^{-1}0 = x^{k-1}, 0 = x^{k-1}$, which contradicts the minimality of k . Hence $\eta x (=y)$ is not a unit. Again, since B is local, therefore, ηx is nilpotent. The proof for $x\eta$ is similar, and the lemma is proved.

Next, we prove the following

THEOREM 1. *Let (B, \times) be any local binary algebra, and let $B = \{0, 1, \zeta_2, \zeta_3, \dots, \zeta_r, \eta_1, \dots, \eta_s\}$, each ζ_i is a unit, each η_i is nilpotent. Suppose $\widehat{\cdot}$ is any $0 \rightarrow 1$ cyclic permutation of B such that $1^{\widehat{\cdot}} = \zeta_2, \zeta_2^{\widehat{\cdot}} = \zeta_3, \dots, \zeta_{r-1}^{\widehat{\cdot}} = \zeta_r$, but otherwise $\widehat{\cdot}$ is entirely arbitrary. Then the algebra $(B, \times, \widehat{\cdot})$ (species (2,1)) is primal.*

PROOF. Let $\zeta^{\wedge n}$ denote $(\dots((\zeta^{\wedge})^{\wedge})^{\wedge}\dots)^{\wedge}$ (n iterations). Since ζ^{\wedge} is a *cyclic* permutation of B , therefore

$$\zeta \zeta^{\wedge} \zeta^{\wedge 2} \dots \zeta^{\wedge r+s} = 0.$$

Hence 0 (and with it $0^{\wedge}, 0^{\wedge 2}, \dots, 0^{\wedge r+s}$) is expressible in terms of the primitive operations \times, \wedge . Therefore all the elements (= constant functions) of B are expressible in terms of \times, \wedge . Next, we show that the characteristic function $\delta_a(x)$ is so expressible. Thus, let α be any element of B . Choose N so large that $\eta^N = 0$ for all nilpotent elements of B . Now, since ζ^{\wedge} is a *cyclic* permutation of B , there therefore exists an integer m such that $\alpha^{\wedge m} = 0$. Recalling that \wedge is defined by the ordering (3), it is readily verified that

$$(7) \quad \delta_a(x) = (x^{\wedge m+1} x^{\wedge m+2} \dots x^{\wedge m+r})^{rN}.$$

In verifying (7) we make use of Lagrange's Theorem and Lemma 1. Now, to prove the theorem, let $f: B \times \dots \times B \rightarrow B$ be any mapping from B^k to B . By what we have just proved, and using (6) and (4), it readily follows that the right-side of (6) is expressible in terms of the operations \times, \wedge, \sim , and hence $f(x_1, \dots, x_k)$ is expressible in terms of \times, \wedge, \sim . Since, however, ζ^{\sim} is the inverse of the *cyclic* permutation ζ^{\wedge} , therefore, $\zeta^{\sim} = \zeta^{\wedge r+s}$ (see (3)). Hence $f(x_1, \dots, x_k)$ is expressible in terms of the primitive operations \times, \wedge , only and the theorem is proved.

Next, we investigate the independence of local binary algebras. To this end, let $B_i = (B_i, \times)$ be a local binary algebra, and let the order (= number of elements) of B_i be n_i ($i = 1, \dots, t$). For each B_i , define a permutation \wedge of B_i as in theorem 1 (i.e., \wedge is as in (3)). We now have the following

THEOREM 2. *Suppose B_1, \dots, B_t are local binary algebras of distinct orders, and suppose that \wedge is a permutation of B_i as prescribed in Theorem 1. Then the algebras $\{(B_1, \times, \wedge), \dots, (B_t, \times, \wedge)\}$ are independent.*

PROOF. Let us first consider the algebras (B_1, \times, \wedge) and (B_2, \times, \wedge) . Let

$$(8) \quad |_{12}(\zeta) = \begin{cases} 1 & (\text{in } B_1) \\ 0 & (\text{in } B_2) \end{cases}; \quad |_{21}(\zeta) = \begin{cases} 0 & (\text{in } B_1) \\ 1 & (\text{in } B_2) \end{cases}.$$

We shall now construct *expressions* (built up from \times, \wedge) $|_{12}(\zeta), |_{21}(\zeta)$ satisfying (8) above. To this end, let N be chosen so that

$$(9) \quad \eta^N = 0 \text{ for all nilpotent elements } \eta \text{ in } B_1 \text{ or } B_2.$$

Now, define

$$(10) \quad E = \zeta \widehat{\zeta} \widehat{\zeta} \dots \widehat{\zeta}^{n-1}, \quad n = \text{larger of the orders of } B_1, B_2,$$

$$(11) \quad r_i = \text{number of units in } B_i \quad (i = 1, 2).$$

We now distinguish three cases.

Case 1: $r_1 > r_2$.

Recalling that $\widehat{}$ is defined for each B_i as in (3), (1), it is easily seen, using (10), (11), (9), Lagrange's Theorem, and Lemma 1, that

$$|_{12}(\zeta) = (E \widehat{E} \widehat{E} \dots \widehat{E}^{r_1})^{r_1 N} = \begin{cases} 1 & (\text{in } B_1) \\ 0 & (\text{in } B_2) \end{cases}.$$

Case 2: $r_1 < r_2$.

As in *Case 1*, we now have

$$|_{21}(\zeta) = (E \widehat{E} \dots \widehat{E}^{r_2})^{r_2 N} = \begin{cases} 0 & (\text{in } B_1) \\ 1 & (\text{in } B_2) \end{cases}.$$

Case 3: $r_1 = r_2$.

Let $n_1 = \text{order of } B_1 < \text{order of } B_2 = n_2$. This is possible since B_1 and B_2 have distinct orders. Again, using Lemma 1, it is easily seen that

$$|_{12}(\zeta) = (E \widehat{E}^{n_1+1} \widehat{E}^{n_1+2} \dots \widehat{E}^{n_1+r_1})^{r_1 N} = \begin{cases} 1 & (\text{in } B_1) \\ 0 & (\text{in } B_2) \end{cases}.$$

Furthermore, since $0 \widehat{} = 1$ (in both B_1 and B_2), therefore

$$(12) \quad |_{ij}(\zeta) = \{(|_{ji}(\zeta)) (|_{ij}(\zeta)) \widehat{}\} \widehat{}; \quad i, j = 1, 2; \quad i \neq j,$$

holds in *both* B_1 and B_2 . Hence, both $|_{12}(\zeta)$ and $|_{21}(\zeta)$ are *expressions*. Clearly, this holds for any pair of distinct algebras B_i, B_j in our set, and we have thus proved that if

$$(13) \quad |_{ij}(\zeta) = \text{def} = \begin{cases} 1 & (\text{in } B_i) \\ 0 & (\text{in } B_j) \end{cases}, \quad 1 \leq i, j \leq t, \quad i \neq j,$$

then every $|_{ij}(\zeta)$ is an expression built up from $\times, \widehat{}, \widetilde{}$. And since $\zeta \widetilde{} = \zeta \widehat{}^{n_i n_j - 1}$, therefore

$$(14) \quad |_{ij}(\zeta) \text{ is an expression (built up from } \times, \widehat{}), i \neq j.$$

Now, let

$$(15) \quad |_i(\zeta) = |_{i1}(\zeta) |_{i2}(\zeta) \dots |_{it}(\zeta) \text{ (no } |_{ii}(\zeta)), i = 1, \dots, t.$$

Then

$$(16) \quad |_i(\zeta) = \begin{cases} 1 & \text{(in } B_i) \\ 0 & \text{(in } B_j) \end{cases}, \quad 1 \leq i, j \leq t, i \neq j.$$

To prove the independence of $\{(B_1, \times, \widehat{}), \dots, (B_t, \times, \widehat{})\}$, let $\varphi_1, \dots, \varphi_t$ be any set of expressions, and define:

$$\psi = \{\varphi_1 |_1(\zeta)\} \times_{\widehat{}} \dots \times_{\widehat{}} \{\varphi_t |_t(\zeta)\}.$$

Now, by (14), (15), (4), it follows that ψ is an expression built up from $\times, \widehat{}, \widetilde{}$, and since $\zeta \widetilde{} = \zeta \widehat{}^m$ ($m = n_1 \dots n_t - 1$, $n_i = \text{order of } B_i$), therefore, ψ is an expression built up from $\times, \widehat{}$. Furthermore, using (16) and (5), we have, $\psi = \varphi_i$ (in each B_i). Hence the algebras $\{B_1, \dots, B_t\}$ are independent, and the theorem is proved.

An easy combination of Theorems 1, 2, and the definition of a primal cluster yields the following

THEOREM 3. *The class $\{\dots, (B_n, \times, \widehat{}), \dots\}$ of all local binary algebras of pairwise distinct orders, where $\widehat{}$ is as in Theorem 2, forms a primal cluster (species (2, 1)).*

Now, let i be any positive integer, and let $(P_i, \times, \widehat{})$ be the basic Post algebra of order i . Here, $P_i = \{0, \varrho_{i-2}, \varrho_{i-3}, \dots, \varrho_1, 1\}$, $\zeta \times \eta = \min(\zeta, \eta)$, where «min» refers to the above ordering, and where $\widehat{}$ is given by

$$(17) \quad 0 \widehat{} = 1, 1 \widehat{} = \varrho_1, \varrho_1 \widehat{} = \varrho_2, \dots, \varrho_{i-2} \widehat{} = 0.$$

In [2], the following theorem was proved.

THEOREM 4. *The class $\{\dots, (P_m, \times, \widehat{}), \dots\}$ of all basic Post algebras of distinct orders forms a primal cluster (of species (2,1)), where $\times, \widehat{}$ are as above.*

Now, suppose m and n are positive integers. Let (B_n, \times, \wedge) and (P_m, \times, \wedge) be as in Theorems 3, 4, and let the orders of B_n and P_m be n and m . We now have the following (compare with [4; 5]).

THEOREM 5. (Principal Theorem). *The class $\{(B_n, \times, \wedge)\}_{n \geq 2} \cup \{(P_m, \times, \wedge)\}_{m \geq 3}$ forms a primal cluster (species (2,1)), where $\{(B_n, \times, \wedge)\}$ and $\{(P_m, \times, \wedge)\}$ are as in Theorems 3, 4, (order of B_n is n , order of P_m is m).*

PROOF. First, observe that $(B_n, \times, \wedge) \cong (P_m, \times, \wedge)$ if and only if $n = m = 2$, and hence no two element-algebras above are isomorphic. This follows since in a local binary algebra, $x^2 = x$ holds if and only if $x = 0$ or $x = 1$. Furthermore, a careful examination of the proof of Theorem 2 shows that, in view of Theorems 3, 4, and the definition of a primal cluster, we will be through if we can show that there exist expressions (built up from \times, \wedge) $|_{ij}(\zeta)$ such that

$$(18) \quad |_{ij}(\zeta) = \begin{cases} 1 & (\text{in } B_i) \\ 0 & (\text{in } P_j) \end{cases}, \quad |_{ji}(\zeta) = \begin{cases} 0 & (\text{in } B_i) \\ 1 & (\text{in } P_j) \end{cases}.$$

Now, let $E = \zeta \zeta^\wedge \zeta^{\wedge 2} \dots \zeta^{\wedge k}$, where $k = \max\{i, j\}$. Let r be the order of the group of units in B_i , and let N be such that $\eta^N = 0$ for all nilpotent elements in B_i . By (3), (17), we have

$$|_{ij}(\zeta) = (((E^\wedge E^\wedge)^{rN})^\wedge)^r = \begin{cases} 1 & (\text{in } B_i) \\ 0 & (\text{in } P_j), \end{cases}$$

$|_{ji}(\zeta)$ as in (12).

Since $\zeta^\wedge = \zeta^{\wedge ij-1}$, both $|_{ij}(\zeta), |_{ji}(\zeta)$ are expressions (built up from \times, \wedge) and the theorem is proved.

3. Applications. In this section, we consider certain classes of binary algebras to which the above theorems apply. First, we have the following [6; p. 228].

Definition 3. Let R be any associative and commutative ring with identity 1. R is called a *local ring* if and only if R is Noetherian and the nonunits of R form an ideal.

We now have the following (compare with Definition 2).

THEOREM 6. *Let R be any finite commutative (associative) ring with identity 1 ($1 \neq 0$). R is a local ring if and only if every element in R is either nilpotent or is a unit.*

PROOF. Let R be a local ring, and let J be the radical of R . Let N be the set of nonunits of R . We claim that $J = N$. Clearly, $J \subseteq N$. Now, suppose $z \in N$. Since N is an ideal and $1 \notin N$, therefore $1 - z \notin N$. Hence, for any x in R , $z \in N$ implies $1 - zx$ is a unit and thus zx is quasi-regular. Therefore, $N \subseteq J$. Hence $N = J =$ set of nilpotent elements in R . The converse is immediate.

COROLLARY 1. *The multiplicative structure (R, \times) of any finite commutative (associative) local ring with identity is a local binary algebra. In particular, the multiplicative structure of any of the following rings is a local binary algebra:*

- (a) $GF(p^k)$; (b) $I/_{(p^k)}$ (= ring of integers, mod p^k (p prime));
- (c) $GF(p^k)[\eta_1, \dots, \eta_i]$ where each η_i is nilpotent, $\eta_i \eta_j = \eta_j \eta_i$, $a \eta_i = \eta_i a$, all i, j , and all a in $GF(p^k)$.

Thus, Theorem 5 applies to all the algebras stated in Corollary 1.

We shall conclude with reference to Foster's results [1; 2]. Indeed, let (G, \times) be any finite group, and let $G' = G \cup \{0\}$, where $y \times 0 = 0 \times y = 0$ ($y \in G'$). As usual, we call the algebra (G', \times) a *group with null*. If, in addition, the above group (G, \times) is *cyclic* of order n , we call (G', \times) an « n -field » [1]. Theorem 1 now has the following corollary which contains Foster's results [1; Theorems 29 and 32].

COROLLARY 2. *Let (G', \times) be any finite group with null, and let $\hat{}$ be any cyclic $0 \rightarrow 1$ permutation of G' . Then the algebra $(G', \times, \hat{})$ is primal (species (2,1)). In particular, $(F_n, \times, \hat{})$ is primal for any « n -field » (F_n, \times) .*

Similarly, by taking $B_n = F_n =$ « n -field » in Theorem 3 and Theorem 5, we obtain the following corollary which contains Foster's results [2; Theorems 4.1 and 4.3].

COROLLARY 3. (a) *The set $\{(F_2, \times, \hat{}), (F_3, \times, \hat{}), \dots\}$ of all « n -fields » endowed with any cyclic $0 \rightarrow 1$ permutation $\hat{}$ of each F_n , forms a primal cluster (species (2,1)). More generally, (b) if $(P_m, \times, \hat{})$ is as in Theorem 5, then*

$$\{(F_2, \times, \hat{}), (F_3, \times, \hat{}), \dots\} \cup \{(P_3, \times, \hat{}), (P_4, \times, \hat{}), \dots\}$$

forms a primal cluster.

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