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# Fubini-type theorems

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# FUBINI-TYPE THEOREMS

by ROBERT S. STRICHARTZ (*)

## § 1. Introduction.

Given a class of functions on a product space $M \times N$, it is a natural problem to try to characterize those functions in terms of their restrictions to sections $\{m\} \times N$ or $M \times \{n\}$ for all $m \in M$, $n \in N$. Fubini's theorem does this for Lebesgue integrable functions on a space with a $\sigma$-finite, positive product measure, $L^1(M \times N, \mu \times \nu)$. It says that a measurable function $f$ is in $L^1(M \times N, \mu \times \nu)$ if and only if the restriction of $|f|$ to $\mu$-almost every section $\{m\} \times N$ is $\nu$-integrable, and their integral, regarded as a function on $N$ is $\mu$-integrable. Its integral is in fact the $\mu \times \nu$ integral of $|f|$. This implies a similar characterization of $L^p(M \times N, \mu \times \nu)$ for $p < \infty$.

In [7] we proved a Fubini-type theorem for the Banach spaces $L^p_\alpha(E_n)$ of Bessel potentials of order $\alpha$ of $L^p$ functions in Euclidean $n$-space, for $\alpha \geq 0$ and $1 < p < \infty$. For $\alpha = k$, an integer, these spaces coincide with the usual Sobolev spaces of functions in $L^p$ with weak derivatives of order $\leq k$ in $L^p$. For $\alpha$ not an integer they form a natural class of « fractional Sobolev spaces ». The precise definition of $L^p_\alpha(E_n)$ is the class of functions of the form $G_\alpha * \varphi$ for some $\varphi \in L^p(E_n)$, where $G_\alpha$ is the function whose Fourier transform is $(1 + |\xi|^2)^{-\alpha/2}$. The $L^p_\alpha(E_n)$ norm of $f$ is the $L^p$ norm of $\varphi$, [1, 2]. The theorem we proved is the following: *Fubini-type Theorem for $L^p_\alpha$*: Let $e_1, \dots, e_n$ be any basis for $E_n$, $n \geq 2$, and denote by $(x_1, \dots, x_n)$ the coordinates of $x \in E_n$ with respect to this basis. Then a function $f \in L^p(E_n)$ is in $L^p_\alpha(E_n)$, $\alpha \geq 0$, $1 < p < \infty$, if and only if for each $j = 1, \dots, n$, the following holds: for almost every $(x_1, \dots, \widehat{x_j}, \dots, x_n) \in E_{n-1}$ the function

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$F_{x_1, \dots, \hat{x}_j, \dots, x_n}(x_j) = f(x_1, \dots, x_n) \in L^p_\alpha(E_1)$ and $\| F_{x_1, \dots, \hat{x}_j, \dots, x_n} \|_{L^p_\alpha(E_1)}$ is in $L^p(E_{n-1})$. Furthermore, the $L^p_\alpha(E_n)$ norm of $f$ is equivalent to the sum of these $n$ $L^p$ norms.

Now there is another class of spaces, which we shall denote $\Lambda^p_\alpha(E_n)$, which for $\alpha$ not an integer may also be considered « fractional Sobolev spaces ». They may be defined as follows:

Let $1 \leq p \leq \infty$ $\alpha = k + \beta$, $k$ a non-negative integer and $0 < \beta \leq 1$. Then $f \in L^p(E_n)$ is in $\Lambda^p_\alpha(E_n)$ provided $f$ and all its derivatives of order $\leq k$ are in $L^p$ and satisfy

$$\iint \frac{| g(x+y) + g(x-y) - 2g(x) |^p}{| y |^{n+\beta p}} \, dx \, dy \leq M^p < \infty.$$

The sum of the smallest such $M$ and the $L^p$ norms of $f$ and its derivatives is the $\Lambda^p_\alpha$ norm. If $p = \infty$ replace the integral by $\sup | y |^{-\alpha} | g(x+y) + g(x-y) - 2g(x) | \leq M < \infty$.

The spaces $L^p_\alpha$ and $\Lambda^p_\alpha$ coincide if and only if $p = 2$ (see [8], where they are denoted $L_{p\alpha}$ and $\Lambda(\alpha; p, p)$ respectively). Thus we already have a Fubini-type theorem for $\Lambda^2_\alpha$. The main goal of this paper is to extend this result to $\Lambda^p_\alpha$ for $1 \leq p \leq \infty$:

***Fubini-type theorem for*** $\Lambda^p_\alpha$: Let $e_1, \dots, e_n$ and $(x_1, \dots, x_n)$ be as above. Then a function $f \in L^p(E_n)$ is in $\Lambda^p_\alpha(E_n)$, $\alpha > 0$, $n \geq 2$, $1 \leq p \leq \infty$ if and only if for each $j = 1, \dots, n$ the following holds: for almost every $(x_1, \dots, \widehat{x_j}, \dots, x_n) \in E_{n-1}$ the function $F_{x_1, \dots, \hat{x}_j, \dots, x_n}(x_j) = f(x_1, \dots, x_n) \in \Lambda^p_\alpha(E_1)$ and $\| F_{x_1, \dots, \hat{x}_j, \dots, x_n} \| \Lambda^p_\alpha(E_1)$ is in $L^p(E_{n-1})$. Furthermore, the $\Lambda^p_\alpha(E_n)$ norm of $f$ is equivalent to the sum of these $n$ $L^p$ norms.

We shall prove this theorem in § 2. In § 3 we give some remarks showing the relationship between this result and other results. In § 4 we give an application of the Fubini-type theorem for $L^p_k(E_n)$ to resolve the following problem of Lions [4]: for which open sets $\Omega \subset E_n$ are the $C^\infty$ functions of compact support in $\Omega$ dense in $L^p_k(\Omega)$?

## § 2. Proof of the main theorem.

We lean heavily on the work of Taibleson [8], especially Theorem 4, p. 421. We summarize his results as follows:

LEMMA 1: For $f \in L^p(E_n)$ denote by $f(x, y)$, $x \in E_n$, $y > 0$ its Poisson integral $f(x, y) = C_n \int_{E_n} f(x - t) \frac{y}{(| t |^2 + y^2)^{n+1/2}} \, dt$. Then $f \in \Lambda^p_\alpha(E_n)$ if and only if, for some $k > \alpha$, $f \in L^p(E_n)$ and $\iint |y^{k-\alpha-1/p} D^k f(x,y)|^p\, dx\, dy \leq M^p < \infty$ for every derivative $D^k f$ of order $k$ of $f$. For $f \in \Lambda_\alpha^p(E_n)$ it is sufficient to consider only certain derivatives; for instance $\left(\frac{\partial}{\partial y}\right)^k$ alone will do, or all derivatives involving only the $x$ variables. The best constant $M$ plus the $L^p$ norm of $f$ gives a norm equivalent to $\|f\|_{\Lambda_\alpha^p}$. If $p = \infty$ the same results hold if the integral is replaced by ess sup $|y^{k-\alpha} D^k f(x,y)| \leq M < \infty$.

We will need a slight improvement of these results which is not given in [8]:

LEMMA 2: $f \in \Lambda_\alpha^p(E_n)$ if and only if $f \in L^p(E_n)$ and, for some $k > \alpha$, $\iint \left| y^{k-\alpha-1/p} \left(\frac{\partial}{\partial x_j}\right)^k f(x,y) \right|^p dx\, dy \leq M^p < \infty$ for $j = 1, \dots, n$. A similar result holds for $p = \infty$. Thus if $\alpha = \beta + m$, $0 < \beta \leq 1$, $m$ an integer $\geq 1$, then $f \in \Lambda_\alpha^p(E_n)$ if and only if $f \in L^p$ and $\left(\frac{\partial}{\partial x_j}\right)^m f \in \Lambda_\beta^p$ for $j = 1, \dots, n$.

PROOF: Assume $\iint \left| y^{k-\alpha-1/p} \left(\frac{\partial}{\partial x_j}\right)^k f(x,y) \right| dx\, dy < M^p \leq \infty$ for $j = 1, \dots, n$. It follows from Lemma 4b, p. 419 of [8], that $\iint \left| y^{k+k_0-\alpha-1/p} D^{k_0} \left(\frac{\partial}{\partial x_j}\right)^k f(x,y) \right| dx\, dy < C M^p$ for any derivative $D_x^{k_0}$ of order $k_0$ involving only the $x$ variables. But if $k_0 \geq (n-1)k$ then every derivative of order $k + k_0$ can be expressed as $D^{k_0}\left(\frac{\partial}{\partial x_j}\right)^k$ for some $j$. Thus by Lemma 1, $f \in \Lambda_\alpha^p$.

PROOF OF THE MAIN THEOREM: Let $f \in \Lambda_\alpha^p(E_n)$ with $\alpha = \beta + m$, $0 < \beta \leq 1$, $m$ an integer. It suffices to shows that for almost every $x' = (x_2, \dots x_n)$, $F_{x'}(x_1) = f(x_1, x') \in \Lambda_\alpha^p(E_1)$ and $\|F_{x'}\|_{\Lambda_\alpha^p(E_1)}$ is in $L^p(E_{n-1})$. In fact it suffices to show $\iint \left| \left(\frac{\partial}{\partial x_1}\right)^k f(x_1, x') \right|^p dx_1\, dx' \leq M^p < \infty$ and

$$\iiint \left| \frac{\left(\frac{\partial}{\partial x_1}\right)^k f(x_1 + t, x') + \left(\frac{\partial}{\partial x_1}\right)^k f(x_1 - t, x')}{|t|^{1+\beta p}} - \frac{2\left(\frac{\partial}{\partial x_1}\right)^k f(x_1, x')}{|t|^{1+\beta p}} \right|^p dt\, dx_1\, dx' \leq M^p < \infty \tag{1}$$

for $k = 0, 1, \dots, m$. This is a consequence of the definition of $\Lambda_\alpha^p(E_1)$ and the fact that if $\frac{\partial}{\partial x_1} f(x)$ is an $L^p$ derivative of $f$ as a function on $E_n$, then for almost every $x'$, $\frac{\partial}{\partial x_1} f(x_1, x')$ is an $L^p$ derivative of $f(x_1, x')$ as a function on $E_1$. [5, p. 57].

Now $\left(\frac{\partial}{\partial x_1}\right)^k f \in L^p(E_n)$ because $f \in \Lambda_\alpha^p(E_n)$. Thus it remains to prove the last inequality. We use the estimate of the $L^p$ norm of a second difference of a function in terms of its Poisson integral derived in [8] (the formula on top of p. 426):

$$\left(\iint \left| \left(\frac{\partial}{\partial x_1}\right)^k f(x_1 + t, x') + \left(\frac{\partial}{\partial x_1}\right)^k f(x_1 - t, x') - 2\left(\frac{\partial}{\partial x_1}\right)^k f(x_1, x')\right|^p dx_1\, dx'\right)^{\frac{1}{p}}$$

$$\leq 4\int_0^t y \left\| \left(\frac{\partial}{\partial x_1}\right)^k \left(\frac{\partial}{\partial y}\right)^2 f(x, y)\right\|_p dy + t^2 \left\| \left(\frac{\partial}{\partial x_1}\right)^k \left(\frac{\partial}{\partial y}\right)^2 f(x, t)\right\|_p$$

$$+ 2t^2 \left\| \left(\frac{\partial}{\partial x_1}\right)^{k+1} \left(\frac{\partial}{\partial y}\right) f(x, t)\right\|_p .$$

We take the $L^p$ norm with respect to $|t|^{-1-\beta p}\, dt$ and use the triangle inequality to dominate the $p$-th root of the expression in (1) by the sum of three terms:

$$4\left(2\int_0^\infty \left|\int_0^t y \left\| \left(\frac{\partial}{\partial y}\right)^2 \left(\frac{\partial}{\partial x_1}\right)^k f(x, y)\right\|_p dy\right|^p t^{-1-\beta p}\, dt\right)^{1/p}$$

$$+\left(2\int_0^\infty \int \left| t^{2-\beta-1/p} \left(\frac{\partial}{\partial y}\right)^2 \left(\frac{\partial}{\partial x_1}\right)^k f(x, t)\right|^p dx\, dt\right)^{1/p}$$

$$+ 2\left(2\int_0^\infty \int \left| t^{2-\beta-1/p} \left(\frac{\partial^2}{\partial y\, \partial x_1}\right) \left(\frac{\partial}{\partial x_1}\right)^k f(x, t)\right|^p dx\, dt\right)^{1/p} .$$

The last two terms are finite because $\left(\frac{\partial}{\partial x_1}\right)^k f \in \Lambda_\beta^p(E_n)$. The first term is handled as follows: Consider the integral operator $T\varphi(s) = \int_0^s \varphi(t)\, t^{-1+\alpha+1/p} s^{-\alpha-1/p}\, dt$.

The kernel is homogeneous of degree $-1$, so the operator is bounded in $L^p$ provided $\int_0^1 t^{-1+\alpha}\, dt < \infty$, i.e. provided $\alpha > 0$, [3, Chapter 9]. Applying this result to $\varphi(t) = \left\| \left(\frac{\partial}{\partial y}\right)^2 \left(\frac{\partial}{\partial x_1}\right)^k f(x,t) \right\|_p t^{2-\alpha-1/p}$ we see that the first term is dominated by a constant multiple of the second.

The same argument works for $p = \infty$ if we replace the $L^p$ norms by sups.

Conversely, suppose $\| F_{x_1, \dots, \hat{x}_j, \dots, x_n} \|_{\Lambda_\alpha^p(E_1)}$ is in $L^p(E_{n-1})$ for $j = 1, \dots, n$. Since the $\Lambda_\alpha^p(E_1)$ norm dominates the $L^p(E_1)$ norm we have $f \in L^p(E_n)$. Thus by Lemma 2 it suffices to show

$$\iint \left| y^{k-\alpha-1/p} \left(\frac{\partial}{\partial x_j}\right)^k f(x,y) \right|^p dx\, dy \leq M^p < \infty$$

for $j = 1, \dots, n$ and $k = m + 2$. We do the case $j = 1$, $m = 0$, the others being almost identical.

We use the fact that for each fixed $x'$ and $y$, $\left(\frac{\partial}{\partial x_1}\right)^2 P(x_1, x', y)$ is an even function with mean value zero in $x_1$, where $P$ is the Poisson kernel $C \frac{y}{(|x|^2 + y^2)^{n+1/2}}$. Thus

$$\left(\frac{\partial}{\partial x_1}\right)^2 f(x_1, x', y) = \iint \left(\frac{\partial}{\partial x_1}\right)^2 P(t_1, x' - t', y) f(x_1 - t_1, t')\, dt_1\, dt'$$

$$= \frac{1}{2} \iint \left(\frac{\partial}{\partial x_1}\right)^2 P(t_1, x' - t', y) [f(x_1 + t_1, t') + f(x_1 - t_1, t') - 2f(x_1, t')]\, dt_1\, dt'.$$

Taking the $L^p$ norm in $x_1$ and using Minkowski's inequality we get

$$\left\| \frac{\partial^2 f}{\partial x_1^2}(\cdot, x' y) \right\|_p \leq \frac{1}{2} \iint \left| \frac{\partial^2 P}{\partial x_1^2}(t_1, x' - t', y) \right| \cdot \| f(\cdot + t_1, t') + f(\cdot - t_1, t') - 2f(\cdot, t') \|_p\, dt_1\, dt'.$$

Then we take the $L^p$ norm in $x'$:

$$\left\| \frac{\partial^2 f}{\partial x_1^2}(\cdot, y) \right\|_p \leq \frac{1}{2} \int \left[ \int \left| \frac{\partial^2 P}{\partial x_1^2}(t, x', y) \right| dx' \right] \cdot \left[ \int \| f(\cdot + t, t') + f(\cdot - t, t') - 2f(\cdot, t') \|_p^p\, dt' \right]^{1/p} dt_1$$

Let $\varphi(t_1) = \left( \int \frac{\| f(\cdot + t_1, t') + f(\cdot - t_1, t') - 2f(\cdot, t') \|_p^p}{|t_1|^{1+\alpha p}} dt' \right)^{1/p}$.

Then our hypothesis is precisely that $\varphi \in L^p$. On the other hand we have

$$\iint \left| y^{2-\alpha-1/p} \left( \frac{\partial}{\partial x_j} \right)^2 f(x, y) \right|^p dx\, dy \leq$$

$$\frac{1}{2} \int \left| \left( \int \left| \frac{\partial^2 P}{\partial x_1^2}(t_1, x', y) \right| dx' \right) y^{2-1/p-\alpha} |t_1|^{1/p+\alpha} \varphi(t_1)\, dt_1 \right|^p dy$$

so it suffices to show that the integral operator

$$\int_0^\infty K(t_1, y)\, \varphi(t_1)\, dt_1$$

is bounded in $L^p$, where

$$K(t_1, y) = y^{2-1/p-\alpha}\, t_1^{1/p+\alpha} \int \left| \frac{\partial^2 P}{\partial x_1^2}(t_1, x', y) \right| dx'.$$

Now

$$\frac{\partial^2 P}{\partial x_1^2}(t_1, x', y) = C_1 \frac{y}{(t_1^2 + |x'|^2 + y^2)^{n+3/2}} + C_2 \frac{y t_1^2}{(t_1^2 + |x|^2 + y^2)^{n+5/2}}$$

so

$$\int \left| \frac{\partial^2 P}{\partial x_1^2}(t_1, x', y) \right| dx' \leq \frac{C_1 y}{(t_1^2 + y^2)^2} + \frac{C_2 y t_1^2}{(t_1^2 + y^2)^3}$$

Thus it suffices to handle the kernels

$$K_1(t_1, y) = \frac{y^{3-1/p-\alpha}\, t_1^{1/p+\alpha}}{(t_1^2 + y^2)^2}$$

and

$$K_2(t_1, y) = \frac{y^{3-1/p-\alpha}\, t_1^{2+1/p+\alpha}}{(t_1^2 + y^2)^3}$$

Both are homogeneous of degree $-1$ so we need $\int_0^\infty \frac{t_1^\alpha}{(1 + t_1^2)^2} dt_1$ and $\int \frac{t_1^{2+\alpha}}{(1 + t_1^2)^3} dt_1$ finite, which is certainly the case in the range $0 < \alpha \leq 1$ (remember we took $\alpha = \beta$, $m = 0$).

Again the case $p = \infty$ is a simple modification.

§ 3. **Remarks.**

1) The above theorems referred to the decomposition of $E_n$ as a product of $E_1$ and $E_{n-1}$. Corresponding to the decomposition $E_n = E_K \times E_{n-k}$ for any $k$, $1 \leq k \leq n-1$, we have a similar theorem taking the $\Lambda_\alpha^p$ (resp. $L_\alpha^p$) norm with respect to the $E_k$ variables and then the $L^p$ norm in the remaining variables, and doing this for sufficiently many such decompositions so that the $E_k$ variables span the entire space. This may be proved by successive applications of the above theorems.

2) These results should be contrasted with the known restriction theorems with loss of smoothness [6, 8]:

RESTRICTION THEOREM: Let $V^k$ be any $k$-dimensional affine linear subvariety of $E_n$, for $1 \leq k \leq n-1$. Suppose $\beta = \alpha - \frac{n-k}{p} > 0$. Then every function in $\Lambda_\alpha^p(E_n)$ and $L_\alpha^p(E_n)$ (provided $1 < p < \infty$) has a well defined restriction to $V^k$ which is in $\Lambda_\beta^p(V^k)$. The restriction map is in both cases continuous and onto, and there exist corresponding continuous linear extension operators from $\Lambda_\beta^p(V^k)$ to $\Lambda_\alpha^p(E_n)$ and $L_\alpha^p(E_n)$.

3) The restriction theorem has a dual statement. A distribution $T$ on $V^k$ regarded as a $k$ dimensional Euclidean space may be regarded as a distribution on $E_n$ supported on $V^k$ (but not all distribusions supported on $V^k$ arise in this way). Then $T \in \Lambda_\beta^p(V^k)^*$ if and only if $T \in \Lambda_\alpha^p(E_n)^*$ if and only if $T \in L_\alpha^p(E_n)^*$ (provided $1 < p < \infty$). (Here * denotes the dual space — See [2, 8] for a characterization of these spaces).

There is a less precise dual consequence of the Fubini-type theorems:

COROLLARY. Suppose $T$ is a distribution on $E_n$ which can be decomposed as $Tf = \int_{E_{n-1}} T_{x'} F_{x'} \, dx'$ where $F_{x'}(x_1) = f(x_1, x')$ and $T_{x'}$ is a distribution on $E_1$ for each $x' \in E_{n-1}$. If $T_{x'} \in \Lambda_\alpha^p(E_1)^*$ for almost every $x' \in E_{n-1}$ and $| T_{x'} |_{\Lambda_\alpha^p(E_1)^*}$ is in $L^{p'}(E^{n-1})$ then $T \in \Lambda_\alpha^p(E_n)^*$. Similarly for $L_\alpha^p$ provided $1 < p < \infty$.

PROOF. $|Tf| \leq \int_{E_{n-1}} \| T_{x'} \|_{\Lambda_\alpha^p(E_1)^*} \| F_{x'} \|_{\Lambda_\alpha^p(E_1)} \, dx'$. Apply Holder's inequality and the Fubini-type theorem.

4) The proof of the $\Lambda_\alpha^p$ Fubini-type theorem can be simplified in many special cases. The first half follows from the $L_\alpha^p$ Fubini-type theorem and

the $L^p_\alpha$ restriction theorem in case $1 < p < \infty$, and is immediate in case $p = \infty$. The second half is quite simple in case $\alpha < 1$. For then the second difference in the definition of $\Lambda^p_\alpha$ may be replaced by a first difference and we may always write $f(x) - f(y) = [f(x_1, \dots, x_n) - f(y_1, x_2, \dots, x_n)] + \dots$ $\dots + [f(y_1, \dots, y_{n-1}, x_n) - f(y_1, \dots, y_n)]$. No such identity seems to hold for second differences, however.

5) Most of the results of [7] for $L^p_\alpha$ spaces can now be carried over to $\Lambda^p_\alpha$ spaces, usually without the restriction $1 < p < \infty$.

## § 4. A problem of Lions.

It is a simple proposition that the $C^\infty$ functions with compact support in some fixed open set $\Omega \subset E_n$ are dense in $L^p_\alpha(E_n)$ (resp. $\Lambda^p_\alpha(E_n)$) if and only if the complement $\Omega'$ of $\Omega$ supports no non-zero distribution in $L^p_\alpha(E_n)^*$ (resp. $\Lambda^p_\alpha(E_n)^*$) [4].

If $k$ is any positive integer we may define $L^p_k(\Omega)$ to be the space of functions in $L^p(\Omega)$ which have derivatives of order $\leq k$ in $L^p(\Omega)$ (in the sense of distributions on $\Omega$) with the sum of these $L^p$ norm as the $L^p_k(\Omega)$ norm. We may ask when the $C^\infty$ functions with compact support in $\Omega$ are dense in $L^p_k(\Omega)$? Lions [4] shows under certain additional hypotheses, that the answer is the same as before, namely if and only if $\Omega'$ supports no non-zero distribution in $L^p_k(E_n)^*$. Using the Fubini-type theorem we will establish this result in general.

Let $L^p_k(E_n \mid \Omega)$ denote the space of restrictions to $\Omega$ of functions in $L^p_k(E_n)$ with the factor space norm $L^p_k(E_n \mid \Omega) = L^p_k(E_n)/\{f \in L^p_k(E_n) : f \equiv 0 \text{ on } \Omega\}$. Let $\mathring{L}^p_k(\Omega)$ denote the closure of $C^\infty_{\text{com}}(\Omega)$ in $L^p_k(E_n)$. Then we always have the continuous inclusions

$$\mathring{L}^p_k(\Omega) \subseteq L^p_k(E_n \mid \Omega) \subseteq L^p_k(\Omega).$$

THEOREM. *Let* $1 < p < \infty$, $k \geq 1$. *The following three conditions are equivalent:*

1) $\mathring{L}^p_k(\Omega) = L^p_k(E_n \mid \Omega)$

2) $\mathring{L}^p_k(\Omega) = L^p_k(\Omega)$

3) $\Omega'$ *supports no non-zero distribution in* $L^p_k(E_n)^*$.

For the proof we will need two lemmas. Let $\pi_i$ denote the projection of $E_n$ on $E_{n-1}$ given by $\pi_i(x_1, \dots, x_n) = (x_1, \dots, \widehat{x_i}, \dots, x_n)$. In what follows we identify functions which are equal almost everywhere.

LEMMA 1. Let $A$ be a closed set such that $\pi_i(A)$ has positive measure for some $i$. Then $A$ supports a positive distribution in $L_\alpha^p(E_n)^*$ for $1 < p < \infty$ and $\alpha > 1/p$, and in $\Lambda_\alpha^p(E_n)^*$ for $\alpha > 1/p$.

PROOF. Without loss of generality we may assume $A$ compact, and $i = 1$. Let $\varphi(x_2, \ldots, x_n) = \sup\{t : (t, x_2, \ldots, x_n) \in A\}$ and $\psi(x_2, \ldots, x_n) = (\varphi(x_2, \ldots, x_n), x_2, \ldots, x_n)$. These are clearly measurable. Consider the distribution $Tg = \int_{\pi_1(A)} g(\psi(x_2, \ldots, x_n))\, dx_2, \ldots, dx_n$. It is non-zero (because $m(\pi_1(A)) > 0$) and supported in $A$. For each $x' = (x_2, \ldots, x_n)$ let $T_{x'}$ be the delta distribution on $E_1$ at the point $\varphi(x')$. Then $Tg = \int_{\pi_1(A)} T_{x'}\, G_{x'}\, dx'$ where $G_{x'}(x_1) = g(x_1, x')$. Thus the lemma follows from the Corollary to the Fubini-type theorems in § 3, 3, and the well known facts that the $\delta$ distribution is in $L_\alpha^p(E_1)^*$ and $\Lambda_\alpha^p(E_1)^*$ for the given values of $\alpha$ and $p$ (see e.g. [7,8]).

LEMMA 2. Suppose $m(\pi_i(\Omega')) = 0$ for $i = 1, \ldots, n$. Then $L_k^p(\Omega) = L_k^p(E_n)$, $1 < p < \infty, k \geq 1$.

PROOF. Since $m(\Omega')=0$ we have $L_k^p(E_n \mid \Omega') = L_k^p(E_n)$. Since we know $L_k^p(E_n \mid \Omega) \subseteq L_k^p(\Omega)$ we must show the opposite containment. Thus let $f \in L_k^p(\Omega)$. We apply the criterion of the Fubini-type theorem to show $f \in L_k^p(E_n)$. Since $f \in L_k^p(\Omega)$ we have $f \in L^p(\Omega)$ and $\left(\frac{\partial}{\partial x_1}\right)^k f \in L^p(\Omega)$. Thus for almost every $x'$, $F_{x'}(x_1) = f(x_1, x_2, \ldots, x_n) \in L_k^p(\Omega_{x'})$ where $\Omega_{x'} = \{x_1 : (x_1, x') \in \Omega\}$ and $\int \| F_{x'} \|_{L_k^p}^p (\Omega_{x'})\, dx' \leq \|f\|_{L_k^p(\Omega)}^p$. But since $m(\pi_1(\Omega')) = 0$ we have $\Omega_{x'} = E_1$ for almost every $x'$. Thus $F_{x'}(x_1) \in L_k^p(E_1)$ for almost every $x'$ and

$$\int \| F_{x'} \|_{L_k^p}^p (E_1) \leq \|f\|_{L_k^p(\Omega)}^p .$$

Similar results hold replacing $x_1$ by $x_2, \ldots, x_n$. The Fubini-type theorem now applies to $f$ and completes the proof.

PROOF OF THE THEOREM. Let us show first 1) and 3) are equivalent. We note first that $\mathring{L}_k^p(\Omega)$ is closed in $L_k^p(E_n \mid \Omega)$. For taking the Sobolev norm on $L_k^p(E_n)$ we have the $L_k^p(E_n \mid \Omega)$ norm equal to the $L_k^p(E_n)$ norm for functions supported on a compact subset of $\Omega$.

Suppose $\mathring{L}_k^p(\Omega) \neq L_k^p(E^n \mid \Omega)$. Then by the Hahn-Banach theorem there exists a non-zero element in $L_k^p(E_n \mid \Omega)^*$ which annihilates $\mathring{L}_k^p(\Omega)$. But every element of $L_k^p(E_n \mid \Omega)^*$ lifts to an element of $L_k^p(E_n)^*$. Thus there is a non-zero distribution in $L_k^p(E_n)^*$ which annihilates $C_{\text{com}}^\infty(\Omega)$ hence is supported in $\Omega'$.

Conversely, suppose $\overset{\circ}{L}{}_k^p(\Omega) = L_k^p(E_n \mid \Omega)$. Then multiplication by the characteristic function of $\Omega$ is a bounded operator on $L_k^p(E_n)$. For if $f \in L_k^p(E_n)$ then $\chi_\Omega f$ restricted to $\Omega$ is in $L_k^p(E_n \mid \Omega)$ hence in $\overset{\circ}{L}{}_k^p(\Omega)$ hence $\chi_\Omega f \in L_k^p(E_n)$. It follows from the results of [7] that $\Omega'$ must have measure zero, hence $\overset{\circ}{L}{}_k^p(\Omega) = L_k^p(E_n)$. Now if $T$ is in $L_k^p(E_n)^*$ and supported in $\Omega'$ it annihilates $C_{\text{com}}^\infty(\Omega)$ hence is zero.

Since 2) implies 1) it remains to show 3) implies 2). But assuming 3) we have, by Lemma 1, that $m(\pi_i(\Omega')) = 0$ for $i = 1, \dots, n$. Lemma 2 then implies $L_k^p(\Omega) = L_k^p(E_n)$. Since 3) implies 1) we obtain $\overset{\circ}{L}{}_k^p(\Omega) = L_k^p(E_n \mid \Omega) = L_k^p(E_n) = L_k^p(\Omega)$.

NOTE ADDED IN PROOF: We have recently learned that O. V. Besov has given a different proof of the main theorem. See Proc. Stek. Inst. Math. 77 (1965) 37-48.

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