

ANNALI DELLA  
SCUOLA NORMALE SUPERIORE DI PISA  
*Classe di Scienze*

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*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3<sup>e</sup> série*, tome 21,  
n° 3 (1967), p. 349-351

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## THE GEGENBAUER FUNCTION

by W. R. SCHUCANY (\*)

In the notes of the American Mathematical Monthly, [1] G. G. Bilodeau gives a new expression for the Gegenbauer polynomials by employing the concept of a fractional derivative. It is interesting to note that by employing the results obtained in [3], one can obtain a single formula for both the classical polynomials and the Gegenbauer functions ( $C_\beta^\mu$ ) as defined by Bateman [2].

The only definition necessary for clarity is the following:

If i)  $\int_a^x (x-t)^{n-\alpha-1} f(t) dt \in C^{(n)}$  on  $(a, b)$ ,

ii)  $\alpha > 0$  and  $n - \alpha > 0$ ,

where  $n$  is a positive integer, then the Holmgren-Riesz transform of  $f$  is given by

$$I_a^{-\alpha} f(x) = {}_a D_x^\alpha f(x) = \frac{D_x^n}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f(t) dt.$$

Now one solution of the equation

$$(1) \quad (1-x^2)y'' - (2\mu+1)xy' + \beta(\beta+2\mu)y = 0,$$

may be written in its  $H-R$  transform form as

$$(2) \quad y = k(1-x^2)^{-\mu+1/2} {}_1 D_x^\beta (1-x^2)^{\beta+\mu-1/2}, \quad \beta > 0 \quad \text{and} \quad x > 1,$$

(See [3]).

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Pervenuto alla Redazione il 18 Agosto 1966.

(\*) LTV ELECTROSYSTEMS, INC. GREENVILLE DIVISION.

If the arbitrary constant  $k$  equals

$$\frac{(-1)^\beta \Gamma(\mu + 1/2) \Gamma(\beta + 2\mu)}{2^\beta \Gamma(\beta + 1) \Gamma(2\mu) \Gamma(\mu + \beta + 1/2)},$$

then when  $\beta = n$ , equation (2) becomes the standard Rodrigues formula for the Gegenbauer polynomials. Also for positive non-integer values of  $\beta$ ,  $\mu > -1/2$  and  $\mu \neq 0$ , equation (2) can be shown to be equivalent to

$$y = C_\beta^\mu(x) = \frac{\Gamma(\beta + 2\mu)}{\Gamma(\beta + 1) \Gamma(2\mu)} F\left[\beta + 2\mu, -\beta; \mu + 1/2; \frac{1-x}{2}\right]$$

for  $x \in (1, 3)$ , where  $F[a, b; c; z]$  is the Hypergeometric function. (See [2]).

Consider (2) when  $\beta \neq n$ , an integer, and note that when  $n = |[-\beta]|$ ,

$${}_1D_x^\beta (1-x^2)^{\beta+\mu-1/2} = \frac{D_x^n}{\Gamma(n-\beta)} \int_1^x (x-t)^{n-\beta-1} (1-t^2)^{\beta+\mu-1/2} dt.$$

Letting  $z = \frac{t-1}{x-1}$ , the right hand side of the above equation becomes,

$$\begin{aligned} \frac{D_x^n}{\Gamma(n-\beta)} \int_0^1 [x-1+z(1-x)]^{n-\beta-1} [z(1-x)]^{\beta+\mu-1/2} [2-z(1-x)]^{\beta+\mu-1/2} (x-1) dz \\ = \frac{(-1)^{n-\beta} 2^{\beta+\mu-1/2}}{\Gamma(n-\beta)} D_x^n (1-x)^{n+\mu-1/2} \int_0^1 (1-z)^{n+\beta-1} z^{\beta+\mu-1/2} \\ \cdot \left[1-z\left(\frac{1-x}{2}\right)\right]^{\beta+\mu-1/2} dz. \end{aligned}$$

Then setting  $w = \frac{1-x}{2}$  we have,

$$\begin{aligned} {}_1D_x^\beta (1-x^2)^{\beta+\mu-1/2} &= \frac{(-1)^\beta 2^{\beta+2\mu-1} \Gamma(\beta + \mu + 1/2)}{\Gamma(n + \mu + 1/2)} \\ &\cdot D_w^n \{w^{n+\mu-1/2} F[-\beta - \mu + 1/2, \beta + \mu + 1/2; n + \mu + 1/2; w]\}, \\ &= \frac{(-1)^\beta 2^{\beta+2\mu-1} \Gamma(\beta + \mu + 1/2)}{\Gamma(\mu + 1/2)} \left(\frac{1-x}{2}\right)^{\mu-1/2} F\left[-\beta - \mu + 1/2, \beta + \mu + 1/2; \right. \\ &\qquad\qquad\qquad \left. \mu + 1/2; \frac{1-x}{2}\right]. \end{aligned}$$

Now applying a linear transformation formula we get

$$\begin{aligned} & {}_1D_x^\beta (1-x^2)^{\beta+\mu-1/2} \\ &= \frac{(-1)^\beta 2^\beta \Gamma(\beta+\mu+1/2)}{\Gamma(\mu+1/2)} (1-x^2)^{\mu-1/2} F\left[\beta+2\mu, -\beta; \mu+1/2; \frac{1-x}{2}\right]. \end{aligned}$$

Therefore

$$\begin{aligned} & \frac{(-1)^\beta \Gamma(\mu+1/2) \Gamma(\beta+2\mu)}{2^\beta \Gamma(\beta+1) \Gamma(2\mu) \Gamma(\mu+\beta+1/2)} (1-x^2)^{-\mu+1/2} {}_1D_x^\beta (1-x^2)^{\beta+\mu-1/2} \\ &= \frac{\Gamma(\beta+2\mu)}{\Gamma(\beta+1) \Gamma(2\mu)} F\left[\beta+2\mu, -\beta; \mu+1/2; \frac{1-x}{2}\right] = C_\beta^\mu(x). \end{aligned}$$

Hence the single expression (2) with the given value of  $k$  is a valid formula for the Gegenbauer polynomials, and with the generality afforded by the  $H-E$  transform the expression may be interpreted as including the associated function as well.

#### REFERENCES

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