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ON PERIOD RELATIONS FOR ABELIAN INTEGRALS
ON ALGEBRAIC CURVES

by A. ANDREOTTI and A. L. MAYER (*)

In primo luogo non dovra il Poeta moderno aver letti, né legger mai gli Autori antichi Latini o Greci. Imperocché nemeno gli antichi Greci o Latini hanno mai letti i moderni.

B. Marcello. Il Teatro alla Moda

Let $H_g$ be the Siegel upper half plane of rank $g > 1$ and $\Gamma$ be the modular group [26]. The space $V_g = H_g/\Gamma$ represents the space of moduli for principally polarised abelian varieties. The set of Jacobians, i.e. the moduli space of curves of genus $g$ is open and dense in a $3g - 3$ dimensional analytic subspace $M_g$ of $V_g$.

Let $\mathcal{J}$ be the counter image of $M_g$ in $H_g$. Riemann raised the question of writing a set of equations for $\mathcal{J}$ by analytic functions on $H_g$. This problem is meaningful for any $g \geq 4$ because then $\mathcal{J}$ is a proper analytic subset of $H_g$.

For $g = 4$, $\mathcal{J}$ is of codimension one and Schottky [25] was able to write a polynomial in the «theta-nulls» non identically zero and vanishing on $\mathcal{J}$, so that $\mathcal{J}$ appears as an irreducible component of the set of zeros of that polynomial.

In this paper we consider the following problem:

Let $X$ be a minimal positive polar divisor of the principally polarised abelian variety $A$, let $S(X)$ be the singular set of $X$ and introduce the following invariant $r(A) = \text{codimension of } S(X) \text{ in } X$.

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Let $V^r_g$ be the subset of $V_g$ where $r(A) \leq r$. $V^r_g$ is an analytic subset of $V_g$ and $V^{r-1}_g$ is of codimension one in $V_g$. We give here a proof of the fact that $M_g$ is an irreducible component of $V^3_g$ (§ 1, 2 and 3).

By virtue of the geometric significance of the sets $V^r_g$ we are able to write the equations of those sets (on the set over which $X$ is irreducible) in terms of thetanulls and their derivatives. We thus obtain (when $r = 3$) a generalisation of Schottky’s result for all values of $g \geq 4$ (§ 4).

At the end of the paper an outline of a procedure of Wirtinger is given which eventually may give more explicit computations and does not involve thetanulls’ derivatives.

All the sets $V^r_g$ (on the open subset where $X$ is irreducible) are shown to be algebraic sets.

1. General remarks on algebraic curves.

1. The canonical image. a) Let $C$ be a complete irreducible algebraic curve of genus $g$ defined over an algebraically closed field $k$. Let $\Omega^1$ be the sheaf of germs of holomorphic differentials on $C$ and let $\omega_1, \ldots, \omega_g$ be a basis for the vector space $H^0(C, \Omega^1)$. The canonical map

$$\Phi : C \rightarrow P_{g-1}(k)$$

is defined by

$$\Phi(x) = (\omega_1(x), \ldots, \omega_g(x)).$$

A change of basis in $H^0(C, \Omega^1)$ changes the map by an homography of $P_{g-1}(k)$ into itself. This map has the following properties (cf. [4]).

i) $\Phi$ is a morphism of $C$ onto an algebraic non singular curve $\Gamma$ of $P_{g-1}(k)$ not contained in any proper subspace of $P_{g-1}(k)$.

ii) If $C$ is not hyperelliptic $\Phi$ is an isomorphism of $C$ onto $\Gamma$. If $C$ is hyperelliptic $\Phi$ is of degree 2 and $\Gamma$ is a rational curve.

iii) If $C$ is not hyperelliptic the hypersurfaces of order $l \geq 1$ of $P_{g-1}(k)$ cut out on $\Gamma$ the complete linear series $|lK|$ where $K$ is the canonical divisor on $\Gamma$.

b) We assume in the sequel that $C$ is not hyperelliptic. From iii) and Riemann-Roch theorem one deduces that the linear system $\Sigma$ of all quadrics containing $\Gamma$ has projective dimension

$$d = \frac{1}{2} g (g + 1) - (3g - 3) - 1 = \frac{1}{2} (g - 2) (g - 3) - 1.$$
Let us represent a quadric \( \Sigma \) in \( P_{g-1}(k) \) (the \( x_i \)'s being homogeneous coordinates in that space) by the point of \( P_g(k) \), \( q = \frac{1}{2} g (g + 1) - 1 \), with homogeneous coordinates \( a_{ij} \).

The linear group \( GL(g, k) \) acts on \( P_g(k) \) by

\[
(M, (a_{ij})) \rightarrow M(a_{ij}) M^{-1}
\]

for any matrix \( M \in GL(g, k) \) where \( (a_{ij}) \) denotes the matrix of the numbers \( a_{ij} \).

The space \( P_g(k) \) decomposes then into \( g \) orbits \( W_r, 1 \leq r \leq g \), where

\[
W_r = \{(a_{ij}) \in P_g(k) | \text{ rank } (a_{ij}) = r \}.
\]

If \( \Omega_r \) is the stabiliser of a point in \( W_r \) e. g.

\[
\Omega_r = \left\{ M \in GL(g, k) \mid M \left( \begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array} \right) M^{-1} = \sigma M \left( \begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array} \right) \text{ for some } \sigma \in k^* \right\}
\]

then \( W_r = GL(g, k)/\Omega_r \) and one easily computes then the dimension of \( W_r \):

\[
\dim W_r = gr - \frac{1}{2} r (r - 1) - 1.
\]

The Zariski closure of \( W_r \) is the algebraic variety \( Y_r = \bigcup_{r \leq g} W_r \) and thus has the same dimension as \( W_r \).

**Lemma 1.** In the linear system \( \Sigma \) there are no quadrics of rank \( \leq 2 \). Moreover the quadrics of \( \Sigma \) of rank \( \leq 4 \) are represented in \( P_g(k) \) by an algebraic variety of dimension \( \geq g - 4 \) (non empty if \( g \geq 4 \)).

The quadrics of rank \( g - 3 \) being reducible, the first assertion follows from i). Since \( \Sigma \) is of codimension \( 3g - 3 \) in \( P_g(k) \) the variety \( Y_4 \cap \Sigma \) is of codimension \( \leq 3g - 3 \).

c) Quadrics of \( \Sigma \) of rank 4. Without loss of generality we may assume that the quadric \( Q \) of rank 4 has the equation \( x_1^2 x_2 - x_3 x_4 = 0 \). It is thus the projection from a projective space \( P_{g-5} \) (its singular set) of a non singular quadric of projective 3 space. This last is ruled by two distinct pencils of lines so that \( Q \) is ruled by two pencils \( |P_{g-3}| \) and \( |P_{g-3}^e| \) of projective spaces of dimension \( g - 3 \) (\( x_1 = \lambda x_3 \), \( x_2 = \lambda x_4 \) and \( x_1 = \lambda x_4 \), \( x_3 = \lambda x_2 \) for \( \lambda \in k \cup \infty \)).

Let \( \widetilde{Q} \) be the monoidal transform of \( Q \) with center \( P_{g-5} \) and let \( \pi : \widetilde{Q} \rightarrow Q \) be the natural projection. The variety \( \widetilde{Q} \) is non singular and \( \pi^{-1}(P_{g-3}) = S \) is a divisor on \( \widetilde{Q} \). The proper transforms of the two pencils
Let $P_{g-3}$ and $P'_{g-3}$ give two linear pencils $|D|$ and $|D'|$ on $\widetilde{Q}$. Let $\widetilde{\Gamma}$ be the proper transform of $\Gamma'$; then $\pi | \widetilde{\Gamma}$ is an isomorphism of $\widetilde{\Gamma}$ onto $\Gamma$. We set, identifying $\Gamma'$ with $\hat{\Gamma}$:

$$G_0 = S \cdot \Gamma, \quad g^1_p = |D| \cdot \Gamma, \quad g^1_q = |D'| \cdot \Gamma.$$ 

and remark that since $P_{g-3} + P'_{g-3}$ is a hyperplane section of $Q$ we must have

$$|G_0 + g^1_p + g^1_q| = |K|.$$ 

We have therefore the following

**Lemma 2.** Let $Q$ be a quadric of $\Sigma$ of rank 4. Then $Q$ determines

i) two linear series $g^1_p$ and $g^1_q$ without fixed points cut out on $\Gamma$ by the two ruling pencils $|P_{g-3}|$ and $|P'_{g-3}|$.

ii) a divisor $G_0 \geq 0$ whose support is in the set where the vertex of $Q$ meets $\Gamma$

such that

$$|g^1_p + g^1_q + G_0| = |K|.$$ 

d) Quadrics of $\Sigma$ of rank 3. We may assume $Q = \{x_0^2 + x_1 x_3 = 0\}$ so that $Q$ is the projection of an irreducible plane conic from a vertex $P_{g-4}$. Thus $Q$ is ruled by a single pencil $|P_{g-3}|$ of projective spaces of dimension $g-3$. One has the following

**Lemma 3.** Let $Q$ be a quadric of $\Sigma$ of rank 3. Then $Q$ determines

i) a linear series $g^1_p$ without fixed points cut out on $\Gamma$ by the ruling pencil $|P_{g-3}|$.

ii) a divisor $G_0 \geq 0$ whose support is in the set where the vertex of $Q$ meets $\Gamma$

such that

$$|2g^1_p + G_0| = |K|.$$ 

e) Conversely one has the following

**Lemma 4.** Let $G_0$ be a positive divisor on $\Gamma$ and $g^1_p, g^1_q$ two linear series without fixed points such that

$$|g^1_p + g^1_q + G_0| = |K|.$$ 

Then there is a quadric $Q$ in $\Sigma$ of rank $\leq 4$ such that $G_0, g^1_p, g^1_q$ are determined by $Q$ as in lemmas 2 and 3.
The rank of $Q$ is 4 or 3 according as $g_p^1 = g_q^1$ or $g_p^1 = g_q^1$. If one of the series $g_p^1$ or $g_q^1$ is complete then the quadric $Q$ is unique.

**Proof.** Let $G_p^1$, $G_p^2$ (resp. $G_q^1$, $G_q^2$) be two distinct divisors of $g_p^1$ (resp. $g_q^1$). Their supports are disjoint since $g_p^1$ (resp. $g_q^1$) has no fixed point. Let $l_{a,b}(X) = 0$ be the unique hyperplane which cuts out the canonical divisor $G_p^a + G_0 + G_q^b$ for $a = 1, 2$, $b = 1, 2$. The rational function on $\Gamma$

$$h = l_{11}(x) l_{22}(x) l_{12}^{-1}(x) l_{21}^{-1}(x)$$

is not identically zero and well defined on $\Gamma$ (lemma 1). Moreover it has no zeros nor poles, thus it is a constant $c \neq 0$. The quadric

$$Q = l_{11}(x) l_{22}(x) - cl_{12}(x) l_{21}(x) = 0$$

satisfies the requirement of the lemma. It is the unique quadric of that sort if say $g_p^1$ is complete because then for any $G_q^1 \in g_q^1$ the specialty index $i(G_q^1 + G_0) = 2$ and thus $G_q^1 + G_0$ determines uniquely the $P_{g-3}'$ of the pencil $|P_{g-3}'|$ corresponding to that divisor. When $G_q^1$ varies in $g_q^1$, $P_{g-3}'$ describes $|P_{g-3}'|$ and thus $Q$ being the set theoretic union of those $P_{g-3}'$ is uniquely determined.

**2. Special curves of genus $g \geq 1$**

We assume now that the curve $C$ carries a linear series $g_h^1$ of dimension 1 and degree $h$ without fixed points. If $h \leq g - 1$ this linear series is special. Let $D, D'$ be two distinct positive divisors of $g_h^1$. Since $g_h^1$ has no fixed points $D$ and $D'$ have disjoint supports and there exists a non-constant rational function $f$ on $C$ such that

$$(f) = \text{the divisor of } f = D' - D.$$  

Let $E = H^0(C, \Omega^1)$ and let

$$F = \{ \omega \in E \mid (\omega) \geq D \}.$$  

We define a linear map

$$\lambda : F \rightarrow E$$

by sending each element $\omega \in F$ into the element $f\omega$ of $E$. This map has the property that if $H$ is any subspace of $F$ such that $\lambda (H) \subset H$ then necessarily $H = 0$. In fact if $\omega \in H$ and $\omega \neq 0$, one has

$$(\omega) \geq D, (f\omega) \geq D, (f^2\omega) \geq D, \ldots, (f^h\omega) \geq D, \ldots.$$  

This for large enough $h$ (e. g. $h > 2g - 2$) is absurd.
b) We have to study the following situation. Let $E$ be a finite dimensional vector space over a field $k$, $F$ a subspace of $E$ and

$$\lambda : F \to E$$

a linear map of $F$ into $E$. We say that $\lambda$ is irreducible if for a linear subspace $H \subseteq F$ we have

$$\lambda(H) \subset H \implies H = 0.$$

An irreducible map is certainly injective since for $H = \ker \lambda$, $\lambda(H) = 0 \subset H$ and thus $\ker \lambda = 0$. If a basis $e_1, \ldots, e_r$ of $F$ is completed into a basis $e_1, \ldots, e_r, e_{r+1}, \ldots, e_s$ of $E$ the map $\lambda$ is fully described by the action on the basis of $F$:

$$\begin{align*}
\lambda(e_\alpha) &= \sum_{\beta=1}^{r} \sigma_{\alpha\beta} e_\beta \\
1 &\leq \alpha \leq r
\end{align*}$$

i.e. by the matrix $\sigma = (\sigma_{\alpha\beta})$.

Assume $\lambda$ irreducible and consider the sequence of spaces

$$F, F \cap \lambda(F) = F_1, F_1 \cap \lambda(F_1) = F_2, \ldots$$

One has $\dim F_1 < \dim F$ otherwise $\lambda(F) = F$ and $F = 0$, similarly $\dim F_2 < \dim F_1$ otherwise $\lambda(F_1) = F_1$ and $F_1 = 0$, etc. There exists therefore an integer $\mu$ such that

$$F_\mu \neq 0, \quad F_\mu \cap \lambda(F_\mu) = 0.$$

**Lemma 5.** Under the specified assumptions one can find $\mu + 1$ subspaces $E_1, \ldots, E_{\mu+1}$ in $F$ such that

i) $F = \bigoplus_0^\mu \lambda^\alpha(E_\alpha) \bigoplus_0^{\mu-1} \lambda^\alpha(E_2) \bigoplus \ldots \bigoplus E_{\mu+1}$

ii) $F + \lambda(F) = \bigoplus_0^{\mu+1} \lambda^\alpha(E_\alpha) \bigoplus_0^{\mu} \lambda^\alpha(E_2) \bigoplus \ldots \bigoplus \lambda^\alpha(E_{\mu+1})$.

**Proof.** If $\mu = 0$ we get

$$F + \lambda(F) = F \oplus \lambda(F).$$
By induction we may assume
\[ F_\ell = \bigoplus_{\ell=0}^{\ell'} \lambda^\ell (B_1) \bigoplus \lambda^{\mu-1} (B_2) \bigoplus \ldots \bigoplus \lambda^{\mu+1} (B_{\mu+1}) \]
\[ F_\ell + \lambda (F_\ell) = \bigoplus_{\ell=0}^{\ell'+1} \lambda^\ell (B_1) \bigoplus \lambda^{\mu'} (B_2) \bigoplus \ldots \bigoplus \lambda^{\mu+1} (B_{\mu'+1}) \]
where \( \mu' = \mu - \varphi \).

Since \( F_\ell = F_{\ell-1} \cap \lambda (F_{\ell-1}) \) both \( F_\ell \) and \( \lambda (F_\ell) \) are contained in \( \lambda (F_{\ell-1}) \).

Let \( B_{\mu'+2} \) be a complement in \( \lambda (F_{\ell-1}) \) of \( F_\ell + \lambda (F_\ell) \). We thus have
\[ \lambda (F_{\ell-1}) = \bigoplus_{\ell=0}^{\ell'+1} \lambda^\ell (B_1) \bigoplus \ldots \bigoplus \lambda^\ell (B_{\mu'+1}) \bigoplus B_{\mu'+2} \]

We set
\[ E_1 = \lambda^{-1} (B_1), \ldots, E_{\mu'+2} = \lambda^{-1} (B_{\mu'+2}) \]
so that
\[ F_{\ell-1} = \bigoplus_{\ell=0}^{\ell'+1} \lambda^\ell (E_1) \bigoplus \ldots \bigoplus \lambda^\ell (E_{\mu'+1}) \bigoplus E_{\mu'+2} \]
and we see that
\[ F_{\ell-1} + \lambda (F_{\ell-1}) = \bigoplus_{\ell=0}^{\ell'+2} \lambda^\ell (E_2) \bigoplus \ldots \bigoplus \lambda^\ell (E_{\mu'+2}) \bigoplus \lambda^\ell (E_{\mu'+2}) \]

**Corollary 1.** Choosing a proper basis in \( F \) and completing it to a proper basis of \( E \) the matrix \( \sigma \) of the map \( \lambda \) can be given the form
\[ \sigma = \begin{pmatrix} U_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & U_1 & 0 \end{pmatrix} \]
where each \( U_\alpha \) is a \( r_\alpha \times s_\alpha \) rectangular matrix with \( s_\alpha \geq r_\alpha \) of the form \((0, 1)\).

Note that \( l \leq \dim E - \dim F \).

**Corollary 2.** With respect to a choice of the basis as in the previous corollary the \( \frac{1}{2} r (r + 1) \) elements
\[ Q_{\alpha\beta} = e_\alpha \lambda (e_\beta) - e_\beta \lambda (e_\alpha) \]
where \( 1 \leq \alpha < \beta \leq r \)

of the symmetric tensor product of \( E \) with itself are linearly independent.
In fact if $\sum C_{\alpha \beta} Q_{\alpha \beta} = 0$, since $e_1$ occurs only in $Q_1$, we must have $\sum C_{\alpha \beta} (\epsilon_\beta) = 0$. Since $\lambda$ is injective $C_{\alpha \beta} = 0 \forall \beta$. Since $e_2$ occurs only in $Q_2$, we must have $\sum C_{\alpha \beta} (\epsilon_\beta) = 0$ and thus $C_{2\beta} = 0 \forall \beta$. Continuing in this way we get $C_{\alpha \beta} = 0 \forall \alpha, \beta$.

c) Returning to the situation described at the beginning, by application of the previous lemma and corollaries we obtain the following

**Lemma 6** Let $\Gamma$ be non hyperelliptic canonical curve of genus $g \geq 4$. Let $D$ and $D'$ be two disjoint positive divisors on $\Gamma$, linearly equivalent and of degree $h \leq g - 1$. Let $f$ be a rational function on $\Gamma$ with $(f) = D' - D$ and let $(l_0, ..., l_\alpha)$ be a basis of the linear forms on $P_{g-1}(k)$ with $(l_0) \geq D$. If $(l'_0, ..., l'_{\alpha})$ is a basis of linear forms on $P_{g-1}(k)$ with $(l'_0) \geq D'$ such that

$$l'_a = fl_a$$

then $\Gamma$ lies in the algebraic variety $\Phi$ of $P_{g-1}(k)$ defined by

$$\text{rank} \begin{pmatrix} l_1 & \ldots & l_r \\ l'_1 & \ldots & l'_{r'} \end{pmatrix} \leq 1$$

and the $\frac{1}{2} r (r - 1)$ quadratic forms $l'_a l'_\beta - l_\alpha l'_\beta (\alpha < \beta)$ are linearly independent.

**Remark** $r = \text{dim}_k (F) =$ specialty index of $D$.

3. **Curves with a $g_3^1$** a) If a non hyperelliptic curve $C$ carries a linear series $g_3^1$, this must be complete and without fixed points (otherwise $C$ would be hyperelliptic). If the characteristic of the groundfield $k$ is $\neq 3$ one can construct a curve of this type for any value of $g \geq 3$. For instance for $g = 3k - 1$ we can take for $C$ the normalisation of the plane curve

$$z^{3k-2} y^3 = x^{3k-1} \prod_{1}^{3k-1} (x - a_i z).$$

For the other values of $g$ we can take the normalisation of the plane curve

$$z^{r-2} y^3 = \prod_{1}^{r} (x - a_i z),$$

whose genus is $r - 1$ or $r - 2$ according to whether $r$ is prime or not to 3.

**Proposition 1.** Let $\Gamma$ be the canonical model of a non hyperelliptic curve $C$ of genus $g \geq 4$ carrying a $g_3^1$. 
Then each divisor of the \( g^1_3 \) on \( \Gamma \) lies on a straight line and these lines describe a ruled surface \( \Phi \) of degree \( g - 2 \) whose equations by a suitable choice of coordinates can be put in the form

\[
\text{rank} \left( \begin{array}{cccc}
x_0 & \ldots & x_{m-1} & x_{m+1} & \ldots & x_{g-2} \\
x_1 & \ldots & x_m & x_{m+2} & \ldots & x_{g-1}
\end{array} \right) \leq 1.
\]

**Proof.** Since the \( g^1_3 \) is complete the index of specialty of each divisor \( D \in g^1_3 \) is \( i(D) = g - 2 \). This means that \( D \) lies on a straight line.

In this case the space \( F \) described in the previous section is of codimension 2 in \( E = H^0(\Gamma, \Omega^1) \). Except in the case \( g = 4 \) where \( \Gamma \) lies on a cone, \( F + \lambda F = E \), since \( i(D + D') \leq g - 4 \) by Clifford's theorem [4]. Thus for the matrix \( \sigma \) in the canonical form of Corollary 1, we must have \( l = 1 \) or \( l = 2 \).

With the notations of lemma 6, the divisor of \( g^1_3 \) being given as \( (f = \text{const.}) \), we recognize that the locus of these lines is the variety \( \Phi \). By the choice of the basis we see that in each case \( F \) can be written in the given form.

**Remark 1.** For \( g \geq 5 \) one has \( 1 \leq m < g - 2 \) so that the surface \( \Phi \) is non singular. For \( g = 4 \), \( \Phi \) could very well be the cone \( x_0 x_2 - x_1^2 = 0 \) but there exists always a curve of genus 4 for which \( \Phi \) is non singular.

Indeed if \( m = g - 2 \), \( \Phi \) is the cone of the straight lines joining the point \((0, \ldots, 0, 1)\) to the points of a rational normal curve in \( X_{g-1} = 0 \). The order of the cone is \( g - 2 \). Since \( \Gamma \) lies on this cone and is of order \( 2g - 2 \) the order of the cone is \( \leq \frac{2g - 2}{3} \) (a hyperplane though the vertex cuts the cone in at most \( \frac{2g - 2}{3} \) generators). Hence \( g \leq 4 \).

For \( g = 4 \) any non singular curve which is the complete intersection of a quadric and a cubic in \( P_3(k) \) is the canonical image of a curve of genus 4.

Except possibly for \( g = 4 \) we can assume \( 1 \leq m \leq \frac{1}{2} (g - 2) \). This invariant was first discussed by Maroni [15].

**Remark 2.** Each quadric through the curve \( \Gamma \) contains the surface \( \Phi \). The \( \frac{1}{2} (g - 2) (g - 3) \) quadrics obtained from the second order minors of the matrix defining \( \Phi \) are linearly independent, and thus span the full system \( \Sigma \) of all quadrics through \( \Gamma \).
PROPOSITION 2. Let $C$ be a non hyperelliptic curve of genus $g \geq 4$ carrying a positive divisor $D$ of degree 3 with $\dim |D| = 1$. Let $E_0$ be a divisor of degree $g - 1$ with $\dim |E_0| = 1$. Then there exists a divisor $E \in |E_0|$ or a divisor $E' \in |K - E_0|$ such that either $E$ or $E'$ is $\sim D$.

PROOF. Since $C$ is not hyperelliptic we can identify $C$ with its canonical image $\Gamma$.

Let $|E_0| = g - 1 = g^1_1 + P_1 + \ldots + P_{g-1-p}$ where $P_1 + \ldots + P_{g-1-p}$ is the fixed part of $|E_0|$ so that $g^1_1$ has no fixed points.

Since $\dim |E_0| = 1$ we have $i(E_0) = 2$ so that $\dim |K - E_0| = 1$.

Let analogously $|K - E_0| = g - 1 = Q_1 + \ldots + Q_{g-1-q}$ where $Q_1 + \ldots + Q_{g-1-q}$ is the fixed part of $|K - E_0|$ and $g^1_1$ has no fixed points.

We construct the quadric $Q \in \Sigma$ corresponding to the data of the series $g^1_1$, $g^1_2$ and of the divisor $G_0 = P_1 + \ldots + Q_{g-1-q}$, as in lemma 4. Let $V$ be the vertex of $Q$. This is a projective space of dimension $g - 5$ or $g - 4$ according as the rank of $Q$ is 4 or 3. Let $A \in |D|$ be a divisor consisting of 3 distinct points and disjoint from the finite set $V \cap \Gamma$. The 3 points of $A$ lie on a straight line $l$ not contained in $V$. Consider the projection with center $V$ on a $P_3(k)$ or $P_2(k)$ (according to the dimension of $V$) not meeting $V$. The projection of $l$ will be a line or a point in the image space. Therefore in any case $l$ lies on a space $P_{g-3}$ of one of the rulings of $Q$. This implies that either $A \in g^1_2$ or $A \in g^1_2$ and this proves our contention.

COROLLARY. If $C$ is a non hyperelliptic curve of genus $g \geq 4$ carrying a $g^1_1$, then for every complete linear series $g^1_{g-1}$ of degree $g - 1$ and dimension 1 one has

either $g^1_{g-1} = g^1_1 + P_1 + \ldots + P_{g-4}$

or $|K - g^1_{g-1}| = g^1_{g-1} = g^1_1 + P_1 + \ldots + P_{g-4}$

where $P_1 + \ldots + P_{g-4}$ is a fixed divisor of degree $g - 4$.

For any integer $s \geq 1$ we denote by $(C)^s$ the $s$-fold symmetric product of $C$. This is a non singular algebraic variety [4].

PROPOSITION 3. Under the same assumptions for $C$, there exists a proper subvariety $S \subset (C)^{(g-4)}$ such that for

$$(P_1 + \ldots + P_{g-4}) \in (C)^{(g-4)} - S$$

the linear series

$$g^1_1 + P_1 + \ldots + P_{g-4}$$

is a complete linear series of degree $g - 1$ and dimension 1.
PROOF. Let $D \in g_2^1$. The above condition on $P_1, \ldots, P_{g-4}$ is equivalent to $i(D + P_1 + \ldots + P_{g-4}) = 2$, i.e. the exceptional set is the set where $i(D + P_1 + \ldots + P_{g-4}) > 2$.

First one can select $P_1, \ldots, P_{g-4}$ such that $i(D + P_1 + \ldots + P_{g-4}) = 2$. In fact $|K - g_2| = g_{g-3}^3$. We can select $P_1$ outside the fixed divisor of this series. Then $P_2 \neq P_1$ outside the fixed divisor of $|g_{g-5}^g - P_1| = g_{g-6}^{g-4}$ and so on. We end up by selecting $P_1, P_2, \ldots, P_{g-4}$ such that $\dim |K - g_3^2 - P_1 - \ldots - P_{g-4}| = 1$. This is what we wanted to prove. Secondly we remark that if $a = (a_1, \ldots, a_g)$ and $b = (b_1, \ldots, b_g)$ are two distinct points on the line containing $D$ on the canonical curve $\Gamma$, the condition $i(D + P_1 + \ldots + P_{g-4}) > 2$ is equivalent to the condition

$$\operatorname{rank} \left( \begin{array}{c}
\omega_1(P_1), \ldots, \omega_1(P_{g-4}) a_1 b_1 \\
\vdots \\
\omega_g(P_1), \ldots, \omega_g(P_{g-4}) a_g b_g 
\end{array} \right) \leq g - 3.$$  

This condition on the cartesian product $C^{g-4}$ defines a proper analytic set $\tilde{S}$ invariant by the action of the symmetric group. Its image $S$ in $(C)^{g-4}$ by the natural map $C^{g-4} \to (C)^{g-4}$ is a proper analytic subset of the space $(C)^{g-4}$.

To a complete linear series $g_{g-1}^{g-1}$ on the canonical curve $\Gamma$ corresponds a unique quadric $Q$ of rank $\leq 4$ through the canonical curve. This quadric is described by the spaces $P_{g-3}$ spanned by the divisors of $g_{g-4}^1$. In particular in the case under consideration we can consider the quadrics of rank $\leq 4$ corresponding to the complete linear series of the form

$$g_2^1 + P_1 + \ldots + P_{g-4}$$

where the $P_i$'s are distinct generic points on $\Gamma$.

**Proposition 4.** Under the same assumptions on $C$ we can choose \( \frac{1}{2} (g - 2)(g - 3) \) complete linear series of degree $g - 1$ and dimension 1 of the form $g_{g-1}^1 + P_1 + \ldots + P_{g-4}$, with distinct $P_i$'s, such that the corresponding quadrics of rank $\leq 4$ are linearly independent and thus span the full system of quadrics through the canonical curve.

**Proof.** Consider the projection map

$$\lambda_{a\beta} : C^{g-2} \to C^{g-4}$$
defined by \( \lambda_{a\beta} (P_1 \times ... \times P_{g-2}) = P_1 \times ... \times \widehat{P}_a \times ... \times \widehat{P}_\beta \times ... \times P_{g-2} \). Let \( \hat{S} \) be the counter image in \( C^{g-4} \) of the set \( S \) defined in proposition 3. The set \( \bigcup_{a \leq \beta} \lambda_{a\beta}^{-1} (S) \) is a proper analytic subset of \( C^{g-2} \) and thus we can select \( P_1, ..., P_{g-2} \) on \( \Gamma' \) such that

i) the points \( P_1, ..., P_{g-2} \) are distinct

ii) for any choice of \( \alpha < \beta, 1 \leq \alpha, \beta \leq g-2 \), the linear series \( g_1^1 + P_1 + ... + \widehat{P}_a + ... + \widehat{P}_\beta + ... + P_{g-2} \) is complete of dimension 1.

iii) no divisor of the series \( g_{2g-5}^{g-3} = |K - g_3^1| \) is \( \geq P_1 + ... + P_{g-2} \).

For any \( \alpha, 1 \leq \alpha \leq g-2 \) there is a unique divisor \( H_\alpha \in g_{2g-5}^{g-3} \) such that

\[ H_\alpha \geq P_1 + ... + \widehat{P}_a + ... + P_{g-2}. \]

The \( g-2 \) divisors \( H_\alpha \) are linearly independent divisors of \( g_{2g-5}^{g-3} \). This can be seen as follows: we represent the divisors of \( g_{2g-5}^{g-3} \) by the points of a projective space \( P_{g-3} \). The sets \( E_\alpha = \{ D \in g_{2g-5}^{g-3} : D \geq P_\alpha \} \) are represented by hyperplanes, and by iii) we have \( g_{2g-5}^{g-3} \cap E_\alpha = \emptyset \). Thus these hyperplanes are linearly independent and therefore the points \( H_\alpha = \bigcap_{\beta \neq \alpha} E_\beta \) are also linearly independent. Let \( D, D' \) be two distinct divisors of \( g_3^1 \) on the canonical curve \( \Gamma' \). Then \( D + H_\alpha \) is the divisor of a linear form \( l_\alpha \) and \( D' + H_\alpha \) is the divisor of another linear form \( l'_\alpha \). The linear forms \( l_\alpha \) are linearly independent because the divisors \( H_\alpha \) are linearly independent. The same is true for the forms \( l'_\alpha \).

Consider the rational functions

\[ h_{a\beta} = \frac{l_\alpha v_\beta}{l_\beta v_\alpha} \quad \text{for} \quad \alpha \neq \beta. \]

These, having no zeros or poles on \( \Gamma' \), are constants \( \neq 0 \) on \( \Gamma' \). Since \( h_{a\beta} h_{\beta\gamma} h_{\gamma\alpha} = 1 \) and \( h_{a\beta} = h_{\beta\alpha}^{-1} \) we can find non zero constants \( C_\alpha \) such that

\[ h_{a\beta} = C_\beta C_\alpha^{-1}. \]

Replacing \( l_\alpha \) by \( C_\alpha l_\alpha \) we may thus assume \( h_{a\beta} = 1 \) so that on \( \Gamma' \)

\[ \text{rank} \left( \begin{array}{c} l_1, l_2, ..., l_{g-2} \\ l_1', l_2', ..., l_{g-2}' \end{array} \right) = 1. \]

This is the equation of the ruled surface \( \Phi \) (cf. lemma 6 and proposition 1). We know that the \( \frac{1}{2} (g-2)(g-3) \) quadrics \( Q_{a\beta} \) are \( \det \left( \begin{array}{c} l_\alpha l_\beta \\ l'_\alpha l'_\beta \end{array} \right) = 0 \) for \( \alpha < \beta 1 \leq \alpha, \beta \leq g-2 \) are linearly independent. By construction the quadric \( Q_{a\beta} = 0 \) corresponds to the linear series \( g_3^1 + P_1 + ... + \widehat{P}_a + ... + \widehat{P}_\beta + ... + P_{g-2} \).
4. We end this section with a proposition which shows that the complete special series \( g_{g-1}^r \) of dimension \( > 1 \) are limiting cases of complete linear series \( g_{g-1}^1 \) of dimension 1.

**Proposition 5.** Let \( g_{g-1}^r \) be a complete linear series of degree \( g-1 \) and dimension \( r \geq 1 \) on the curve \( C \). We can choose a divisor \( D_0 \in g_{g-1}^r \) such that in any neighborhood of \( (D_0) \) in \( (C)_{g-1} \) there is a divisor \( D \) of \( g-1 \) distinct points with \( \dim |D| = 1 \).

**Proof.** Choose \( P_1, \ldots, P_{r-1} \) on \( C \) distinct and such that \( g_{g-1}^1 = |g_{g-1}^{g-1} - P_1 - \ldots - P_{r-1}| \) is of dimension 1. Let \( A_0 \) be the fixed divisor of \( g_{g-1}^r \) so that \( g_{g-1}^1 = A_0 + g_1^1 \), \( l = g - r - s \), \( s = \) degree of \( A_0 \). Let \( D_0 \in g_1^1 \) be a divisor of \( l \) distinct points and consider the map

\[
\lambda: (C)^{r+1} \rightarrow (C)^{g-1}
\]

defined by \( \lambda((Q_1) + \ldots + (Q_{r+s-1})) = (D_0) + (Q_1) + \ldots + (Q_{r+s-1}) \). The image of \( \lambda \) is an irreducible subvariety of \( (C)^{g-1} \) containing the divisor \( D = D_0 + \ldots + D_0 + P_1 + \ldots + P_{r-1} \in g_{g-1}^r \).

If \( (Q_1) + \ldots + (Q_{r+s-1}) \) is generic, (see proposition 3), we see that

\[
|D_0 + Q_1 + \ldots + Q_{r+s-1}| = g_1^1 + Q_1 + \ldots + Q_{r+s-1}.
\]

Given any neighborhood of \( (D) \) in \( (C)^{g-1} \), we can find \( Q_1, \ldots, Q_{r+s-1} \) such that i) \( D_0 + Q_1 + \ldots + Q_{r+s-1} \) consists of distinct points, ii) the complete series \( |D_0 + Q_1 + \ldots + Q_{r+s-1}| \) has dimension 1, iii) \( \lambda((Q_1) + \ldots + (Q_{r+s-1})) \in U \).

5. Hyperscopic curves. In this case the canonical image \( \Gamma \) of \( C \) is a rational twisted curve of \( P_{g-1}(k) \) of degree \( g-1 \). Every complete linear series \( g_{g-1}^1 \) is of the form \( g_1^2 + P_1 + \ldots + P_{g-3} \), and conversely if \( P_1, \ldots, P_{g-3} \) are generic this series is complete. To each one of these series corresponds a quadric of rank 3 through \( \Gamma \) which is the projection of \( \Gamma \) from the space of dimension \( g-4 \) spanned by the images of the points \( P_1, \ldots, P_{g-3} \). Choosing \( g-1 \) linearly independent points \( P_1, \ldots, P_{g-1} \) on \( \Gamma \) the \( \frac{1}{2} (g-1)(g-2) \) quadrics of rank 3 projecting \( \Gamma \) from the space spanned by \( P_1, \ldots, \hat{P}_a, \hat{P}_\beta, \ldots, P_{g-1} \) for \( 1 \leq a, \beta \leq g-1 \), \( z < \beta \), are linearly independent and span the full system of quadrics through \( \Gamma \).

2. Theta functions and theta divisors.

6. The theta function. a) By \( H_g \) we denote the Siegel space of rank \( g \), i.e. \( H_g = \{ z = z^{(g, \cdot)}, z = z, \text{Im} z > 0 \} \), \( z^{(g, \cdot)} \) denoting a \( g \times g \) matrix with
complex elements. Let \( u = (u_1, \ldots, u_g) \) be coordinates in \( \mathbb{C}^g \). For every \( z \in H_g \) the theta function is the following Fourier series:

\[
\theta(u, z) = \sum_{m \in \mathbb{Z}^g} \exp(\pi i (\langle m, u \rangle + 2 \langle m, u \rangle)).
\]

This series has the following properties (cf [6] [12] [13] [27] [29])

i) it is uniformly convergent on any compact subset of \( \mathbb{C}^g \times H_g \)

ii) it has the following periodicity properties

for any \( n, m \in \mathbb{Z}^g \).

iii) it satisfies the « heat equations »

\[
\frac{\partial \theta}{\partial z_{a\beta}} = 2 \pi i (1 + \delta_{a\beta}) \frac{\partial^2 \theta}{\partial u_a \partial u_\beta},
\]

where \( \delta_{a\beta} \) is the Kronecker delta, for \( 1 \leq \alpha \leq \beta \leq g \).

iv) for any \( z_0 \in H_g \), \( \theta(u, z_0) \) is not identically zero but vanishes somewhere in \( \mathbb{C}^g \).

We set \( \Omega(z) = (I, z) \) and we consider the following representation \( \phi \) of \( \mathbb{Z}^g \) as a group of automorphisms of \( \mathbb{C}^g \times H_g \) associating to every vector \( \gamma \in \mathbb{Z}^g \) the map

\[
\phi(\gamma) \equiv \begin{cases} 
(u, z) & \rightarrow (u + \Omega(z)\gamma, z) \\
(z, z) & \rightarrow z.
\end{cases}
\]

The quotient space \( \mathbb{V} = (\mathbb{C}^g \times H_g)/\phi(\mathbb{Z}^g) \) is a complex manifold and we have a commutative diagram of holomorphic maps:

\[
\begin{array}{ccc}
\mathbb{C}^g \times H_g & \xrightarrow{p} & H_g \\
\downarrow \phi & & \downarrow \phi \\
\mathbb{V} & \xrightarrow{i} & H_g
\end{array}
\]

Since \( H_g \) is topologically a cell, \( \mathbb{C}^g \times H_g \) is the universal covering space of \( \mathbb{V} \). For every \( z \in H_g \) let \( A_z \) be the discrete subgroup (of maximal rank) of \( \mathbb{C}^g \) generated by the column vectors of \( \Omega(z) \). Then \( \phi \) is a proper map, and for each \( z \in H_g \), \( \phi^{-1}(z) \) is the complex torus \( \mathbb{C}^g/A_z \).
b) We define

\[ A(\theta) = \{(u, z) \in \mathcal{V} \times H_g \mid \theta(u, z) = 0\} \]

\[ B(\theta) = \left\{(u, z) \in A(\theta) \mid \frac{\partial \theta}{\partial u_n}(u, z) = 0 \text{ for } 1 \leq n \leq g\right\}. \]

Let \( \tau = \operatorname{pr}_H|B(\theta) \), for any integer \( s \) with \( 0 \leq s \leq g - 2 \) we define

\[ B_s(\theta) = \{x \in B(\theta) \mid \dim \tau^{-1} \tau(x) \geq s\} \]

so that \( B_0(\theta) = B(\theta) \).

The sets \( A(\theta) \) and \( B(\theta) \) are analytic subsets of \( \mathcal{V} \times H_g \).

**Lemma 7.** For any \( s, 0 \leq s \leq g - 2 \), the sets \( B_s(\theta) \) are analytic subsets of \( \mathcal{V} \times H_g \).

This is almost an immediate consequence of a theorem of Remmert ([24] Satz 17). We remark that the sets \( A(\theta) \) and \( B_s(\theta) \) are invariant under the action of \( \theta(\mathbb{Z}^{\infty}) \).

**Proposition 6.** i) The set \( A(\theta) \) is of pure dimension \( \frac{1}{2} g(g + 1) + g - 1 \) and \( \operatorname{pr}_H(A(\theta)) = H_g \).

ii) For any \( s, 0 \leq s \leq g - 2 \), the sets \( \operatorname{pr}_H(B_s(\theta)) \) are analytic and \( \operatorname{pr}_H(B_s(\theta)) \) is a proper analytic subset of \( H_g \).

**Proof.** The first part of the proposition is a consequence of the property iv) of the \( \theta \)-function. Moreover \( A(\theta) \) being non void and the set of zeros of a holomorphic function in \( \mathcal{V} \times H_g \) is of codimension one in that space.

To prove the second part we first remark that \( \mu \) being a local isomorphism \( \mu(B_s(\theta)) \) is analytic. Since \( \tilde{\mu} \) is proper it follows that \( \tilde{\mu} \mu(B_s(\theta)) \) is an analytic subset of \( H_g \). It remains to prove that \( \operatorname{pr}_H(B_s(\theta)) \neq H_g \). This is a straightforward consequence of the following two lemmas:

**Lemma 8.** Let \( \mathcal{V} \) be a complex space with countable topology and let \( \tilde{\omega} : \mathcal{V} \to U \) be a proper holomorphic surjective map of \( \mathcal{V} \) onto an open subset \( U \subset \mathbb{C}^n \). Then there exists an open subset \( V \subset U \) and a holomorphic section \( s : V \to \mathcal{V} \) (i.e. \( \tilde{\omega} \cdot s(u) = u \forall u \in V, V \neq \emptyset \)).

**Proof.** We may assume \( U \) connected; also since \( \tilde{\omega} \) is proper we may assume \( \mathcal{V} \) irreducible. If \( \mathcal{S}(\mathcal{V}) \) is the singular set of \( \mathcal{V} \) either \( \tilde{\omega}(\mathcal{S}(\mathcal{V})) = U \) or it is a proper analytic subset of \( U \); in this case we may replace \( U \) with
and \( \mathcal{V} \) with \( \mathcal{V}' = \phi^{-1}(\mathcal{S}(\mathcal{V})) \). By this procedure we see that we can assume that \( U \) is connected and \( \mathcal{V}' \) is a connected manifold. Also by analogous procedure we see that it is not restrictive to assume that the rank of the map \( \phi \) is constant in \( \mathcal{V}' \). Thus we need only show that under these conditions the rank of \( \phi \) equals \( n \) in \( \mathcal{V}' \). If the rank of \( \phi \) is strictly \( < n \) then each point \( x \in \mathcal{V}' \) has a neighborhood \( N(x) \) such that \( U = \phi(N(x)) \) is nowhere dense by virtue of a lemma of Remmert ([24], p. 348-350). Cover \( \mathcal{V}' \) with a countable union \( \{ N(x_i) \}_{i \in \mathbb{N}} \) of such neighborhoods then \( \phi^{-1}(\mathcal{V}') = \bigcup_{i \in \mathbb{N}} \phi^{-1}(N(x_i)) \). This is absurd by a well known theorem of Baire.

The second lemma is a unicity theorem for the Cauchy problem of the heat equations.

Let \( U \) be open and connected in \( \mathbb{C}^\nu \) and \( V \) be open and connected in \( \mathbb{C}^{\nu+1} \); let \( u = (u_1, \ldots, u_\nu) \) be holomorphic coordinates in \( \mathbb{C}^\nu \) and let \( z = (z_\alpha) \), \( 1 \leq \alpha \leq \nu, 1 \leq \beta \leq \nu, \ z = z \), be holomorphic coordinates in \( \mathbb{C}^{\nu+1} \).

The «heat equations» in \( U \times V \) are a special case of a system of partial differential equations in the unknown function \( v \) of the form

\[
\begin{align*}
\frac{\partial^2 v}{\partial u_j \partial u_k} &= \sum a_{jk}^\nu (u, z) \frac{\partial v}{\partial z_\alpha} + \sum b_{j\beta}^\nu (u, z) \frac{\partial v}{\partial u_\beta} + c_{jk} (u, z) v \\
1 \leq j, k \leq \nu
\end{align*}
\]

where the \( a \)'s, \( b \)'s, \( c \)'s are holomorphic functions in \( U \times V \).

**Lemma 9.** Let \( u = s(z) \) be a holomorphic section of \( U \times V \rightarrow V \) and let \( v = v(u, z) \) be a holomorphic solution of (I) in \( U \times V \). If

\[
v(s(z), z) = 0, \quad \frac{\partial v}{\partial u_i} (s(z), z) = 0, \ 1 \leq i \leq \nu
\]

then \( v \) is identically zero.

**Proof.** In \( \mathbb{C}^\nu \times V \) we can perform the change of coordinates

\[
\begin{align*}
u' &= u - s(z) \\
z' &= z.
\end{align*}
\]

Then the system (I) is changed into a system of the same type. In the new system of coordinates the section \( s \) is reduced to \( u = 0 \). Without loss
of generality we may thus assume \( s(z) = 0 \). Let

\[ v(u, z) = \sum_{a \in \mathbb{N}^g} a_a(z) u^a \]

be the Taylor expansion of \( v(u, z) \) near \( u = 0 \).

Substituting in (I) we get:

\[
\sum_{a \in \mathbb{N}^g} a_a(z) \frac{\partial^2 v}{\partial u^i \partial u^k} = \sum_{a \in \mathbb{N}^g} a_a(z) \frac{\partial v}{\partial z_\alpha} u^a + \sum_{b \in \mathbb{N}^g} b_b(u, z) \sum_{a \in \mathbb{N}^g} a_a(z) \frac{\partial w}{\partial u^i} a_{b_a(z) u^a}.
\]

Let \( \mathfrak{g} \) be the ideal generated by \( u_1, \ldots, u_g \) in the ring of formal power series \( \mathbb{C}[u_1, \ldots, u_g] \). By the assumption, for any \( z_0 \in \mathfrak{g} \) we have \( v(z_0, u) \in \mathfrak{g}^2 \).

Equations (II) imply that if \( v(z_0, u) \in \mathfrak{g}^k \) for \( k \geq 2 \), then \( v(z_0, u) \in \mathfrak{g}^{k+1} \). Hence for any \( z_0 \in \mathfrak{g} \), \( v(z_0, u) \in \bigcap_{k \geq 1} \mathfrak{g}^k = 0 \).

The sets \( N_s = \text{pr}_{H_g}(B_s(\theta)) \) will be called the \textit{ramification sets of order} \( s \) in \( H_g \), \( 0 \leq s \leq g - 2 \). There are the obvious inclusions

\[ N_{g-2} \subset \cdots \subset N_1 \subset N_0 = H_g. \]

We note that this filtration of \( H_g \) by analytic sets is invariant under the action of the modular group. This follows from the theory of transformation of theta functions (cf. n. 15).

7. \textit{The theta divisor on a Jacobian variety.} a) Let \( X \) be the Riemann surface of the algebraic curve \( C \) and let \( \gamma_1, \ldots, \gamma_{2g} \) be a basis of \( H^1(X, \mathbb{Z}) \) with the intersection matrix

\[
\begin{pmatrix}
\langle \gamma_i, \gamma_j \rangle \\
0 & I
\end{pmatrix}.
\]

Let \( \omega_1, \ldots, \omega_g \) be a basis of \( H^0(C, \mathcal{O}_1^1) \) so normalised that the period matrix

\[
\left( \int_{\gamma_i} \omega_j \right)
\]

has the form \( (I, z) \). As is well known \( z \) is a \( g \times g \) matrix with complex elements such that \( \text{Im}(z) > 0 \). The matrix \( z \) represents thus a point of the Siegel space \( H_g \). We denote by \( J(C) \) the complex torus \( \mathbb{C}^g / \Lambda_z \). If \( D \) is a divisor of degree 0 on \( C \) and \( \sigma \) is a differentiable 1-chain on \( X \) such that \( D = \sigma \partial \sigma \) then the point \( \lambda(D) = \left( \int_{\sigma} \omega_1, \ldots, \int_{\sigma} \omega_g \right) \in \mathbb{C}^g \) is well determined by \( D \) modulo the elements of \( \Lambda_z \). The map \( \lambda \) defines therefore
a map (that we will still denote by $\lambda$) of the group $G_0$ of all divisors on $X$ of degree 0 on $J(C)$. Abel's theorem asserts that $\lambda(D) = 0$ if and only if $D$ is linearly equivalent to zero: $D \equiv 0$. Moreover the image of $\lambda$, $\lambda(G_0)$ is the whole torus $J(C)$ so that $J(C)$ is isomorphic to the group of classes of divisors (by linear equivalence) of degree zero. The torus $J(C)$ is thus the «Jacobian variety» of $C$.

b) Let $P_0$ be a fixed point on $C$. For every $P \in C$ we set

$$\psi(P) = \left( \int_{P_0}^P \omega_1, \ldots, \int_{P_0}^P \omega_g \right) \text{ mod } A_z,$$

In this way we define a holomorphic map $\psi : C \to J(C)$. This map is one to one (if $g \geq 1$) and an isomorphism of $C$ onto $\psi(C)$. If $C^g$ represents the tangent space to $J(C)$ at the origin and $P_{g-1}(C)$ represents the set of lines of $C^g$ through the origin, by associating to every point $p \in \psi(C)$ the tangent line to $\psi(C)$ at $p$ translated to the origin of $J(C)$ we define a map $\sigma : \psi(C) \to P_{g-1}(C)$ analogous to the «Gauss map». It is worth noticing that the canonical map $\Phi$ described in section 1 is nothing else than the composition of the map $\psi$ just defined and the map $\sigma$, $\Phi = \sigma \circ \psi$.

By linearity we can extend the map $\psi$ to the whole $G_0$. In particular for every $h > 0$ we obtain a holomorphic map

$$\pi : (C)^{(h)} \to J(C)$$

from the $h$-fold symmetric product of $C$ into $J(C)$ given by

$$\pi(P_1 + \ldots + P_n) = \left( \sum_{i=1}^{h} \int_{P_0}^{P_i} \omega_1, \ldots, \sum_{i=1}^{h} \int_{P_0}^{P_i} \omega_g \right) \text{ mod } A_z.$$

If $K$ is the canonical divisor of $C$, $\pi(K) = x \in J(C)$ is a well determined point of $J(C)$ which depends only on the choice of $P_0$.

c) The function of $u \theta(u, z)$ can be viewed as a holomorphic section of a line bundle over $J(C)$. Its divisor $\theta$ is thus defined on $J(C)$; it is a non empty holomorphic divisor. We recall the following theorem due essentially to Riemann.

**Theorem of Riemann.** i) The map $\pi : (C)^{(g)} \to J(C)$ is surjective.

ii) The image of the map $\pi : (C)^{(g-1)} \to J(C)$ up to a translation by a point $a \in J(C)$ with $2a = x$ is the theta divisor $\theta$. 


iii) The section $\theta(u, z)$ vanishes of order one on $(\theta)$.

For the proof of this theorem one can see [13] [30].

Only the last condition iii) is not explicitly proved in the literature. It can be seen as follows. The support of $(\theta)$ is an irreducible variety because of ii). If $\theta(u, z)$ vanishes on it of order $\mu$ then for any $c \in J(C)$ the induced section $\theta(\psi(x) - c)$ on $C$ either is identically zero or vanishes on each one of its zeros of order $\geq \mu$. But if $P_1, \ldots, P_{g-1}$ are distinct and chosen in such a way that $P_0 + \ldots + P_{g-1}$ is non special then for $c = \pi(P_1 + \ldots + P_{g-1}) + a$ the section $\theta(\psi(x) - c)$ is not identically zero and vanishes to the first order at the points $P_0, \ldots, P_{g-1}$ [13]. This implies that $\mu = 1$. We can now translate by $a$ the map $\pi$ so that $\pi((C)_g) = (\theta)$. Since $\theta(-u, z) = \theta(u, z)$ the involution $x \mapsto -x$ on $J(C)$ changes $(\theta)$ into itself. If $D$ is a divisor on $C$ of degree $g - 1$ then by this choice of the map $\pi$ we obtain

$$\pi(D) = -\pi(K - D)$$

i.e. the involution $x \mapsto -x$ on $(\theta)$ corresponds to the involution on the set of complete linear series of degree $g - 1$ which associates to each such series the residual series with respect to the canonical one.

Given a point $P_1 + \ldots + P_{g-1} \in (C)_g$ we denote by $i(P_1 + \ldots + P_{g-1})$ the speciality index of the divisor $P_1 + \ldots + P_{g-1}$.

**Proposition 7.** (cf. [16]) The subset of $(C)_g$:

$$S_r = \{P_1 + \ldots + P_{g-1} \in (C)_g \mid i(P_1 + \ldots + P_{g-1}) \geq r\}$$

is the subset of $(C)_g$ where the jacobian of the map $\pi$ has rank $\leq g - r$.

**Proof.** Let $P_1 + \ldots + P_{g-1} \in (C)_g$ and let $V$ be an open set on $C$ such that $P_i \in V$ for $1 \leq i \leq g - 1$ and on which there exists a holomorphic function $t$ with the following properties

i) at each point $a \in V$, $t - t(a)$ is a local parameter at $a$

ii) for $a, b \in V$, $a \neq b$, $t(a) \neq t(b)$.

The open set $(U)$ is a neighborhood of $P_1 + \ldots + P_{g-1}$ in $(C)_g$. Let $t_i$ denote the lifting to the Cartesian product $U^{g-1}$ of the function $t$ on the (i)-th factor. The elementary symmetric functions of the $t_i$'s $\Phi_1 = t_1 + \ldots + t_{g-1}$, $\Phi_2 = t_1 \ldots t_{g-1}$ can be taken as local coordinates on the set $(U)$.

One has

$$d\Phi_1 \wedge \ldots \wedge d\Phi_{g-1} = \Pi_{i < j} (t_i - t_j) \, dt_1 \wedge \ldots \wedge dt_{g-1}.$$
If $\mathbf{x}_\alpha \ 1 \leq \alpha \leq g$ are coordinates on the universal covering $C^g$ of $J(C)$ then the map $\pi$ is given by

$$\pi (a_1 + \ldots + a_{g-1}) = \left( \mathbf{x}_\alpha = \int_{P_0}^{a_1} \omega_\alpha + \ldots + \int_{P_0}^{a_{g-1}} \omega_\alpha + \text{const} \right)_{1 \leq \alpha \leq g}.$$

If $\omega_\alpha = A_\alpha (t) \, dt$ on $U$, the jacobian matrix $\frac{\partial (\mathbf{x}_1, \ldots, \mathbf{x}_g)}{\partial (\Phi_1, \ldots, \Phi_{g-1})}$ is easily computed in terms of the functions $t_i$ and given by the matrix

$$\left\{ \prod_{i<j} (t_i - t_j) \right\}^{-1} \left( (A_k (t_i))_{1 \leq k \leq g} \right)_{1 \leq i \leq g-1, 1 \neq i \neq j},$$

at a point where $t_i = t_j \ \forall \ i, j \ i \neq j$. At a particular point $a_1 + \ldots + a_{g-1} = n_1 P_1 + \ldots + n_r P_r$ with $n_i = g - 1$ with $P_i \neq P_j$ if $i \neq j$, if we set $t_i = t (P_i)$, the rank of the Jacobian matrix $\frac{\partial (\mathbf{x})}{\partial (\Phi)}$ is the same as the rank of the matrix

$$\frac{1}{\prod_{i<j} (t_i - t_j)} \begin{pmatrix} A_1 (P_1), \ldots, \frac{d^{n_i-1} A_1 (P_1)}{dt_1^{n_i-1}}, \ldots, A_1 (P_r), \ldots, \frac{d^{n_i-1} A_1 (P_r)}{dt_1^{n_i-1}} \\ \vdots \\ A_g (P_1), \ldots, \frac{d^{n_i-1} A_g (P_1)}{dt_1^{n_i-1}}, \ldots, A_g (P_r), \ldots, \frac{d^{n_i-1} A_g (P_r)}{dt_1^{n_i-1}} \end{pmatrix}.$$

The rank of this matrix is thus equal to $g - i (a_1 + \ldots + a_{g-1})$.

In particular it follows from the above proposition that the sets $S_\alpha$ are algebraic subsets of $(C)^{g-1}$. One has $S_1 = (C)^{g-1}$ and $S_2$ is a proper algebraic subset of $(C)^{g-1}$. From lemmas 1-4 and proposition 5 if $g \geq 4$ $S_2$ is non empty and contains always a point $D = P_1 + \ldots + P_{g-1}$ with distinct $P_i$'s and with $\dim | D | = 1$.

**Lemma 7.** Let $C$ be non hyperelliptic of genus $g \geq 4$. Let $D = P_1 + \ldots + P_{g-1}$ be a point of $S_2$ with the properties

i) the points $P_i$ are distinct

ii) $\dim | D | = 1$.

In a neighborhood of $D$, $S_2$ is of pure dimension $g - 3$.

**Proof.** Let $h : C^{g-1} \rightarrow (C)^{g-1}$ be the natural map from the cartesian to the symmetric product of $C$. Since $h$ has finite fibers it is enough to prove the statement for the set $\Sigma_2 = h^{-1} (S_2)$.
Let $V_i$ be mutually disjoint neighborhoods of the points $P_i$ and let $U = \Pi V_i$. Let $t = t_i$ be local parameters in $V_i$ and let $\omega_a = A_a(t_i) \, dt_i$ on $V_i$. We have

$$\Sigma_2 \cap U = \{ t_1 \times \ldots \times t_{g-1} \in U \mid \text{rank} \left( A_{a}(t_i)_{1 \leq a \leq g} \right)_{1 \leq i \leq g-1} \leq g - 2 \}.$$ 

Since in the space of $g \times (g - 1)$ matrices those of rank $\leq g - 2$ are a subset of codimension 2, it follows that at each point of $\Sigma_2 \cap U$ the dimension of $\Sigma_2$ is $\geq g - 3$. Suppose, if possible, that one irreducible component of $\Sigma_2 \cap U$ has dimension $g - 2$. At a non singular point $t_1^0 \times \ldots \times t_{g-1}^0$ of it, by renumbering the coordinates, it will have an equation of the form $t_{g-1} = g(t_1, \ldots, t_{g-2})$ where $g$ is holomorphic in a neighborhood of $t_1^0 \times \ldots \times t_{g-2}^0$. Now the rank of the matrix $(A_{a}(t_i))_{1 \leq a \leq g}$ is $g - 2$ since $g - 2$ generic points on the canonical curve are linearly independent. Moreover since $C$ is not hyperelliptic $(A_{a}(t_{g-1})) = (A_{a}(t_i))$ for $1 \leq i \leq g - 2$. By the definition of $\Sigma_2$ there will exist $g - 2$ holomorphic functions $k_a(t_1, \ldots, t_{g-2})$, $1 \leq a \leq g - 2$, such

$$A_{a}(t_{g-1}) = \sum_{a=1}^{g-2} k_a(t_1, \ldots, t_{g-2}) A_{a}(t_a) \quad \text{for} \quad t_{g-1} = g(t_1, \ldots, t_{g-2}).$$

Two at least of these $k_a$ must be $\neq 0$ in a neighborhood of $t_1^0 \times \ldots \times t_{g-2}^0$, for instance $k_1$ and $k_2$.

Taking derivatives of the above relation with respect to $t_1$ and $t_2$ one sees that the space spanned by the points $\left( A_{a}(t_1), \ldots, (A_{a}(t_{g-1}) \right)$ contains also the points $\left( \frac{dA_{a}(t_1)}{dt_1}, \frac{dA_{a}(t_2)}{dt_2} \right)$. It follows that for any choice of $t_1, \ldots, t_{g-2}$ in small neighborhoods of $t_1^0, \ldots, t_{g-2}^0$ respectively the corresponding points $Q_1, Q_2, \ldots, Q_{g-2}$ on $C$ have the property $i(2Q_1 + 2Q_2 + Q_3 + \ldots + Q_{g-2}) \geq 1$. This is impossible since $Q_1, \ldots, Q_{g-2}$ are generic, as one sees, by specializing the $Q_j$ to a common non-Weierstrass point.

**Proposition 8.** Let $C$ be non hyperelliptic of genus $g \geq 4$. Then

a) the set $\pi(S_2)$ is the singular set of $(\theta)$, and is of pure dimension $g - 4$,

b) the points $\pi(D)$, where $D = P_1 + \ldots + P_{g-1}$ is such that

$$P_i = P_j \quad \text{if} \quad i = j, \quad \dim |D| = 1,$$

are dense in $\pi(S_2)$.
c) at each one of these points the multiplicity of \((0)\) is 2 and the quadratic equation
\[
\sum \frac{\partial^2 \theta}{\partial u_i \partial u_j} (\pi(D)) X_i X_j = 0
\]
is the equation in \(P_{g-1}(C)\) of the quadric of rank \(\leq 4\) through the canonical curve corresponding to the series \(|D|\).

**Proof.** From proposition 5 follows that the points \(\pi(D)\) are dense in \(\pi(S_2)\). From the previous lemma it follows that at a point \(D\) the local rank (in the sense of Remmert [24]) of the map \(\pi|S_2\) is \(g - 4\). Thus \(\pi(S_2)\) is of pure dimension \(g - 4\). From proposition 7 we deduce that the singular set of \((\theta)\) is contained in \(\pi(S_2)\). It is also known (see [17]) that at each point \(\pi(D)\) \((\theta)\) has a singular point of multiplicity 2. This statement will be reobtained in the course of the present proof. We lift the map \(\pi\) to the cartesian product \(C^{g-1}\) and use the same notations as in proposition 7 and lemma 7.

In a neighborhood \(U = \bigcup U_i\) of \(D \in C^{g-1}\) the map \(\pi\) will have equations of the form
\[
w_a = w_a(t_1, ..., t_{g-1}) = \int_0^{t_1} \omega_a + ... + \int_0^{t_{g-1}} \omega_a + \text{const}, 1 \leq a \leq g.
\]

Since \(\pi(C^{g-1}) = (\theta)\) we get the identity \(\theta(w(t_1, ..., t_{g-1})) = 0\), and therefore at any point \(c \in U\) we get the conditions
\[
\theta(w(c)) = 0; \left(\frac{\partial \theta}{\partial t_i}\right)_c = 0, 1 \leq i \leq g - 1; \left(\frac{\partial^2 \theta}{\partial t_i \partial t_j}\right)_c = 0 1 \leq i, j \leq g.
\]

The first condition restates the fact \(\pi(C^{g-1}) = (\theta)\). The second reads explicitly as follows:
\[
\begin{align*}
\sum \frac{\partial \theta}{\partial u_a}(w(c)) A_a(c_i) &= 0 \\
1 \leq i \leq g - 1.
\end{align*}
\]

If we denote by \(M_a(c)\) up to sign the minor determinants of order \(g - 1\) extracted from the matrix \((A_a(c_i))_{1 \leq a, i \leq g}^{1 \leq a, i \leq g-1}\) by deleting the \(a\)-th column, we get a set of holomorphic functions on \(U\). Moreover \(q(c) = \frac{\partial \theta}{\partial u_a}(w(c)) M_a(c)^{-1}\) is meromorphic and independent of \(a\). From lemma 7
and proposition 7 we deduce that $(\Sigma_2$ being of codimension 2) this function is holomorphic and different from zero outside of a set of codimension 2. It follows that $\varphi(c)$ is holomorphic and $\neq 0$ in the whole of $U$.

Now if $c = c^0 \in \Sigma_2$ at the point $D \in U$, $M_{\alpha}(c^0) = 0$ and thus the point $\nu(c^0) = \pi(D)$ is singular on $\langle \theta \rangle$.

If all second derivatives of $\theta(u)$ vanish at $u = \nu(c^0)$ it would follow from $\frac{\partial^2 \theta}{\partial u^2}(\nu(c)) = \varrho(c)\, M_{\alpha}(c)$ that at $c^0$, $(dM_{\alpha}(c))_{\alpha} = 0$ for $1 \leq \alpha \leq n$.

Let $|D| = g_k + P_{k+1} + \ldots + P_{g-1}$ where $g_k$ has no fixed points. From this condition follows that for $1 \leq l \leq h$ the points $P_1, \ldots, \widehat{P}_l, \ldots, P_h$ on the canonical curve span a space of dimension $h - 2$ which contains also the point $P_l$. It follows that the points $P_1, \ldots, \widehat{P}_l, \ldots, P_h, \ldots, P_{g-1}$ span a space of dimension $g - 3$ containing $P_l$.

But from the conditions $(dM_{\alpha})_{\alpha} = 0$ we see that in this space of dimension $g - 3$ is also contained the tangent line to the canonical curve at $P_l$.

This argument could be repeated for any choice of $P_1 + \ldots + P_h \in \Sigma_1$ since the point $\nu(c^0)$ is not changed (Abel’s theorem). Now when $P_1 + \ldots + P_h$ describe $\Sigma_1$ the space spanned by $P_1, \ldots, P_{g-1}$ is a space of dimension $g - 3$ describing one of the rulings of a quadric of rank $\leq 4$ through the canonical curve. The pencil of these rulings would cut, outside of a fixed divisor, the series of divisors $2(P_1 + \ldots + P_h)$. This is impossible by Bertini’s theorem. Therefore $\nu$ is a double point.

Now at $\nu(c^0)$, since the first derivatives of $\theta$ vanish, we deduce that

$$\left\{ \frac{\partial^2 \theta}{\partial u^2}(\nu(c^0)) A_{i,\alpha} c_i^0 A_{\beta,\alpha} = 0 \right\}$$

Note that the bilinear form $H(X, Y) = \sum \frac{\partial^2 \theta}{\partial u^2}(\nu(c^0)) X_{\alpha} Y_{\beta}$ is not identically zero (since $\nu(c^0)$ is of multiplicity 2 on $\langle \theta \rangle$) and that the previous condition says

$$H(\Sigma \lambda_i A \cdot c_i^0, \Sigma \lambda_j A \cdot c_j^0) = 0$$

for any choice of the $\lambda$’s. This means that the space of dimension $g - 3$ spanned by the points $P_1, \ldots, P_{g-1}$ on $\Gamma$ is contained in the quadric $H(X, X) = 0$. When the divisor $P_1 + \ldots + P_{g-1}$ varies in $|D|$ the point $\nu(c^0)$ does not change while the space spanned by $P_1, \ldots, P_{g-1}$ describes the rulings $P_{g-3}$ of the quadric corresponding to the complete linear series $|D|$. This achieves the proof.
Corollary. Let $C$ be a non hyperelliptic curve containing a $g_3$, then the singular set of $(\theta)$ consists of two irreducible components

a) the set with generic point $\pi (g_3^1 + P_1 + \ldots + P_{g-4})$ where $P_1, \ldots, P_{g-4}$ are generic points on $C$

b) the image of the previous set by the involution $x \mapsto -x$ on $J(C)$ whose generic point is $\pi (| K - (g_3^1 + P_1 + \ldots + P_{g-4}) |)$. These two components are distinct if $g > 4$. For $g = 4$ every algebraic curve of genus 4 (non hyperelliptic) contains a $g_3^1$ and the two above components are reduced to two points which possibly may coincide.

Moreover the tangent cones to $(\theta)$ at the singular points give a system of quadrics of rank $\leq 4$ through the canonical curve $\Gamma$ which spans the full system of quadrics through $\Gamma$.

Proof. This is a direct consequence of the previous proposition, of propositions 3, 4, and 5 and the corollary of proposition 2. The only thing that remains to be seen is the fact that, if $g > 4$, the two components of the singular set are distinct. Unless $g = 4$ and $| 2g_3^1 | = | K |$, (which occurs only in the special case when the quadric through $\Gamma$ is a cone), $\dim | 2g_3^1 | \leq 2$ by Clifford's theorem. Then if $D$ and $D'$ and two divisors in $g_3^1$, $\mu (D + D' + P_1 + \ldots + P_{g-4}) \leq 3$ if the $P_j$ are sufficiently generic (cf. Lemma 1 of [17]). So we cannot have

$$| D + P_1 + \ldots + P_{g-4} | = | K - D' - Q_1 - \ldots - Q_{g-4} |$$

for any choice of $Q_1, \ldots, Q_{g-4}$. Thus $\pi (g_3^1 + P_1 + \ldots + P_{g-4})$ lies only in component a).

In the case $C$ is hyperelliptic the set $S_2$ is of pure dimension $g - 2$ since every generic complete $g_3^{1}_{g-1}$ is of the form $g_3^1 + P_1 + \ldots + P_{g-3}$. Again $\pi (S_2)$ is the singular set of $(\theta)$. This is irreducible and of dimension $g - 3$. As before one proves that the tangent cone at the point $\pi (g_3^1 + P_1 + \ldots + P_{g-3})$, for $P_1, \ldots, P_{g-3}$ generic on $C$, is the quadric of rank 3 that projects the canonical curve $\Gamma$ from the points $P_1, \ldots, P_{g-3}$. These quadrics generate the full system of quadrics through the canonical curve $\Gamma$.

3. The modular space of polarized Jacobians.

8. Teichmüller space. Let $X_0$ be a standard model of a topological oriented surface of genus $g$. A Teichmüller surface is the data of a Riemann surface $X$ and a homotopy class of orientation preserving homeomorphisms $f: X_0 \rightarrow X$. The set of Teichmüller surfaces is the Teichmüller space $\mathcal{T}$. 
We fix on \( X_0 \) a basis for the fundamental group of \( X_0 \) made of \( 2g \) closed paths \( \sigma_1, \ldots, \sigma_{2g} \) with intersection matrix \( \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \). Then for any Teichmüller surface we can take the cycles \( f(\sigma_i) = \gamma_i \) as a basis for the first homology group as we did in n. 7. Choosing on \( X \) a normalised basis for the space \( H^0(X, \Omega^1) \) we can compute the corresponding period's matrix \((I, z)\) as explained in n. 7. We thus obtain a natural map

\[ \lambda : \mathcal{C} \rightarrow \mathbb{H}_g \]

associating to the Teichmüller surface \((X; [f])\) the point \( z \in \mathbb{H}_g \).

From the theory of Teichmüller spaces we borrow the following facts. On the space \( \mathcal{C} \) one can introduce a structure of a connected complex manifold of dimension \( 3g - 3 \) such that \( \lambda \) is a holomorphic map with discrete fibers.

For the theory of Teichmüller spaces the reader is referred to L. Ahlfors, L. Bers and H. Rauch (cf. the bibliography at the end of this paper).

9. The space of Jacobians. Let \( J = \lambda(\mathcal{C}) \). This is the subset of \( \mathbb{H}_g \) representing Jacobians of algebraic curves of genus \( g \). This space will be called the Jacobi space. Let us consider on \( \mathbb{H}_g \) the Zariski topology (the closed sets are the analytic subsets of \( \mathbb{H}_g \)) and let \( \overline{J} \) be the closure of the set \( J \) in the Zariski topology. Since \( \mathbb{H}_g \) is a Stein manifold the Zariski closure \( \overline{J} \) is the analytic set

\[ \{ z \in \mathbb{H}_g \mid f(z) = 0 \ \forall f \ \text{holomorphic on} \ \mathbb{H}_g \ \text{with} \ f(J) = 0 \} . \]

Since \( J = \lambda(\mathcal{C}) \) and \( \mathcal{C} \) is an irreducible manifold, it follows that \( \overline{J} \) is an irreducible analytic subset of \( \mathbb{H}_g \). In n. 6 we introduced the analytic sets \( B_4(\theta) \) in \( \mathcal{C}^g \times \mathbb{H}_g \) and the analytic sets \( N_4 = \text{pr}_{\mathbb{H}_g}(B_4(\theta)) \) in \( \mathbb{H}_g \). From proposition 8 one deduces the inclusion

\[ \overline{J} \subset N_{g-4} . \]

We want to prove the following

**Theorem 1.** The Zariski closure of the Jacobi space \( J \) is an irreducible analytic set of dimension \( 3g - 3 \). It coincides with the unique irreducible component of the ramification set \( N_{g-4} \) containing \( J (g \geq 4) \).

**Proof.** a) We first prove that \( \overline{J} \geq 3g - 3 \). This is a consequence of the following form of the implicit function theorem (cf. [24]):
Let $U$ be an open subset of $\mathbb{C}^n$ containing the origin and $f: U \to \mathbb{C}^p$ a holomorphic map, $f(0) = 0$. Suppose that the Jacobian of $f$, $Df(x)$ has constant rank $k$ for all $x \in U$. Then there are local biholomorphisms $h$ and $g$ of $\mathbb{C}^n$ and $\mathbb{C}^p$ respectively such that
\[ g \circ f \circ h(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0). \]

Indeed since $\lambda$ has discrete fibers the rank of the Jacobian of $\lambda$ must be $3g - 3$ on an open (dense) subset of $\mathbb{C}$. Therefore the image $\lambda(\mathbb{C})$ cannot be contained in an analytic set of dimension $< 3g - 3$.

To complete the proof it is enough to show that at some point $z_0 \in J$ the dimension of the ramification set $N_{g-4}$ is exactly $3g - 3$.

By definition $N_{g-4} = pr_{H_g}(B_{g-4}(\theta))$. Let $M$ be an irreducible component of $N_{g-4}$. There exists an irreducible component $A$ of $B_{g-4}(\theta)$ such that $pr_{H_g}(A) = M$. In fact the counter image in $B_{g-4}(\theta)$ of $M$ consists at most of denumerably many irreducible components of $B_{g-4}(\theta)$. The projection map being the product of a local isomorphism with a proper map, the projection of each one of these components is an analytic set. One at least of these sets must be $M$ since by the Baire theorem $M$ cannot be a countable union of proper analytic subsets. We need:

**Lemma 8.** At each simple point $z_0 \in M$ any tangent vector $[dz_{a\beta}]$ to $M$ satisfies the conditions
\[ \sum_{\alpha \leq \beta} \frac{\partial}{\partial z_{a\beta}} (u_0, z_0) dz_{a\beta} = 0 \]
for any $(u_0, z_0) \in pr_{H_g}^{-1}(z_0) \cap A$.

**Proof of the Lemma:** Let $S(M)$ be the singular set of $M$ and $S(A)$ the singular set of $A$. The set $A' = A - pr_{H_g}^{-1}(S(M)) - S(A)$ is a connected manifold. The projection on the space $H_g$ gives a holomorphic map of the connected manifold $A'$ into the connected manifold $M - S(M)$. It is of maximal constant rank on an open dense subset $A''$ of $A'$. Moreover the projection of $A''$ is dense in $M$. If we prove the statement for the points $(u_0, z_0) \in A''$, by continuity we deduce the statement at any other point $(u_0, z_0)$ with $z_0 \in M - S(M)$ and $(u_0, z_0) \in pr_{H_g}^{-1}(z_0) \cap A$.

Let $(u_0, z_0) \in A''$. By the quoted implicit function theorem, we can find parametric equations of $A''$ in a neighborhood of $(u_0, z_0)$ of the form
\[ \begin{cases} \ u = u(y, t) \\ \ z = z(t) \end{cases} \]
where $t$ is in a neighborhood of the origin in $\mathbb{C}^r (s = \dim M)$ and $y$ in a neighborhood of the origin in $\mathbb{C}^r (r = \dim A - \rank p^*H_\bar{g}, A^r)$. Since $A \subset B_{t-1} (\theta) \subset B_0 (\theta)$ we must have
\[
\theta ((u, y), z (t)) = 0, \quad \frac{\partial \theta}{\partial u_i} ((u, y), z (t)) = 0 \quad \text{for} \quad 1 \leq i \leq g.
\]

Differentiating the first condition with respect to $t$ and making use of the second set of conditions we get:
\[
\sum_{\alpha \leq \beta} \frac{\partial \theta}{\partial u_i} (u (y, t), z (t)) \frac{\partial z_{\alpha \beta}}{\partial t_i} = 0.
\]

For $y = 0$, $t = 0$ and $dz_{\alpha \beta} = \sum_l \lambda_i \left( \frac{\partial z_{\alpha \beta}}{\partial t_i} \right)_{i = 0}$ we get the desired result since when the $\lambda_i$'s vary, $|dz_{\alpha \beta}|$ describes the space of tangent vectors to $M$ at $z_0$ because $z = z (t)$ are parametric equations of $M$ in the neighborhood of $z_0$.

To complete the proof it is enough to show that if $M$ is any one of the irreducible components of $N_{g-4}$ containing $J$ at some point $z_0$, its dimension is $\leq 3g - 3$.

Let $z_0 \in J$ now be a point corresponding to a Jacobian variety of a non hyperelliptic curve $C$ carrying a $g^1_2$. We do not know a priori if $z_0$ is simple.

Let $A$ and $-A$ (possibly $A = -A$, if $g = 4$) be the two irreducible components of the singular set of $\{ \theta (u, z_0) = 0 \}$ on $J (C)$. Then $A$ must contain one of the irreducible components of the counter image of $A$ or $-A$ in $C^g$, the universal covering $J (C)$. Call that one $\tilde{A}$.

By virtue of the corollary of proposition 8 we can choose $\frac{1}{2} (g - 2) (g - 3)$ distinct points $(u_0^{(a)}, z_0)$ for $1 \leq a \leq \frac{1}{2} (g - 2) (g - 3)$ on $\tilde{A}$ such that the corresponding quadrics of rank $\leq 4$ through the canonical image $\Gamma$ of $C$ are linearly independent. By proposition 8 these quadrics have the equations
\[
\sum \frac{\partial^2 \theta}{\partial u_i \partial u_j} (u_0^{(a)}, z_0) X_i X_j = 0 \quad \text{or, equivalently by the « heat equations »}
\sum \frac{\partial \theta}{\partial z_{\alpha \beta}} (u_0^{(a)}, z_0) X_\alpha X_\beta = 0.
\]

Let $\{z_\nu\}_{\nu \in \mathbb{N}}$ be a sequence of non singular points on $M$ such that $\lim_{\nu \to \infty} z_\nu = z_0$.
We show now that the sequence \( \{z_\alpha\}_{\alpha \in \mathcal{N}} \) can be so chosen that we can lift it to sequences \( \{u^{(a)}, z_\alpha\}_{\alpha \in \mathcal{N}} \), for \( 1 \leq \alpha \leq \frac{1}{2} (g - 2) (g - 3) \), such that
\[
\lim_{\nu \to \infty} (u^{(a)}_{\nu}, z_{\nu}) = (u^{(a)}_0, z_0).
\]

Let \( \mu = \text{pr}_{H_g} \big/ A \). At each point \( x \in \mathcal{A} \) one can consider the local rank \( r_\mu(x) \) of this map according to Remmert and Stein, that is the integer
\[
r_\mu(x) = \dim \mathcal{A} - \dim_{x} \mu^{-1} \mu(x).
\]
This function of \( x \) is lower semicontinuous on \( \mathcal{A} \) (cf. [24] theorem 15). At \( x_a = (u^{(a)}_0, z_0) \) we have \( r_\mu(x_a) = \dim \mathcal{A} - (g - 4) \).

By definition of \( B_{g-4}(\theta) \) at each point \( x \in \mathcal{A} \), \( r_\mu(x) \leq \dim \mathcal{A} - (g - 4) \). On an open neighborhood \( U(x_a) \) of \( x_a \) in \( \mathcal{A} \) we must have \( r_\mu(x) = r_\mu(x_a) \forall x \in U(x_a) \forall x_a \).

Now \( \dim \mathcal{M} = \sup_{x \in \mathcal{A}} r_\mu(x) = \dim \mathcal{M} \), \( X_a \) is a holomorphic map «without degeneracy», (i.e. of constant rank \( r_\mu \)).

Let \( V \) be a neighborhood of \( z_0 \) in \( H_g \) such that \( V \cap \mathcal{M} = Y_1 \cup \ldots \cup Y_k \) decomposes in \( k \) irreducible components \( Y_i \), \( 1 \leq i \leq k \), one for each irreducible germ of \( \mathcal{M} \) at \( z_0 \). One of the irreducible components \( X \) of \( \mathcal{A} \cap \text{pr}_{H_g}^{-1}(V) \) must contain the set \( \mathcal{A} \). The projection of \( X \) on \( H_g \), since it is an irreducible analytic set of dimension \( \dim \mathcal{M} \), must coincide with one of the \( Y_i \)'s. Call it \( Y \).

Let \( U'(x_a) \) be the connected component containing \( x_a \) in \( U(x_a) \cap \mathcal{X} \). Then
\[
\mu : U'(x_a) \to Y
\]
is a holomorphic map without degeneracy of constant rank \( r \) into a connected irreducible complex space \( Y \) of dimension \( r \). By a theorem of Remmert ([24] theorem 28) \( \mu \) is an open map.

This is true for any \( \alpha \), \( 1 \leq \alpha \leq \frac{1}{2} (g - 2) (g - 3) \), thus the set \( \bigcap_{\alpha} \mu(U'(x_a)) \) is an open neighborhood of \( z_0 \) in \( Y \). Any non-singular point of \( Y \) in that neighborhood can be lifted to a point in each one of the sets \( U'(x_a) \).

Since the neighborhoods \( U(x_a) \) can be chosen arbitrarily small our assertion follows.

(e) Let \( g = \frac{1}{2} (g - 2) (g - 3) \). The \( \frac{1}{2} g (g + 1) \) matrix
\[
\begin{pmatrix}
\frac{\partial \theta}{\partial z_{11}}(u^{(1)}_{\nu}, z_{\nu}), \ldots, \frac{\partial \theta}{\partial z_{gg}}(u^{(1)}_{\nu}, z_{\nu}) \\
\frac{\partial \theta}{\partial z_{11}}(u^{(q)}_{\nu}, z_{\nu}), \ldots, \frac{\partial \theta}{\partial z_{gg}}(u^{(q)}_{\nu}, z_{\nu})
\end{pmatrix}
\]
for \( v \to \infty \) tends to a matrix of rank \( q \). For \( v \) large enough the rank of this matrix will be \( q \). But then, for those \( v \)'s the conditions of the previous lemma define a space of tangent vectors of dimension \( \frac{1}{2} g(g+1) - \frac{1}{2} (g-2)(g-3) = 3g - 3 \). This shows that at those points the dimension of \( M \) is \( \leq 3g - 3 \) and this concludes the proof.

**Remark.** Let \( E \) be the subset of \( J \) representing hyperelliptic curves. One can easily show that the Zariski closure \( \overline{E} \) of \( E \) is an irreducible analytic set.

By a similar and simpler argument one can prove that \( \overline{E} \) is of dimension \( 2g-1 \) and coincides with the unique irreducible component of the ramification set \( N_{g-3} \) containing \( E \).

**10. a)** The group of automorphisms of \( X_0 \) (cf. n. 8) acts on the Teichmüller space \( \mathcal{C} \) by

\[
\alpha : \langle x, [f] \rangle \mapsto \langle x, [f \circ \alpha] \rangle,
\]

for \( \alpha \in \text{Aut}(X_0) \). If \( \alpha \) is homotopic to the identity the action of \( \alpha \) on \( \mathcal{C} \) is the identity. If \( N = \{ \alpha \in \text{Aut}(X_0) \mid \alpha \text{ homotopic to the identity} \} \) the action of \( \text{Aut}(X_0) \) on \( \mathcal{C} \) reduces to the action of the group \( A = \text{Aut}(X_0)/N \) (note that \( N \) is a normal subgroup of \( \text{Aut}(X_0) \)). Let \( H \) be the subgroup of \( \text{Aut}(X_0) \) defined by

\[
H = \{ \alpha \in \text{Aut}(X_0) \mid \alpha : H_1(X_0, \mathbb{Z}) \to H_1(X_0, \mathbb{Z}) \text{ is the identity} \}.
\]

This is another normal subgroup of \( \text{Aut}(X_0) \). Setting \( \Gamma = H/N \subset A \). One sees that \( \Gamma \) acts freely on \( \mathcal{C} \) (cf. Rauch [21], lemma 2) and the manifold \( T = \mathcal{C}/\Gamma \) represents the classes of « Torelli surfaces ». The natural map \( \lambda : \mathcal{C} \to H_2 \) can be factored through the natural map \( \mathcal{C} \to \mathcal{C}/\Gamma = T \) and a map of degree \( 2 \) of \( T \) onto \( J \). This last map (as it follows from Torelli's theorem) is obtained by dividing \( T \) by the action of the involutory automorphism \( \tau \) of \( T \) corresponding to an orientation preserving diffeomorphisms of \( X_0 \) which changes the sign of the basis \( \sigma_1, \ldots, \sigma_g \) (as homology basis). The fixed points of this automorphism of \( T \) are the points corresponding to hyperelliptic curves. These are known facts in the theory of the Teichmüller space. It follows that the Jacobi space \( J \) is in one to one correspondence with the normal space \( T/\tau \). In fact \( J \) is non singular at a point corresponding to a non hyperelliptic curve. In particular \( J \) is locally irreducible. From the theorem of Remmert ([24], theorem 28) it follows that if \( Y \) is
the Zariski closure of \( J \) in \( H_g \) (i.e. the irreducible component of the ramification set \( N_{g-4} \) containing \( J \)) then the map \( \lambda : \mathcal{C} \rightarrow Y \) is an open map, i.e. \( J \) is an open subset of \( Y \).

b) From the proof of Theorem 1 we also deduce the following remark. At a point \( z_0 \) of a Zariski open (non empty) subset of \( J \) the tangent space to \( Y \) at \( z_0 \) is a \( 3g - 3 \) dimensional space with the equations

\[
\sum_{\alpha \leq \beta} a_{\alpha \beta} \, dx_{\alpha \beta} = 0
\]

where \( \sum a_{\alpha \beta} X_{\alpha} X_{\beta} = 0 \) describe the full linear system of quadrics through the canonical curve corresponding to the point \( z_0 \).

**Proof.** Let \( S(N_{g-4}) \) be the singular set of the ramification set \( N_{g-4} \). The set \( A = J - J \cap S(N_{g-4}) - E \) is Zariski open in \( J \) and non empty. Let \( \mu = \frac{1}{2} g (g + 1) - (3g - 3) \) and let \( A_{\mu} \) be the part of \( A \) over \( A \) (with the notations of the proof of theorem 1). Let \( A_{\mu}^\mu \) be the \( \mu \)-th fibered product of \( A_{\mu} \) over \( A \). For any point \((\mu^1, ..., \mu^g; z_0) \in A_{\mu}^\mu \) we consider the \( \mu \times \times \frac{1}{2} g (g + 1) \) matrix \( \left( \frac{\partial \theta}{\partial x_{\alpha}} (\mu^\alpha, z_0) \right)_{1 \leq \alpha \leq \mu, 1 \leq \beta \leq 2g} \). Let \( C \) be the analytic subset of \( A_{\mu}^\mu \) where the rank of that matrix is \( < \mu \). Let \( \pi = \text{pr}_{H_g} | A_{\mu}^\mu \). Consider the set \( D = \{ z_0 \in A | \pi^{-1} (z_0) \subset C \} \); for \( z_0 \in A - D \) the desired requirements are satisfied. One has thus to prove that \( D \) is analytic. This is actually possible; however we can more simply remark that \( D \) is contained in the subset \( D' = \{ z_0 \in A | \text{dim} \, \pi^{-1} (z_0) \geq \mu (g - 4) \} \) and that \( (a) \) \( D' \) is analytic as is proved by the usual arguments using Remmert's theory of holomorphic maps; \( b) \) \( D' \subset A \) as it follows from the proof of theorem 1.

**Remark.** The above statement is to be considered as a weak form of a known theorem of Rauch [20] which says that at any point \( z_0 \in J \) not representing a hyperelliptic curve, \( 3g - 3 \) of the local coordinates \( z_{\alpha \beta} \) on \( H_g \) can be taken as local coordinates on \( J \) at \( z_0 \) provided the corresponding quadratic differentials \( \omega_{\alpha \beta} \) are linearly independent.

4. The equations of the ramification sets.

11. The Kummer variety. a) We have remarked that the functional equations

\[
\Phi (u + Lu + zm) = e^{-\pi i (\text{mzm} - 2 \text{mu})} \Phi (u)
\]
for \( m, n \in \mathbb{Z}^g \) where \( \Phi \) is a holomorphic function on \( \mathbb{C}^g \times \mathcal{H}_g \) has a one-dimensional space of solutions \( k\theta (u, z), k \in \mathbb{C}, \) and that these solutions represent the holomorphic sections of the line bundle \( F \) on the torus \( \mathbb{C}^g / \Lambda \), corresponding to the factor of automorphy on \( \mathbb{C}^g \):

\[
J_\gamma (u, z) = e^{-i \pi (m z m + 2 m u)}
\]

for \( \gamma = (n, m) \in \mathbb{Z}^{2g} \). Replacing the line bundle \( F \) with \( F^l \) one is lead to consider the functional equations

\[
\Phi (u + In + zm) = e^{-i \pi (m z m + 2 m u)} \Phi (u)
\]

These admit a \( 19 \)-dimensional space of solutions (the theta functions of order \( l \)). A basis for that space of solutions is given by the functions

\[
\theta_l [\mu] (u, z) = \sum_{m \in \mathbb{Z}^g} e^{\pi i \left( \frac{m + \mu}{l} \right)^2 \left( \frac{m + \mu}{l} \right)^2 \left( \frac{m + \mu}{l} \right)^2 u}
\]

where \( \mu \) describes a system of representatives of \( \mathbb{Z}^g / \mathbb{Z}^g \).

In particular for \( l = 2 \) one has a \( 2^g \) dimensional space of theta functions of second order.

For any choice of \( c \in \mathbb{C}^g \) the function

\[
\Phi_c (u) = k\theta (u + c, z) \theta (u - c, z), k \in \mathbb{C},
\]

is a theta function of second order and therefore a linear combination of the functions \( \theta_2 [\mu] (u, z) \) of the basis considered above.

An easy computation gives actually the useful identity

\[
\theta (u + c, z) \theta (u - c, z) = \sum_{\mu} \theta_2 [\mu] (c, z) \theta_2 [\mu] (u, z).
\]

Let \( L \) be the vector space of theta functions of the second order in which we choose the basis given by the elements \( \theta_2 [\mu] (u, z) \). The subset of \( L \) represented by the functions \( \Phi_c \) has thus the parametric equations

\[
\lambda_\mu = k \theta_2 [\mu] (c, z), k \in \mathbb{C}, c \in \mathbb{C}^g.
\]

Let \( P_t (\mathbb{C}), t = 2^g - 1 \), be the « projectification » of \( L \) in which the \( \lambda_\mu \)'s are accordingly taken as homogeneous coordinates. Then the the equations (3) for \( k \neq 0 \) and fixed \( z \in \mathcal{H}_g \) represent the general point of an algebraic subvariety of \( P_t (\mathbb{C}) \) which is called the Kummer (or Wirtinger) variety. This can be seen as follows.
First we remark that for fixed \( z \) and for any given \( a \in \mathbb{C}^g \) we can find a point \( c \in \mathbb{C}^g \) such that \( \vartheta(a + c, z) \neq 0 \). This is because \( \vartheta(u, z) \) is not identically zero as function of \( u \). Thus there always exists a theta function of second order which is different from 0 at \( a \). This means that the map

\[
\chi : \mathbb{C}^g \rightarrow P_t(\mathbb{C})
\]
defined by formulae (3) is a holomorphic map. By the periodicity conditions this map factors through a holomorphic map of \( \mathbb{C}^g/\Lambda_z \) in \( P_t(\mathbb{C}) \). It follows that the image of \( \chi \) is an irreducible compact analytic subset of \( P_t(\mathbb{C}) \), i.e. an irreducible algebraic variety.

b) Our preliminary object is the study of the map \( \chi \).

**Lemma 9.** For any given \( z \in H_g \) the set of points \( u_0 \in \mathbb{C}^g \) such that \( \vartheta(u, z) \) and \( \vartheta(u - 2u_0, z) \) are coprime at each point \( u \in \mathbb{C}^g \) is everywhere dense in \( \mathbb{C}^g \). At each one of these points \( \chi \) is of rank \( g \).

**Proof.** On the torus \( \mathbb{C}^g/\Lambda_z \) let \( A_1 \cup \ldots \cup A_k \) be the decomposition into irreducible components of the set \( \{ \vartheta(u) = 0 \} \). Select \( p_i \in A_i \). Let \( E' = \{ \vartheta(p_i + c) = 0 \} \cup \ldots \cup \{ \vartheta(p_k + c) = 0 \} \) and let \( E \) be its counter image in \( \mathbb{C}^g \). If \( -2u_0 \notin E \) then the sets \( \{ \vartheta(u) = 0 \} \) and \( \{ \vartheta(u - 2u_0) = 0 \} \) intersect in a set of codimension \( \geq 2 \) and thus \( \vartheta(u) \) and \( \vartheta(u - 2u_0) \) are coprime everywhere. Moreover the set \( \{ u_0 \in \mathbb{C}^g \mid -2u_0 \notin E \} \) is everywhere dense.

Let \( u_0 \) be chosen as indicated and let us denote by \( \vartheta^{(0)}, \ldots, \vartheta^{(t)} \) the \( t+1 \) theta functions of second order of a basis for \( L \). The rank of the jacobian of the map \( \chi \) at \( u_0 \) equals \( g \) if and only if the matrix

\[
J = \begin{pmatrix}
\vartheta^{(0)} & \cdots & \vartheta^{(t)} \\
\frac{\partial \vartheta^{(0)}}{\partial u_1} & \cdots & . \\
. & \cdots & . \\
. & \cdots & \frac{\partial \vartheta^{(t)}}{\partial u_2}
\end{pmatrix}
\]
is of rank \( g + 1 \) at \( u_0 \) (cf. Conforto's book, pg. 144).

Suppose, if possible, that the rank of \( J \) at \( u_0 \) is \( \leq g \) so that we have a relation of the form

\[
a_0 \vartheta^{(a)}(u_0) = \sum_i a_i \frac{\partial \vartheta^{(a)}}{\partial u_i}(u^0)
\]
for all \( \alpha, 0 \leq \alpha \leq t \), and with constants \( a_i \) not all zero. From (2) for any choice of \( c \) we get

\[
a_0 \Phi_c(u_0) = \sum_i a_i \frac{\partial \Phi_c}{\partial u_i}(u_0)
\]
From this relation, dividing both sides by the non identically zero function in \( c \), \( \Phi_c(u_0) \), we get, setting \( h(u) = \sum a_i \frac{\partial}{\partial u_i} \log \theta(u) \),

\[
\sigma_0 = h(u^0 + c) + h(u^0 - c)
\]

Now \( h(u^0 + c) \), as a function of \( c \), is holomorphic outside \( \{ \theta(u^0 + c) = 0 \} \) while \( h(u^0 - c) \) is holomorphic outside \( \{ \theta(u^0 - c) = 0 \} \) thus \( h(u^0 + c) \) and \( h(u^0 - c) \), by the choice of \( u_0 \), are both holomorphic outside of a set of co-dimension 2 and thus holomorphic everywhere. Hence the function \( h(u) \) is holomorphic. From the periodicity conditions for \( \theta \) we get

\[
(h) \quad [h(u + e_h) = h(u), \quad (\beta) \quad [h(u + ze_h) = h(u) - 2\pi i \alpha , \quad 1 \leq h \leq g.]
\]

when \( e_h = (0, \ldots, 1, \ldots, 0) \). This implies that the first partial derivatives of \( h(u) \) are periodic and therefore constants so that \( h(u) \) must be a linear function of \( u \):

\[
h(u) = h_0 + h_1 u + \ldots + h_g u_g.
\]

From (a) we deduce then that \( h_i = \ldots = h_g = 0 \) and then by (b), that \( a_i = \ldots = a_g = 0 \). Hence \( h = 0 \) and therefore also \( a_0 = 0 \) (1).

**Lemma 10.** If \( z \in H_g \) is such that \( \theta(u, z) \) is irreducible then for any \( v_0 \in C^g \) which is not a period \( (v_0 \notin \Lambda_2) \theta(u, z) \) and \( \theta(u - v_0, z) \) are coprime.

**Proof.** If not \( \theta(u)/\theta(u - v_0) \) must be a non vanishing holomorphic function of the form \( \exp(Q(u)) \) where \( Q(u) \) is a polynomial of first order in the \( u \)'s. The periodicity conditions then lead to the conclusion \( v_0 \in \Lambda_2 \) (cf. for the detailed argument [12], pg. 196).

**Lemma 11.** The set

\[
[z \in H_g \mid \theta(u, z) \text{ is reducible}]
\]

is an analytic set and is contained in the ramification set \( N_{g-2} \)

**Proof.** Let \( D \) be a positive divisor on the torus \( C^g/\Lambda_2 \) and let us denote by \( C_1(D) = \frac{1}{2\pi i} \int_{\partial D} H \overline{du} \) the unique \((1,1)\) form with constant coefficients (i.e. harmonic) in the Chern class of the line bundle associated to \( D \).

---

(1) This proof follows the argument of C.L. Siegel given in Conforto's book [12], pg. 160.
From the theory of intermediary functions on the torus $T = \mathbb{C}^g / \Lambda_x$ we know that (cf. [29], ch. VI.)

(i) $C(D)$ has integral periods

(ii) The associated hermitian form $\langle u, H u \rangle$ is non-negative definite and if $\Phi(u)$ is an intermediary function with divisor $D$ there exists an entire function $G(u)$ and a constant $C > 0$ such that

$$| \Phi(u) e^{G(u)} | \leq C e^{\vartheta H u}$$

(iii) If $D_1, D_2$ are any two positive divisors then

$$\int T C_1(D_1)^h C_1(D_2)^{g-h} = h! (g-h)! n_h$$

where $n_h$ is an integer $\geq 0$.

(iv) If $D$ is the divisor of $\theta(u, z)$ on $T$ then one has

$$\int T C_1(D)^g = g!,$$ and $H$ is positive definite.

Suppose now that the divisor $D$ of $\theta$ is reducible $D = D_1 + D_2$ with $D_1$ and $D_2$ holomorphic and non-empty. Let us choose coordinates $u$ (by a linear transformation in $\mathbb{C}^g$) such that the hermitian forms $H$ and $H_1$ of $D$ and $D_1$ respectively are in diagonal form. Since $H = H_1 + H_2$, $H_2$ being the hermitian form associated to $D_2$ we see that $H_2$ is also in diagonal form. From (ii) and (iv) we get $\Sigma n_h = 1$ so that for a certain $l$, $n_l = 1$, and $n_h = 0$ for $h \neq l$.

It follows then, by renumbering the coordinates, that one must have $H_1 = \text{diag}(\epsilon_1, \ldots, \epsilon_l, 0, \ldots, 0)$, $\epsilon_l > 0$, and consequently $H_2 = \text{diag}(0, \ldots, 0, \epsilon_{l+1}, \ldots, \epsilon_g)$, $\epsilon_j > 0$. From (ii) it then follows that $D_1$ is the divisor of an intermediary function $\psi_1(u) = \psi_1(u_1, \ldots, u_l)$ independent of $u_{l+1}, \ldots, u_g$, while $D_2$ is the divisor of an intermediary function $\psi_2(u) = \psi_2(u_{l+1}, \ldots, u_g)$ independent of $u_1, \ldots, u_l$.

In particular $\text{supp} \ D_1 \cap \text{supp} \ D_2 \neq \emptyset$ so that the divisor of $\theta$ must have a singular set of dimension $\geq g - 2$. Furthermore, in the spaces $C^l$ and $C^{g-l}$ given by the coordinates $u_1, \ldots, u_l$ and $u_{l+1}, \ldots, u_g$ the lattices $\Lambda_x \cap C^l$ and $\Lambda_x \cap C^{g-l}$ are of maximal rank, giving complex tori $T_1 = C^l / (\Lambda_x \cap C^l)$ and $T_2 = C^{g-l} / (\Lambda_x \cap C^{g-l})$. The restrictions of the forms $H_1$ and $H_2$ to $T_1$ and $T_2$ are positive definite and thus give polarizations of $T_1$ and $T_2$. 

A. Andreotti and A. L. Mayer: On period relations for
Now $(A_z \cap C^i) \oplus (A_z \cap C^{g-1})$ is of finite index $r$ in $A$

$$r = \left( \int_{T_1} C_1(D_1)^{g-1} \right) \left( \int_{T_2} C_1(D_2)^{g-1} \right) = \text{order}(T_1 \cap T_2).$$

By Cor. 3 No. 5 VI of [29] we see that $T_1 \cap T_2$ contains only the identity, since $D = D_1 + D_2$ is left invariant by no translations other than the trivial one. Thus the polarizations induced by the forms $H_i$ and $H_2$ on the tori $T_1$ and $T_2$ are principal. So by choosing coordinates properly, one sees that the matrix $z$ is equivalent under the modular group $\Gamma_g$ to a matrix $\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$ with $z_1 \in H_t$ and $z_2 \in H_{g-1}$. Now for $1 \leq l \leq g - 1$ we have embeddings $j_l : H_t \times H_{g-1} \to H_g$ by letting $j_l(z_1, z_2) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$. It is clear that $S = \bigcup (\Gamma \cdot \text{Im}(j_l))$ is the set of all $z$ for which $[u : \theta(u, z) = 0]$ is reducible. Now given $z \in \text{Im}(j_l)$ there are only a finite number of cosets $M \Gamma_t \times \Gamma_{g-1}$ with $M \in \Gamma_g$, $Mz \in \text{Im}(j_l)$ where $\Gamma_t$ is the modular group of $H_t$. This follows from the finiteness of the isotropy subgroups of the modular groups, together with the fact that the divisor $D$ may be uniquely decomposed into a finite number of irreducible components. Thus $S$ is locally finite, and so is an analytic set, invariant under the modular group.

Let

$$E = \{ z \in H_g \mid \theta(u, z) \text{ is irreducible} \}.$$

We have seen that $E$ contains the Zariski open subset of $H_g$, $E' = H_g - N_{g-2}$. Note that the Jacobi space $J$ is contained in $E'$. From lemmas 9 and 10 we get in particular the following

**Corollary.** For any $z \in E$ the jacobian of the map $\chi$ is of maximal rank ($= g$) at any point $u_0 \in C^g$ which is not a half period $u_0 \in \frac{1}{2} A_z$. 

**Lemma 12.** For any $z \in E$, if $u, v \in C^g$ have the same image under $\chi$ then either $u + v$ or $u - v$ is a period.

**Proof.** For some $q \in C^*$ we must have $\theta_2[\mu](u) = q \theta_2[\mu](v) \forall \mu$. Thus $\Phi_c(u) = q \Phi_c(v) \forall c \in C^g$, i.e. $\theta(u + c) \theta(u - c) = q \theta(v + c) \theta(v - c)$. Since $\theta(u + c)$ is irreducible it must divide either $\theta(v + c)$ or $\theta(v - c)$. In the first case by lemma 10 $u - v$ is a period, in the second $u + v$ is a period.
Let us denote for any \( z \in \mathcal{H}_g \) by \( T_g(z) \) the torus \( \mathbb{C}^g/A_z \) and by \( K_g(z) \) the corresponding Kummer variety. From the previous lemmas one deduces easily the following

**Proposition 9.** For any \( z \in \mathcal{C} \) the Kummer variety \( K_g(z) \) is a holomorphic image of the torus \( T_g(z) \) by a map of degree 2 which is of maximal rank everywhere except at the \( 2^{2g} \) points \( u_0 \in T_g(z) \) of order 2 (\( 2u_0 = 0 \)); \( K_g(z) \) has only \( 2^{2g} \) isolated singular points and its order (as a projective variety) is \( g! \cdot 2^{g-1} \).

For the last statement see [28] § 1. It is worth noticing that the degree of the map \( \chi : T_g(z) \rightarrow K_g(z) \) for \( z \in \mathcal{C} \) may be greater than 2, for instance for \( g = 2 \) and \( z = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \) that map is of degree 4.

12. The equations of the Kummer variety. To get a set of equations for the Kummer variety we give here a procedure which is inspired by a similar one given by C. L. Siegel in [28]. We assume \( g \geq 2 \).

Let us consider a generic projection of \( K_g(z) \) onto a projective subspace \( P_{g+1} \) of \( P_t(\mathbb{C}) \) of dimension \( g + 1 \). If \( X = (x_0, \ldots, x_{g+1}) \) are homogeneous coordinates in \( P_t(\mathbb{C}) \) and \( P_{g+1} \) respectively, the projection is given by equations of the form

\[
x = S \lambda
\]

where \( S \) is a \((g + 2) \times (t + 1)\) matrix with elements \( s_{ij} \) that will be considered as indeterminates. The center of projection \((S \lambda = 0)\) is a generic projective subspace of \( P_t(\mathbb{C}) \) of codimension \( g + 1 \) and thus does not meet \( K_g(z) \). The projection is therefore well defined on \( K_g(z) \) and moreover it will be generally one to one. The image of \( K_g(z) \) under this projection is therefore an irreducible hypersurface of degree \( g! \cdot 2^{g-1} \) provided \( z \in \mathcal{C} \). Over the field \( \mathbb{C}(s) \) of rational functions in the \( s_{ij} \) the equation \( f(x_0, \ldots, x_{g+1}) = 0 \) of that hypersurface is given by equating to zero a homogeneous polynomial in the \( x's \) of degree \( g! \cdot 2^{g-1} \). This polynomial is uniquely determined up to a constant non zero factor by the condition

\[
f(S \lambda(u, z)) = 0 \quad \forall u \in \mathbb{C}^g
\]

where \( \lambda(u, z) \) is given by \( \lambda_u = \theta_z[\mu](u, z) \), i.e. the parametric equations of \( K_g(z) \).

The equation \( f(x) = 0 \) over any purely transcendental extension of \( \mathbb{C}(s) \) will be called a normal equation of \( K_g(z) \).

Let us denote by \( \alpha_0, \ldots, \alpha_{g} \) the coefficients of the generic homogeneous polynomial of degree \( g! \cdot 2^{g-1} \) in the variables \( x_0, \ldots, x_{g+1} \).
Let

\[ f(S \lambda (u, z)) = \sum_{\sigma \in \mathbb{N}^g} f_\sigma (x, s, z) u^\sigma \]

be the Taylor expansion of \( f(S \lambda (u, z)) \) as a function of the \( u \)'s. The coefficients \( f_\sigma (x, s, z) \) are linear forms in the \( x \)'s whose coefficients are polynomials with rational coefficients in the indeterminates \( s_{ij} \) and the « thetanulls » \( \theta_n \) for \( n \in \mathbb{N}^g \), and \( \sigma_i \) ranging over a set of representatives of \( \mathbb{Z}^g/2\mathbb{Z}^g \). Condition (1) is therefore equivalent to the system of homogeneous linear equations in the unknowns \( a_0, \ldots, a_\varphi \)

\[ \begin{cases} \left( \frac{\partial^{l+\cdots+r\mu} \theta_n}{\partial u_1^{l_1} \cdots \partial u_g^{l_g}} \right)_{u=0} = 0 \\ \sigma_i \in \mathbb{N}^g \end{cases} \]

for \( r \in \mathbb{N}^g \), and \( \mu \) ranging over a set of representatives of \( \mathbb{Z}^g/2\mathbb{Z}^g \). Condition (1) is therefore equivalent to the system of homogeneous linear equations in the unknowns \( x_0, \ldots, x_\varphi \)

\[ \begin{cases} f_\sigma (x, s, z) = 0 \\ \sigma \in \mathbb{N}^g \end{cases} \] (2)

We now make use of the following

**Lemma 13.** Let \( z_0 \in H_g \) and \( \Phi (u, z_0) \) be a theta function of order \( l \). If \( \Phi (u, z_0) \) vanishes at \( u = 0 \) with all its partial derivatives of order \( \leq lg! \cdot 3^{g-1} \), then \( \Phi (u, z_0) \) is identically zero.

**Proof.** With the theta functions of order 3 one obtains a biregular projective imbedding of the torus \( T_g (z_0) \) (cf. [12] pg. 159). The image manifold is an algebraic variety of order \( g! \cdot 3^g \).

Suppose \( \Phi (u, z_0) \) not identically zero. Then the positive divisor of \( \Phi (u, z_0) \) is transformed by the projective imbedding into an algebraic positive cycle \( C \) of degree \( lg! \cdot 3^{g-1} \). Let \( p \) be the image of \( u = 0 \). By the assumption \( p \) is a point of \( C \) of multiplicity \( \geq lg! \cdot 3^{g-1} + 1 \). This contradicts the theorem of Bézout.

As a consequence of this lemma we deduce the fact that in the system (2) one needs only to consider the system

\[ \begin{cases} f_\sigma (x, s, z) = 0 \\ \sigma = (\sigma_1, \ldots, \sigma_g) \mid \sigma \mid = \Sigma \sigma_i \\ \mid \sigma \mid \leq \sigma (g) \end{cases} \] (3)

where \( \sigma (g) \) is a bound depending only on \( g \).

Since the system (3) has a unique solution up to a multiplicative factor \( \neq 0 \) the rank of the matrix of the coefficients must be \( g \). For any choice of \( \varphi \) rows, say for \( \sigma = \sigma_1, \ldots, \sigma_\varphi \), we can consider the determinants...
of the minors of order $q_i, M_i^A(\tau, z), 0 \leq i \leq q, A = (\sigma_1, ..., \sigma_q)$, extracted from the matrix of those $q$ rows. Choosing properly the sign for the determinants $M_i^A$ we see that the coefficients $a_i$ of the normal equation of $K_g(z)$ are proportional to these minors $M_i^A$. These minors are therefore either all zero or they can be taken as coefficients for the normal equation. For at least one system $A$ of $q$ rows the latter is the case.

Introducing a set of new indeterminates $\eta_A, A = (\sigma_1, ..., \sigma_q), |\sigma_i| \leq \sigma(q)$, we can set

$$a_i = \sum \eta_A M_i^A(\tau, z)$$

for the coefficients of the normal equation of $K_g(z)$. We thus get a normal equation for $K_g(z), z \in \mathcal{C}$

$$(4) \quad f(\eta, s; z; x) = 0$$

in which the coefficients are polynomials over $\mathbb{Q}$ in $\eta, s$ and the thetanulls $C(r, \mu, z)$.

From this fact one deduces the following

**Proposition 10.** There exists a finite set of homogeneous polynomials in the $\lambda_i$'s $[g_h(\tau, \lambda)]_{h \in H}$ whose coefficients are polynomials over $\mathbb{Q}$ in the thetanulls $C(r, \mu, z)$, such that for any $z \in \mathcal{C}$ the set of equations

$$\{g_h(z, \lambda) = 0\}$$

define in $P_1(\mathbb{C})$ the variety $K_g(z)$ as the set of their common zeros.

**Proof.** We set $x = S\lambda$ in the normal equation (4) and rewrite the left hand side as a polynomial in the $s_i$ and $\eta$:

$$f(\eta, s, z, S\lambda) = \sum g_h(z, \lambda) \omega_h(s, \eta)$$

where $\omega_h(s, \eta)$ are the different monomials in the $s_i$ and the $\eta_i$'s. The coefficients $g_h(z, \lambda)$ of those monomials satisfy the requirements of this proposition.

13. The equations of the ramification sets. We have defined in section 6 a sequence of analytic subsets $N_s$ of $H_g$ invariant under the action of the modular group for $0 \leq s \leq g - 2$ giving a filtration of $H_g$:

$$N_{g-2} \subset N_{g-3} \subset ... \subset N_1 \subset N_0 \subset H_g.$$ 

**Proposition 11.** For each $s$ with $0 \leq s \leq g - 2$ one can find a finite set of homogeneous polynomials with rational coefficients in the thetanulls
PROOF. We consider first the set of equations

\[ K = \{ g_h(z, \lambda) = 0 \} \]

of the Kummer variety given by proposition 10. We add to this the system of equations

\[
\begin{align*}
L &= \left\{ \sum \lambda_\mu \theta_\mu [\mu] \left( 0, z \right) = 0, \\
& \quad \sum \lambda_\mu \frac{\partial^2 \theta_\mu [\mu]}{\partial u_\alpha \partial u_\beta} (0, z) = 0 \quad \text{for} \quad 1 \leq \alpha \leq \beta \leq g \right\}
\end{align*}
\]

and the system

\[
M = \left\{ \sum_{1 \leq i \leq s} w^{(i)}_\mu \lambda_\mu = 0 \right\}
\]

in which the coefficients \( w^{(i)} \) are indeterminates.

Let \( z \) be fixed in \( \mathcal{E} \) and let \( \lambda^* = (\lambda^*_\mu) \) be a non trivial solution of \( K \) and \( L \). From the system \( K \) we deduce the existence of a point \( c^* \in C^g \) such that

\[ \lambda^*_\mu = \theta_\mu [\mu] (c^*, z) \]

with some \( q \equiv 0 \). We thus get

\[
\sum \lambda^*_\mu \theta_\mu [\mu] (u, z) = q \Phi,c^* (u) = q \theta (u + c^*, z) \theta (u - c^*, z).
\]

Since \( \lambda^* \) is a solution of \( L \) we then obtain \( \theta^2 (c^*, z) = 0 \) and \( \frac{\partial^2 \theta}{\partial u_\alpha \partial u_\beta} (c^*, z) = 0 \), i.e.

\[ \theta (c^*, z) = 0 \quad \text{and} \quad \frac{\partial \theta}{\partial u_\alpha} (c^*, z) = 0. \]

This means that \( c^* \) is a singular point of \( \theta (u, z) = 0 \) in \( C^g \).

Eliminating \( \lambda \) from \( K, L, M \) by the Kronecker procedure we obtain a finite system of polynomials \( \{ P_{\alpha} \}_{\alpha \in A} \) homogeneous in the coefficients of the given equations and defined over \( \mathbb{Q} \). These polynomials are thus rational polynomials in the thetanulls and the indeterminates \( w^{(i)} \). If \( z \in N_* \cap \mathcal{E} \) the \( P_{\alpha} \)'s vanish for any choice of the \( w^{(i)} \) and conversely since the vanishing of them implies that the singular set of \( \theta = 0 \) is of dimension \( \geq s \) (the map from \( T_g(z) \) to \( K_g(z) \), being of degree 2 with finite fibers, does not affect the dimensions).
To obtain the desired equations we have only to expand each $P_a$ as a polynomial in the $w$'s and take for $p_y$ the coefficients of these polynomials.

14. a) The equations $K$ and $L$ which by the elimination procedure described before give the equations over $E$ of the ramification set, have for coefficients polynomials in the thetanulls $G(r, \mu, z)$. These as function of $z$ are not, if $|r| > 0$, modular forms with respect to any subgroup $G$ of finite index in the modular group $\Gamma$.

Our first goal is to show that the system $L$ can be replaced by an equivalent one in which the coefficients are modular form with respect to a group $G$ of finite index in $\Gamma$. We first prove the following

**Lemma 14.** For any $z \in H_\beta$ the matrix of the system $L$

$$L = \begin{pmatrix} \theta_2[\mu](0, z) \\ \frac{\partial^2 \theta_2[\mu]}{\partial u_a \partial u_\beta}(0, z) \end{pmatrix}$$

is of maximal rank $= 1 + \frac{1}{2}g(g + 1)$.

**Proof.** Suppose that at a point $z_0 \in H_\beta$ the rank of $L$ is $< 1 + \frac{1}{2}g(g + 1)$. We must have therefore a linear relation

$$a_0 \theta_2[\mu](0, z_0) = \sum_{i \leq k} a_{ik} \frac{\partial^2 \theta_2[\mu]}{\partial u_i \partial u_k}(0, z_0)$$

valid for every $\mu$ with $a_0$ and $a_{ik}$ constants not all zero.

Multiplying these relations by $\theta_2[\mu](c, z_0)$ and summing over $\mu$ we then get

$$a_0 \left[\theta(u + c, z_0) \theta(u - c, z_0)\right]_{u=0} = \sum_{i \leq k} a_{ik} \frac{\partial^2}{\partial u_i \partial u_k} \left[\theta(u + c, z_0) \theta(u - c, z_0)\right]_{u=0}.$$

Taking into account the fact that $\theta(u, z) = \theta(-u, z)$, we obtain

$$a_0 \left[\theta(c, z_0)\right]^2 = \sum_{i \leq k} a_{ik} \frac{\partial^2}{\partial c_i \partial c_k} \left[\theta(c, z_0)\right]^2.$$

For the function $\psi(c) = \left[\theta(c, z_0)\right]^2$ there are the periodicity conditions:

$$\psi(c + z_0 m) = e^{2\pi i (m_0 m + z_0 n)} \psi(c)$$
for any \( m \in \mathbb{Z}^g \). Therefore from the above linear relations we obtain

\[
\sum_{i \leq k} a_{ik} \left( m_i \frac{\partial \theta}{\partial c_i}(c, z_0) + m_k \frac{\partial \theta}{\partial c_i}(c, z_0) \right) = 2\pi i \sum_{i \leq k} a_{ik} m_i m_k \theta(c, z_0).
\]

Take all \( m \)'s equal zero except \( m_i = m_k = 1 \). Identifying the coefficients in the Fourier expansion on both sides we get for any \( p \in \mathbb{Z}^g \)

\[
a_{ik} (p_i + p_k) = a_{ik}.
\]

Therefore \( a_{ik} = 0 \). Since \( \theta(c, z_0) \) is not identically zero we must also have \( a_\emptyset = 0 \) and the lemma is proved.

\( b) \) From the theory of transformation of theta functions we borrow the following fact:

There is a subgroup \( G \) of finite index in \( \Gamma \) such that for any \( \gamma \in G \):

\[
\gamma : z \mapsto (Az + B)(Cz + D)^{-1}
\]

each one of the functions \( g(u, z) = \theta_\gamma [\mu](u, z) \) satisfies the functional equation:

\[
(1) \quad g((Cz + D)^{-1}u, (Az + B)(Cz + D)^{-1}) = \frac{1}{e^{2\pi i T_{tr}(Cz + D)^{-1} C_{u} g(u, z)}}
\]

where \( \det (Cz + D)^{\frac{1}{2}} \) is a determined branch of this square root and where \( e \) is an eighth root of unity.

This theorem can be found in C. L. Siegel \[28\], pg. 395; with the notations of that paper if one takes \( T = 2I \) one has \( \theta(x, 0, 2z, 2u) = \theta_\gamma [x](u, z) \) and the group \( G \) is the «Theta gruppe der Stufe \( T \».

Going back to the system \( L \) by virtue of the lemma 14 and Cramer's rule on can replace the system \( L \) by a system having the same set of solutions and of the form

\[
\{ \sum_{1 \leq a \leq s} \lambda_a \mathcal{D}_{\mu a}(z) = 0
\]

where \( \mathcal{D}_{\mu a}(z) \) are the determinants of the minors of maximal rank extracted from the matrix \( \mathcal{L} \).

Now we remark that the functions \( \theta_\gamma [\mu](u, z) \) are even functions of \( u \) so that their first derivatives vanish at \( u = 0 \). \( \mathcal{D}_{\mu a} \) thus appear as Wronskian determinants extracted from the Wronskian matrix for the functions \( \theta_\gamma [\mu](u, z) \). From the functional equation (1) and a known property of Wronskians of automorphic forms we deduce that the functions \( \mathcal{D}_{\mu a}(z) \) sati-
sfy a transformation formula of the form:

\[ \mathcal{D}_\mu ((Az + B)(Cz + D)^{-1}) = \alpha_0 \det(Cz + D)^{k/2} \mathcal{D}_\mu (z) \]

for any \( y \in G \), where \( \alpha_0 \) is an eighth root of unity depending on \( y \) and \( k \) is an integer. Both \( \alpha_0 \) and \( k \) are independent of \( \mu \) and \( \alpha \).

The system \( L \) has thus the same solutions as the system \( L' \)

\[ L' = \left( \sum_{1 \leq n \leq s} \lambda_n \mathcal{D}_\mu (z) \right)^8 = 0. \]

In this the coefficients are now modular forms with respect to \( G \).

c) For the system \( K \) representing the equations of the Kummer variety a system of equations with \( G \)-modular forms as coefficients valid on a certain Zariski open set \( U \) of \( H_g \) was given by Wirtinger [31]. An outline of his method will be given in the last section. From this we deduce that the ramification sets \( \mathcal{N}_s \) on \( U \) can be defined by a set of equations obtained by equating to zero a set of \( G \)-modular forms.

d) The method of C. L. Siegel based on the normal equation of \( K_g (z) \) has however the advantage of giving the equation on a very explicit set \( \mathcal{E} \subset H_g \) and not on an unspecified Zariski open set \( U \) of \( H_g \). Moreover some geometrical facts can be proved along this line. As an instance we derive here from this method the following

**Proposition 12.** For any \( s \) with \( 0 \leq s \leq g - 2 \) and for any \( z_0 \in \mathcal{E} \) we can find a Zariski neighborhood \( V(z_0) \) of \( z_0 \) in \( \mathcal{E} \) and a finite set of meromorphic and \( G \)-automorphic functions \( m_a(z) \)\(, 1 \leq a \leq k \) regular on \( V(z_0) \) such that

\[ \mathcal{N}_s \cap V(z_0) = \{ z \in V(z_0) \mid m_a(z) = 0, \text{ for } 1 \leq a \leq k \}. \]

**Proof.** Let \( f(\eta, s, z, x) = 0 \) be the normal equation of \( K_g(z), \eta \in \mathcal{E} \). We remark that all coefficients \( a_i(\eta, s, z) \) (\( \eta \) and \( s \) being considered as indeterminates) are non zero. Moreover the functions

\[ \beta_j(s, z) = \frac{\alpha_j(\eta, s, z)}{\alpha_0(\eta, s, z)} \]

for \( 1 \leq j \leq g \) do not depend on the parameters \( \eta \) and are meromorphic functions of \( s \) and \( z \).

Because of the transformation formula (1) for \( y \in G \) the Kummer varieties \( K_g(z) \) and \( K_g(\gamma z) \) coincide. Therefore on has

\[ \beta_j(s, \gamma z) = \beta_j(s, z) \quad 1 \leq j \leq g. \]
Let \( z_0 \in \mathbb{C} \) and select \( s = s_0 \) integral such that \( \alpha_0 (\eta, s_0, z_0) \neq 0 \). On a Zariski neighborhood \( U(z_0) \) of \( z_0 \) in \( \mathbb{C} \) we then have \( \alpha_0 (\eta, s_0, z) \neq 0 \).

Let \( f = \sum \omega_i (x) \) where \( \omega_i (x) \) are the different monomials in the \( x \)'s of the proper degree. Consider the substitution \( x = x' \) and set

\[
\omega_i (x') = \sum c_{ij} \omega_j (\lambda) \quad c_{ij} \in \mathbb{Z} [s]
\]

where \( \omega_j (\lambda) \) are the different monomials in the \( \lambda \)'s of the proper degree.

We then have

\[
f (\eta, x, z, x') = \alpha_0 (\eta, s, z) \left( \sum \beta_i (s, z) \omega_i (\lambda) \right).
\]

Develop \( \alpha_0, \beta_i \) and \( c_{ij} \) in power series in \( s - s_0 \):

\[
\alpha_0 (\eta, s, z) = \sum \alpha_{0v} (\eta, s_0, z) (s - s_0)^v \quad \beta_i (s, z) = \sum \beta_{i\mu} (s_0, z) (s - s_0)^\mu \quad c_{ij} = \sum c_{ij\nu} (s - s_0)^\nu
\]

so that one obtains

\[
f (\eta, s, z, x') = \sum \left( \sum \alpha_{0v} (\eta, s_0, z) \sum \beta_{i\mu} (s_0, z) c_{ij\nu} (s - s_0)^\nu \right) \omega_j (\lambda).
\]

Since \( f \) is a polynomial in \( s - s_0 \), we deduce that there is an integer \( N \) independent of \( s_0 \) and \( z \) such that the set of equations

\[
K' = \begin{cases} 
\tau \mid \leq N \end{cases} 
\]

has the same solutions as the set of equations \( K \) for any \( z \in U(z_0) \).

Let \( g(z) \) be any \( G \)-modular form of weight \( 4kh \) such that \( g(z_0) \neq 0 \). This is certainly the case if \( k \) is large enough. Replace the system \( L' \) with the system

\[
L'' = \left\{ g(z)^{-1} (\sum \eta_{\mu} (D_{\mu})^{\lambda}) \right\} = 0.
\]

In a Zariski neighborhood \( V(z_0) \) of \( z_0 \in \mathbb{C} \) then, the equations \( K' \) and \( L'' \) have \( G \)-automorphic, meromorphic coefficients, regular on \( V(z_0) \). By the elimination procedure described above we then get on \( V(z_0) \) the conclusion we wanted.

It is known (and it can be derived from the pseudoconcavity of the modular group cf. [5]) that any meromorphic and \( G \)-automorphic function is the quotient of two \( G \)-modular forms. Moreover with a finite set of
G-modular forms one can obtain a one to one holomorphic imbedding $\tau$ of $H_g/G$ into a projective space. By the use of compactification of the modular space $H_g/G$, this fact was first proved by W. Baily [7] and for more general conditions by H. Cartan [11]. It can however be proved directly as is shown in C. L. Siegel [28]. The image of $\tau(H_g/G)$ is contained in an algebraic variety of the same dimension. In particular it follows from the previous proposition that

**Corollary:** For any $0 \leq s \leq g - 2$ each irreducible component of $N_s \cap C$ has an image by $\tau$ which is an algebraic set. In particular the whole Jacobi space in $\tau(H_g/G)$ lies in an algebraic set of dimension $3g - 3$.

15. The Wirtinger method. We give here an outline of Wirtinger's method of writing the equations of the Kummer variety.

a) First we introduce theta functions with characteristic. For $x, y \in \mathbb{Z}^g$ we set

$$\theta(x, y; u, z) = \sum e^{n \left(\frac{x + \frac{y}{2}}{2}\right) \left(m + \frac{z}{2}\right) + \frac{1}{2} \left(m + \frac{z}{2}\right) \left(u + \frac{y}{2}\right)}$$

$$= e^{n \left(\frac{1}{4} t_{xz} x + t_{xy} y \frac{u}{2} + \frac{z}{2} \frac{x}{2} \right)} \theta\left(u + \frac{y}{2} + \frac{z}{2} \frac{x}{2} ; z\right).$$

Since $\theta(x + 2a, y + 2b; u, z) = e^{n t_{ab}} \theta(x, y; u, z)$, up to constant factors $\neq 0$ one obtains all of these functions when $x, y$ run over a set of representatives of $\mathbb{Z}^g/2\mathbb{Z}^g$. One obtains thus $2^{2g}$ linearly independent functions over $C$. Among them those for which $t_{xy}$ is even are even functions of $u$. Their number is $2^{g-1}(2^g + 1)$. Those for which $t_{xy}$ is odd are odd functions of $u$ and their number is $2^{g-1}(2^g - 1)$. Their periodicity properties are as follows:

$$\theta(x, y; u + n + zm, z) = e^{-ni \left(\frac{t_{nm} + t_{mu} - t_{az} + t_{my}}{2}\right)} \theta(x, y; u, z),$$

for $n, m \in \mathbb{Z}^g$. With the notations of n. 11 one has

$$\theta_\epsilon [\mu](u, z) = \theta(\mu, 0; 2u, 2z).$$

Let

$$\mathbf{\gamma}: z \rightarrow \mathbf{\bar{z}} = (Az + B)(Cz + D)^{-1}$$

be a transformation of the modular group.
We let

\[ \hat{u} = \gamma(Cz + D)^{-1} u \]
\[ \hat{x} = \Lambda x + \Lambda y + [\Lambda C] \]
\[ \hat{y} = \Lambda Bx + \Lambda Dy + [\Lambda BD] \]

where for a square matrix \( M \), \([M]\) denotes the vector whose components are the diagonal elements of \( M \).

One has the following transformation formula (cf. C. L. Siegel [28], theorem 8)

\[ \theta(x, y; \hat{u}, \hat{z}) = g \det (Cz + D)^{-1} e^{-\frac{1}{2} \pi i \psi(x, y)} e^{\pi i \psi(u)} \theta(x, y; u, z) \]

where

\[ g \] is a constant of absolute value 1 depending only on \( \gamma \)

\[ \varphi(x, y) = \frac{1}{4} (x A + y C) (Bx + Dy + 2 [BD]) - \frac{1}{4} (xy) \]

We see that by this transformation even functions are changed into even functions. Moreover, if \((x_1, y_1), (x_2, y_2)\) are two systems of characteristics with \( x_1 + x_2 \equiv 0 \pmod{2}, y_1 + y_2 \equiv \sigma \pmod{2} \) then \( \hat{x}_1 + \hat{x}_2 \equiv \sigma \pmod{2} \)
\( \hat{y}_1 + \hat{y}_2 \equiv \tau \pmod{2} \) with \( \sigma \) and \( \tau \) depending only upon \( \sigma, \tau \) and \( \gamma \).

We will need the addition theorem given by the formulae:

\[ \theta(x, y; u + v, z) \theta(x, y; u - v, z) = \sum_{\mu} (-1)^{t(x+y)\mu} \vartheta_2(\mu)(v, z) \vartheta_2(x+\mu)(u, z) \]
\[ 2^\sigma \vartheta_2(\mu)(v, z) \vartheta_2(\mu+\sigma)(u, z) = \sum_{\gamma} (-1)^{t(x+y)\gamma} \vartheta(x, y; v, z) \vartheta(x, y; u-v, z) \]

A special case of (1) is formula (1) of n. 14 and a special case of (2) is formula (2) of n. 11.

b) We consider for given \( x \in H_\theta \) the ideal \( \mathcal{I}(K_\theta^*(z)) \) of all homogeneous polynomials in the ring \( \mathbb{C}[\ldots, \lambda_\mu, \ldots] \) vanishing on \( K_\theta^*(z) \). It is defined by the conditions

\[ \mathcal{I}(K_\theta^*(z)) = \{ p \in \mathbb{C}[\ldots, \lambda_\mu, \ldots] \mid p(\ldots, \vartheta(\mu)(u), \ldots) = 0 \} \]
We want to write a basis for polynomials of degree 4 in that ideal.

**Lemma 15.** The $2g-1 (2g - 1)$ products $\theta_2[0](u) \theta_2[\mu](u)$ are linearly independent at any point $z \in \mathbb{H}_g$ such that

\[ \theta(x, y; 0, z) \equiv 0 \text{ for all } x, y \text{ with } txy \equiv 0 \text{ (mod 2).} \]

**Proof.** Setting $u = v$ in (2) we get the formula

\[ (2)_0 \quad \theta(x, y; 2u, z) \theta(x, y; 0, z) = \sum (-1)^{t(x+y)\mu} \theta_2[\mu](u, z) \theta_2[x+\mu](u, z) \]

Since $\theta(x, y; 0, z) = 0$ if $txy \equiv 1 \text{ (mod 2)}$ and by the assumption (z) on the left hand side we get $2^{2g-1}(2g - 1)$ linearly independent functions. These being linear combinations of the products $\theta_2[\mu](u) \theta_2[\mu](u)$, we get the conclusion.

Our problem, under assumption (a), is thus reduced to write a basis for all quadratic relations among the products $\theta_2[\mu](u) \theta_2[\mu](u)$. From (3) setting $u = v$ we get

\[ (3)_0 \quad 2^g \theta_2[\mu](u, z) \theta_2[\mu+x](u, z) = \sum (-1)^{t(x+y)\mu} \theta(x, y; 0, z) \theta(x, y; 2u, z). \]

It follows that under assumption (a) any quadratic relation among the products $\theta_2[\mu](u) \theta_2[\mu](u)$ gives a quadratic relation for the even theta functions $\theta(x, y, 2u, z)$ and conversely.

**c)** Let

\[ \sum_{\alpha, \beta, u \in \mathbb{H}_g} C(\alpha, \beta; x, y) \theta(\alpha, \beta; u, z) \theta(x, y; u, z) = 0 \iff u \]

be, for given $z \in \mathbb{H}_g$, a quadratic relation among the even theta functions with characteristic. Making use of the periodicity conditions it follows that any relation of that type is a linear combination of relations of the form

\[ \sum_{\alpha, \beta, x, y \in \mathbb{H}_g \atop \beta \equiv -x \text{ (mod 2)}} C(\alpha, \beta, x, y) \theta(\alpha, \beta; u, z) \theta(x, y; u, z) = 0 \iff u. \]

**d)** For $\alpha = \beta = 0$ we are reduced to find all linear relations among the functions $|\theta(x, y; u, z)|^2$ where $txy \equiv 0 \text{ (mod 2)}$. Setting $v = 0$ in (2) we get:

\[ (2)_1 \quad |\theta(x, y; u, z)|^2 = \sum_{\mu} (-1)^{t(x+y)\mu} \theta_2[\mu](0, z) \theta_2[x+\mu](u, z). \]

Let $A$ be the matrix indexed by $(x, y)$ with $txy \equiv 0 \text{ (mod 2)}$ and $\mu$, given by

\[ A = \left( (-1)^{t(x+y)\mu} \theta_2[\mu](0, z) \right). \]
Then all linear relations among the functions \( \{ \theta(x, y; u, z) \}^2, \, \tau xy = 0 \, (\text{mod } 2) \),
are given by the conditions:

\[
\text{rank } (\{ \theta(x, y; u, z) \}^2, A) = \text{rank } A
\]

Note that setting \( v = 0 \) in (3) we get for \( x = 0 \),

\[
(3)_1 \quad 2^g \theta_{\mathbb{Z}}[\mu](0, z) \theta_{\mathbb{Z}}[\mu](u, z) = \sum (-1)^{t_{\mu} y} \theta(0, y; u, z)^2
\]

so that if

\[
(\beta) \quad \theta_{\mathbb{Z}}[\mu](0, z) = 0 \quad \text{for all } \mu
\]

then among the functions \( \{ \theta(x, y; u, z) \}^2, \, \tau xy = 0 \, (\text{mod } 2) \),
there are exactly \( 2^g \) linearly independent. In this case rank \( A = 2^g \).

\( e) \) For \( \sigma = 0 \tau \equiv 0 \) we are reduced to find all linear relations among
the products \( \theta(x, y; u, z) \theta(x, y + \tau; u, z) \) where \( \tau xy, \tau x(y + \tau) \) and therefore
also \( \tau x \), are even.

Changing in (2) \( u \) into \( u + \frac{\tau}{4} \) and \( v \) in \( v + \frac{\tau}{4} \) and setting \( v = 0 \) we get

\[
(2)_2 \quad \theta(x, y + \tau; u, z) \theta(x, y; u, z) = \sum (-1)^{t_{\mu} y} \theta(\mu, \tau; 0, 2z) \theta(\mu + \tau; 2u, 2z).
\]

Since \( \tau \equiv 0 \) among the functions \( \theta(x + \mu, \tau, 2u, 2z) \) for which \( t(x + \mu) \tau \) is
even there are \( 2^{g-1} \) linearly independent ones.

Let \( B \) be matrix indexed by \( (x, y), \tau xy \equiv 0 \, (\text{mod } 2) \), and \( \mu \)

\[
B = \left( (-1)^{t_{\mu} y} \theta(\mu, \tau; 0, 2z) \right).
\]

Then all linear relations of the type under consideration are given by the conditions

\[
\text{rank } (\theta(x, y + \tau; u, z) \theta(x, y; u, z), B) = \text{rank } B.
\]

Note that from (3) changing \( u \) to \( u + \frac{\tau}{4}, \) \( v \) to \( v + \frac{\tau}{4} \), and setting \( v = 0 \)
and \( x = 0 \) we get

\[
(3)_2 \quad 2^g \theta(\mu, \tau; 0, 2z) \theta(\mu, \tau; 2u, 2z) = \sum (-1)^{t_{\mu} y} \theta(0, y + \tau; u, z) \theta(0, y; u, z)
\]

so that if

\[
(\gamma) \quad \theta(\mu, \tau; 0, 2z) = 0 \quad \text{for every } \mu \text{ with } t_{\mu} \tau \equiv 0 \, (\text{mod } 2)
\]

then the rank of \( B \) equals \( 2^{g-1} \).
f) To get all other relations Wirtinger proceeds as follows. For every modular transformation $\gamma$ one writes the relations given in e) in the variables $\hat{u}$ and $\hat{z}$ and notices that by (1) the rank of $B$ is not affected.

Let

$$\sum C(x, y) \theta(x, y + \tau; \hat{u}, \hat{z}) \theta(x, y; \hat{u}, \hat{z}) = 0 \forall u$$

be one of these relations. From the transformation formula (1) we then obtain a relation

$$\sum C(x, y) \varepsilon(x, y, \tau, \gamma) \theta(x, y + \tau; \hat{u}, \hat{z}) \theta(x, y; \hat{u}, \hat{z}) = 0 \forall u$$

where $\varepsilon$ denotes an eighth root of unity. Conversely from a relation of this second type we derive one of the type given in e) at the point $\hat{z}$.

Now the functions $\theta(x, y + \tau; u, z)$ and $\theta(x, y; u, z)$ are both even as are their transforms. For any choice of $(a, q) \equiv 0$ there exists a transformation $\gamma$ such that the sum of the characteristics in the product $\theta(x, y + \tau; u, z) \theta(x, y; u, z)$ is congruent to $(a, q) \pmod{2}$. In this way one obtains all remaining relations.

$g$) These considerations can be summarized as follows:

Let

$$W = \{ z \in H_g \mid \theta(x, y; 0, z) \equiv 0 \text{ for } ^t xy \equiv 0 \pmod{2} \}$$

Let

$$X_{ab} = \{ z \in H_g \mid \text{rank } A = a, \text{rank } B = b \}.$$  

There is a finite number of polynomials of degree 4 in $C[\ldots, \lambda_\mu \ldots]$ whose coefficients are polynomials over $Q\left(e^{\frac{\pi i}{4}}\right)$ in the thetanulls $\theta(\mu, \tau; 0, 2z)$ for $^t \mu \tau \equiv 0 \pmod{2}$ such that for any point $z \in W \cap X_{ab}$ they form a basis for the homogeneous polynomials of degree 4 in the ideal $J(K_g(z))$ of the Kummer variety. The closure of the set $X_{29, 29-1}$ is $H_g$. Now Wirtinger proves that for a certain Zariski open set $U$ of $H_g$ containing the diagonal matrices these equations of 4th order are defining equations for the Kummer variety on $U \cap W$. The Jacobi space $J$ intersects $U$ (cf. [18]) and moreover $J \cap W \neq \emptyset$ by a theorem of Farkas [14]. Combining these results one can write by equating to zero $G$-modular forms the equations of an analytic set having $\hat{J}$ as an irreducible component.
BIBLIOGRAPHY

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