LEIF ARKERYD

A priori estimates for hypoelliptic differential equations in a half-space

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 22, n° 3 (1968), p. 409-424

<http://www.numdam.org/item?id=ASNSP_1968_3_22_3_409_0>
A PRIORI ESTIMATES FOR HYPOELLIPTIC DIFFERENTIAL EQUATIONS IN A HALF-SPACE

by LEIF ARKERYD

0. Introduction.

Our aim is to show that every distribution solution $u$ of a formally hypoelliptic partial differential equation

$$\mathcal{A}u = f \text{ in } \mathbb{R}^n_+,$$

satisfying Dirichlet's boundary conditions

$$D_{j}^l u = 0, j = 0, \ldots, l \text{ on } \mathbb{R}^{n-1},$$

does belong to $C^\infty$, if $f$ does. In analogy with the elliptic case (cf. Arkeryd [1]), it is natural to try to obtain à priori estimates

$$N_1(u) \leq C N_2(\mathcal{A}u) + N_3(u) \tag{0.1}$$

with suitable norms $N_1, N_2, N_3$, with in particular $N_3$ « weaker » than $N_1$. These estimates are proved in two steps:

1°. The inequality (0.1) is established for operators with constant coefficients.

2°. For operators

$$\mathcal{A} = A + \Sigma a_j Q_j,$$

where $A$ and $Q_j$ have constant coefficients, $Q_j$ is weaker that $A$ and $a_j \in C^\infty$, the inequality (0.1) can be obtained from the constant coefficient case 1° if

$$N_2(\mathcal{A}Q_j u) \leq C \sup |a| N_1(u) + C'N_3(u).$$
In Peetre [8] (see also Schechter [9] and Matsuzawa [7])

\[ N_2(u) = \left( \int_{\mathbb{R}^n_+} |u|^2 \, dx \right)^{1/2} \]

is considered, but then (0.1) is not true for all formally hypoelliptic operators; the second step does not always work. Here we use instead

\[ N_2(u) = \inf_{\mathbb{R}^n} \left( \int |A_-^{-1} \tilde{u}|^2 \, dx \right)^{1/2}, \]

if \( A = A_+ \cdot A_- \) is the « canonical » decomposition of \( A \), with \( \inf \) taken over all \( \tilde{u} \in \mathcal{S}'(\mathbb{R}^n) \), satisfying \( \tilde{u} = u \) in \( \mathcal{H}^n_+ \). In the same way we take

\[ N_4(u) = \inf_{\mathbb{R}^n} \left( \int |A_+ \tilde{u}|^2 \, dx \right)^{1/2}. \]

Then step 1° is immediate (cf. [8], [11]) and the main difficulty is to prove 2°. This can be done by use of a commutator lemma analogous to Friedrich's lemma, which follows from the basic estimate

\[ \left| \frac{\partial A_-}{\partial \xi_+} \right| \leq C |A_-| |\xi'|^{-s}, \xi' \in \mathbb{R}^n, |\xi'| \geq M. \]

Let us mention that Hörmander [4] has proved a regularity theorem for operators with constant coefficients and general boundary conditions. He does not, however, use à priori estimates, but explicit formulas for the corresponding Green and Poisson kernels.

The plan of the paper is as follows. Section 1 contains some preliminaries concerning the distribution spaces involved. Section 2 contains the proof of the basic estimate of the Friedrich's type mentioned above. In Section 3 and Section 4 the applications to differential equations are given. Since they are rather routine, we have cut down the exposition to a minimum.

1. Spaces \( H^+_{\mathcal{H}_+}, s \) and \( H^+_{\mathcal{H}_-}, s \).

The Fourier transform of an element \( f \) in one of the Schwartz classes \( \mathcal{S} \) or \( \mathcal{S}' \) (see [10]) is denoted by \( \mathcal{F}f \), the inverse transform by \( \mathcal{F}^{-1} \), \( \mathcal{F} \mathcal{F}f = f \).
We take formally
\[ Ff(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x) \, dx \]
and use the notation
\[ P(D) f = \mathcal{F}Pf, \]
where P is a function on \( \mathbb{R}^n \). The following functions will often be used:
\[ P(\xi) = A(\xi) = \xi_1 + i \left(1 + \sum_{j=2}^{n} \xi_j^2\right)^{1/2}, \]
\[ P'(\xi) = A_1(\xi) = \left(1 + \sum_{j=2}^{n} \xi_j^2\right)^{1/2}. \]
By
\[ A = A(D) = A(D_1, D') = \sum \sigma_a D^a, \quad D^a = (i)^{-r} \frac{\partial}{\partial x_{a_1}} \cdots \frac{\partial}{\partial x_{a_r}} \]
we denote a hypoelliptic differential operator with constant coefficients and write
\begin{equation}
A(\xi, \xi') = a \Pi \left( \xi_1 - \sigma^+ (\xi') \right) \Pi \left( \xi_1 - \sigma^- (\xi') \right) = a A_+ A_- \tag{1.1}
\end{equation}
with \( a = a_1 \cdots 1 \). Here \( m_+ \) is the number of roots \( \sigma^+ \) with positive and \( m_- \) the number of roots \( \sigma^- \) with negative imaginary part. We require, that \( A \) satisfies the root condition, i.e., that \( m_+ \) and \( m_- \) are independent of \( \xi' \) for \( |\xi'| \geq M \). It is no restriction to take \( a = 1 \). We set
\begin{equation}
B(\pm) = \begin{cases} 
A \pm (\xi) & \text{if } |\xi'| \geq M, \\
(\xi_1 + \mp i)^m \pm & \text{if } |\xi'| < M. 
\end{cases} \tag{1.2}
\end{equation}
where the value of \( M \geq M \) will be defined in Section 2. The following norms are used:
\[ \| u \| = \left( \int_{\mathbb{R}^n} |u(x)|^2 \, dx \right)^{1/2}, \quad \| u \|_P = \| P(D) u \|, \quad \| u \|^{*} = \inf \| \tilde{u} \|_P, \]
where \( \inf \) is taken over all \( \tilde{u} \in S' \), whose restrictions to
\[ \mathbb{R}^n_+ = \{ x : x_1 > 0 \} \]
are equal to \( u \), and such that

\[
P(D)\tilde{u} \in L^2.
\]

The notation \( \tilde{u} \) is used below in this sense. Particular norms of this type are

\[
\| u \|_{H^+_{\pm \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \overline{B}_\pm} \equiv \| u \|_{A^+_{\pm \cdot B_\pm}}
\]

\[
\| u \|_{H^{-1}_{\pm \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \overline{B}_\pm^{-1}} \equiv \| u \|_{A^{-1}_{\pm \cdot \overline{B}_\pm^{-1}}}
\]

\[
\| u \|_{r \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \overline{B}_\pm} \equiv \| u \|_{A^+_{\cdot \cdot \cdot \cdot \overline{B}_\pm}}
\]

The corresponding spaces are denoted by

\[
H^+_{\pm \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \overline{B}_\pm}, H^{-1}_{\pm \cdot \cdot \cdot \overline{B}_\pm^{-1}} \text{ and } H_\cdot \cdot \cdot \cdot \cdot \overline{B}_\pm
\]

The space corresponding to \( \| \cdot \|_p \) is denoted by \( H_p \). Paley-Wiener's theorem gives

\[
\| u \|_{\overline{R}_+^{\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \overline{B}_\pm}} \leq \left( \int A^2_{\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \overline{R}_+\cdot (\| F_{\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \overline{B}_\pm^{-1}} (\xi') \|_{\overline{R}_+^{-1}} (\xi')^2 d\xi' \right)^{1/2},
\]

\[
\| u \|_{\overline{R}^{-1}_{\cdot \cdot \cdot \cdot \cdot \overline{B}_\pm^{-1}}} \leq \left( \int A^2_{\cdot \cdot \cdot \cdot \cdot \overline{R}_+\cdot (\| F_{\cdot \cdot \cdot \cdot \cdot \overline{B}_\pm^{-1}} (\xi') \|_{\overline{R}_+}^{-1} (\xi')^2 d\xi' \right)^{1/2}.
\]

The local spaces (cf. 2.5 in [4])

\[
(H^+_{\pm \cdot \cdot \cdot \overline{B}_\pm}, \overline{H}^{-1}_{\pm \cdot \cdot \cdot \overline{B}_\pm^{-1}} \text{ and } (H_\cdot \cdot \cdot \overline{B}_\pm)^{\text{loc}}
\]

correspond to the above spaces. About \( H^+_p \), we need the following fact, which goes back to Hörmander and Lions [6].

**Lemma 1.1.** Let \( c \in C^\infty_0 (\overline{R}_+^p) \). Then

\[
\| cf \|_{\overline{R}_+}^{\cdot \cdot \cdot \cdot \cdot \overline{B}_\pm} \leq \sup | c | \| v \|_{\overline{R}_+}^{\cdot \cdot \cdot \overline{B}_\cdot \cdot \cdot \cdot \overline{B}_\pm} + K \| v \|_{\overline{R}_+}^{\cdot \cdot \overline{B}_\cdot \cdot \cdot \cdot \overline{B}_\pm}
\]

for all \( v \in H^+_\cdot \cdot \cdot \cdot \overline{B}_\pm \), and with the constant \( K \) independent of \( v \).

Next we state some lemmas in \( H^+_\cdot \cdot \cdot \overline{B}_\pm \).
**Lemma 1.2.** $C^\infty_0$ is dense in $H^{A_1B_+}$.  

**Proof:** We prove in Section 2, that \( (2.4) \)
\[
|B_-(\xi) - B_-(\xi + \eta)| \leq C(1 + |\eta|^{1/2} |B_-(\xi)|),
\]
for all $\xi, \eta \in R^n$. Here and below constants are written $C$ and $K$, sometimes with index. As the same inequality holds for $B_+$, it follows that
\[
|B_+(\xi + \eta)| \leq C'(1 + |\eta|^{1/2} |B_+(\xi)|),
\]
and consequently
\[
|A_1^\prime(\xi + \eta) B_+(\xi + \eta)| \leq C'(1 + |\eta|^{1/2 + 1/2} |A_1^\prime(\xi') B_+(\xi)|).
\]

But from this inequality follows that $C^\infty_0$ is dense in $H^{A_1B_+}$. See [5] Remark p. 36 and Theorem 2.2.1).

We now use Lemma 1.2 to approximate elements of $H^{A_1B_+}$ with support in a half-space.

**Lemma 1.3.** Let $u \in H^{A_1B_+}$, $\text{supp } u \subseteq \overline{R^n_+}$. Then $u$ is the limit in $H^{A_1B_+}$ of a sequence $(u_j)_{j=1}^\infty$ of functions

\[
u_j \in C^\infty_0(R^n), \text{ supp } u_j \subseteq R^n_+.
\]

**Proof.** Denote by $\tau_h$ translation by $h$ along the $x_1$-axis. Then
\[
\|\tau_h u - u\|_{A_1B_+}^{A_1B_+} = \left( \int |A_1^\prime B_+|^2 |e^{ib_i} - 1|^2 |Fu|^2 d\xi \right)^{1/2} \leq \leq 2 \left( \int_{R^n_+} |B_+ A_1^\prime Fu|^2 d\xi \right)^{1/2} + \sup_{x_1} |e^{ib_i}| 1 \|u\|_{A_1B_+}.
\]

which can be made arbitrarily small by a suitable choice of $\xi$ and $h$. As the statement of the lemma is already established implicitly for $\tau_h u$ by Lemma 1.2, this ends the proof.

**Remark.** In Lemma 1.3, $B_+$ can be replaced by $B_-^{-1}$ and $R^n_+$ by $\overline{R^n_+} = \{x : x_1 \leq 0\}$.  

**differential equations in a half-space**  

413
By definition, that a function \( u \in H^+_{\mu_+} \) has the boundary values \( B^+ \), means, that there is a \( \tilde{u} \in H^+_{\mu_+} \) with

\[
\tilde{u} = 0 \quad \text{for} \quad x_1 < 0, \quad \tilde{u} = u \quad \text{for} \quad x_1 > 0.
\]

Finally we need

**Lemma 1.4.** A function \( u \) satisfying (1.5) is in \( R \) if and only if it is in \( H^+_{\mu_+} \) and

\[
\frac{u(x_1, x' + h') - u(x_1, x')}{|h'|}
\]

is bounded in \( H^+_{\mu_+} \) independently of \( h' = (h_2, \ldots, h_n) \).

**Proof:** The proof is immediate if we notice that with

\[
\tilde{u} = u \quad \text{for} \quad x_1 > 0, \quad \tilde{u} = 0 \quad \text{for} \quad x_1 < 0,
\]

there is a characterization of \( B^+_1 \tilde{u} \) in \( H^+_{\mu_1} \) by the same kind of difference quotients.

**2. A version of Friedrich's lemma.**

The derivation of the à priori inequality mentioned in Section 0, is for \( m_+ > 0 \) based on a commutator lemma analogous to Friedrich's lemma (see e.g. [2]), which is established in this section. The proof depends on a number of lemmas, for which we need the following estimates of hypoelliptic polynomials:

\[
|A^u(\xi)/A(\xi)| \leq C_1 |\xi|^{-e - s} \quad \text{if} \quad |\xi| \geq M,
\]

(2.1) \[
\left| \frac{\partial A(\xi)}{\partial \xi^r} \right| A(\xi) \leq C |\xi|^{-e} \quad \text{if} \quad |\text{Im} \xi_1| \leq C' |\xi'|, \quad \xi' \in R^{n-1}, \quad |\xi'| \geq M,
\]

\[
|\text{Im} \xi'(\xi')| > C' |\xi'| \quad \text{if} \quad |\xi'| \geq M
\]
for some $c > 0$ and with $A^a(\xi) = D^a A(\xi)$ (see Hörmander [5]).

**Lemma 2.1.** If $\xi$ belongs to the cylinder $|\xi'| \geq M$, then for all $\nu$

$$\left| \frac{\partial A_-(\xi)}{\partial \xi^\nu} A_-(\xi) \right| \leq K |\xi'|^{-b}.$$  

Here $C$ is independent of $\xi$ and $c^2 > b > 0$.

**Proof.** As the coefficients of $A_-(\xi)$ are analytic in $|\xi'| \geq M$ (see [3] p. 289-290), the derivatives $\frac{\partial A_-(\xi)}{\partial \xi^\nu}$ exist. Cauchy’s formula gives

$$\frac{\partial A_-(\xi)}{\partial \xi^\nu} A_-(\xi) = \frac{1}{2\pi i} \int_{-\infty+it}^{\infty+it} \frac{\partial}{\partial \xi^\nu} A(\xi - \tau, \xi') \frac{d\tau}{A(\xi - \tau, \xi') \tau} \min_{1 \leq j \leq m} \text{Im} e^{ij\tau} > \varepsilon > 0.$$  

Take $q$ and $p$ such that

$$qc > 1, \quad \frac{1}{q} + \frac{1}{p} = 1.$$  

Then by (2.1) and with $\varepsilon = C' |\xi'|^r$ we obtain

$$\left| \frac{\partial A_-(\xi)}{\partial \xi^\nu} A_-(\xi) \right| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{C d\sigma}{(|\xi_1 - \sigma| + |\xi'|^r)^{\frac{q}{r}}} \left( \int_{-\infty}^{\infty} \frac{d\sigma}{\sigma + iC' |\xi'|^p} \right)^{\frac{1}{p}} \leq$$

$$\leq C \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} \frac{d\sigma}{|\xi_1 - \sigma| + |\xi'|^r} \right)^{\frac{q}{r}} \left( \int_{-\infty}^{\infty} \frac{d\sigma}{\sigma + iC' |\xi'|^p} \right)^{\frac{1}{p}} \leq$$

$$\leq K |\xi'|^{-b}.$$  

The next lemma compares $A_-(\xi)$ with $A_-(\xi + \eta)$ for small real $\eta$. For technical reasons, we only make that comparison in a cylinder $|\xi'| \geq M_1 \geq 2M$, with $M_1$ so large that

$$|\xi'| \leq 2 |\xi' + \eta'| \leq 4 |\xi'| \quad \text{if} \quad |\eta| \leq |\xi'|^b.$$  

This is the constant $M_1$ mentioned in formula (1.2).
LEMMA 2.2. Take $\xi$ with $|\xi'| \geq M_1$ and $|\eta| \leq |\xi'|^b$. Then

\begin{equation}
|A_-(\xi + \eta)| \leq K' |A_-(\xi)|, \tag{2.2}
\end{equation}

\begin{equation}
|A_-(\xi + \eta) - A_-(\xi)| \leq C |\eta| |\xi'|^{-b} A_-(\xi), \tag{2.3}
\end{equation}

which $K'$ and $C$ independent of $\xi$ and $\eta$.

**Proof.** We write

$$
\log \frac{A_-(\xi + \eta)}{A_-(\xi)} = \int_0^1 \sum_{j=1}^n \eta_j A_j^{-1}(\xi + t\eta) \frac{\partial}{\partial \xi_j} A_-(\xi + t\eta) dt.
$$

The integrand can be estimated by Lemma 2.1. The restrictions on $\eta$ and $M_1$ then give

$$
\left| \int_0^1 \sum_{j=1}^n \eta_j A_j^{-1}(\xi + t\eta) \frac{\partial}{\partial \xi_j} A_-(\xi + t\eta) dt \right| \leq \frac{1}{|\xi'|^b} \int_0^1 |\xi'|^b |\xi' + t\eta'|^b dt \leq K2^b,
$$

and so

$$
|A_-(\xi + \eta)| \leq |A_-(\xi)| e^{K\eta^2} = K' |A_-(\xi)|.
$$

The inequality (2.3) follows from

$$
|A_-(\xi + \eta) - A_-(\xi)| = \left| \int_0^1 \sum_{j=1}^n \eta_j \frac{\partial}{\partial \xi_j} A_-(\xi + t\eta) dt \right| \leq \frac{1}{|\xi'|^b} \int_0^1 |A_-(\xi + t\eta)| |\xi + t\eta|^b dt \leq \frac{1}{|\xi'|^b} \int_0^1 |A_-(\xi)| |\xi'|^b dt = \frac{1}{|\xi'|^b} |A_-(\xi)|.
$$

The estimate that corresponds to (2.3) for $|\eta| > |\xi'|^b$, is much more easily obtained.
LEMMA 2.3. If $|\xi'| \geq M_1$, $|\xi' + \eta'| \geq M_1$ and $|\eta| \geq |\xi'|$, then

$$|A_-(\xi) - A_-(\xi + \eta)| \leq C |\eta|^a |\xi'|^{-\epsilon} |A_-(\xi)|,$$

where $d$ is independent of $\xi$ and $\eta$.

PROOF. Under the given restrictions on $\xi$ and $\eta$, for some $a > 0$ the following inequalities hold:

$$|\varphi_j(\xi')| \leq |\xi'|^a \leq |\eta|^{a/k},$$

$$|\varphi_j(\xi' + \eta')| \leq |\xi' + \eta'|^a \leq C_a |\eta|^{a/k},$$

$$|\text{Im} \varphi_j(\xi')| \geq C |\xi'|^k.$$

Hence

$$|\varphi_j(\xi') - \varphi_j(\xi' + \eta')| \leq C \frac{(1 + |\eta|)^{a/b}}{|\xi'|^c},$$

which gives the desired estimate, when inserted into

$$A_-(\xi) - A_-(\xi + \eta) = \frac{\Pi (\xi_1 - \varphi_j(\xi')) - \Pi (\xi_1 + \eta_1 - \varphi_j(\xi' + \eta'))}{\Pi (\xi_1 - \varphi_j(\xi'))} =$$

$$= \sum_{j=1}^{m-} \frac{\eta_1 - \varphi_j(\xi' + \eta')}{\xi_1 - \varphi_j(\xi')} \frac{\Pi (\xi_1 + \eta_1 - \varphi_j(\xi' + \eta'))}{\xi_1 - \varphi_j(\xi')} .$$

Recalling from (1.2) that

$$B_-(\xi) = \begin{cases} A_-(\xi), & |\xi'| \geq M_1, \\ (\xi + \eta)^{n-}, & |\xi'| < M_1, \end{cases}$$

and using Lemmas 2.2 and 2.3, the main step in the proof of our commutator lemma easily follows.

LEMMA 2.4. There are constants $k$ and $C$ independent of $\xi, \eta \in \mathbb{R}^n$, such that

$$|B_-(\xi) - B_-(\xi + \eta)| \leq C \frac{(1 + |\eta|^{a/k})}{(1 + |\xi' + \eta'|^{a/b})}. \tag{2.4}$$

PROOF. The points $\xi$ and $\xi + \eta$ can be situated inside or outside the cylinder $|\xi'| = M_1$. This gives four cases, which are treated separately.
1°. By Lemmas 2.2 and 2.3, the inequality (2.4) is fulfilled for \(|\xi'| \geq M_1\) and \(|\xi' + \eta'| \geq M_1\).

2°. For \(|\xi'| \geq M_1\) and \(|\xi' + \eta'| < M_1\), write the left-hand side of (2.4) as in the proof of Lemma 2.3.

\[
\frac{B_-(\xi) - B_-(\xi + \eta)}{B_-(\xi)} = \sum_{j=1}^{m} \frac{-\eta_j - \psi_j(\xi') - \frac{i}{\xi_j - \psi_j(\xi')}}{\xi_j - \psi_j(\xi')}.
\]

Each factor can be estimated by

\[
C' \frac{(1 + |\eta|^3)^k'}{(1 + |\xi' + \eta'|^3)^{k'}}
\]

for some \(k'\) and \(C'\), which obviously implies (2.4).

3°. The case \(|\xi'| < M_1, |\xi' + \eta'| \geq M_1\) is treated analogously.

4°. If \(|\xi'| < M_1\) and \(|\xi' + \eta'| < M_1\), the inequality is well-known.

When \(Q\) is weaker than \(A\), we have

\[
\| B^{-1} A_1^* a Q u \| \leq \left\| B^{-1} A_1^* (aB_- - B_- a) \frac{Q}{B} B_+ u \right\| + \left\| A_1^* a \frac{Q}{B} B_+ u \right\|.
\]

Because \(A\) is hypoelliptic, we have

\[
\left| \frac{Q(\xi)}{B(\xi)} \right| \leq C \text{ for } \xi \in R^u
\]

(see [5] p. 102). Then the first term on the right side can be estimated by Lemma 2.4 as follows;

\[
\left| B^{-1} A_1^* (aB_- - B_- a) \frac{Q}{B} B_+ u \right| =
\]

\[
= \left\| B^{-1}(\xi) A_1(\xi') \int F a(\eta) (B_- (\xi - \eta) - B_-(\xi)) \frac{Q(\xi - \eta)}{B(\xi - \eta)} B_+(\xi - \eta) F u(\xi - \eta) d\eta \right\|
\]

\[
\leq C \left\| A_1(\xi') \int F a(\eta) A^k(\eta) A_1^*(\xi') B_+(\xi - \eta) F u(\xi - \eta) d\eta \right\|
\]

\[
\leq C \left\| \int F a(\eta) A^k(\eta) A_1^1(\xi'-\eta') A_1^{* - b}(\xi' - \eta') B_+(\xi - \eta) F u(\xi - \eta) d\eta \right\|
\]

\[
\leq C \int |F a A^k A_1^1 d\eta| || A_1^{* - b} B_+ u ||.
\]
This estimate, together with the inequality
$$\| u \|_{H^{s}, \tau^{-b}} \leq \varepsilon \| u \|_{H^{s}, \tau^{-1}} + C_{\varepsilon} \| u \|_{H^{s}, \tau^{-1-1}}$$
of Ehrling-Nirenberg type, gives

**Theorem 2.1.** Let $a \in C^\infty_c (R^n)$, Then for $\varepsilon > 0$,
$$\| (a B_1 - B_1 a) Q B_1^{-1} u \|_{H^{s}, \tau^{-1}} \leq \varepsilon \| u \|_{H^{s}, \tau^{-1}} + C_{\varepsilon} \| u \|_{H^{s}, \tau^{-1-1}},$$
with $C_{\varepsilon}$ independent of $u \in H^{s, \tau^{-1}}$.

### 3. A priori inequalities for hypoelliptic operators.

**Theorem 3.1.** Let
$$s (x, D) = A (D) + \sum_{j=1}^{m} a_j (x) Q_j (D),$$
where $A (D)$ is hypoelliptic and $Q_1 (D), \ldots, Q_m (D)$ are weaker than $A (D)$. If $a_1 (x), \ldots, a_m (x) \in C^\infty_c (R^n)$ and
$$\sum_j \sup_{x} \| a_j (x) \| < \varepsilon$$
for some sufficiently small $\varepsilon > 0$, then

$$\| u \|_{H^{s}, \tau^{-1}} \leq C (\| s u \|_{H^{s}, \tau^{-1}} + \| u \|_{H^{s}, \tau^{-1-1}})$$

for all $u \in H^{s, \tau^{-1}}$, satisfying the boundary conditions (1.5).

**Proof.** We prove the theorem for $m_{-} > 0$. The modifications in the simpler case $m_{-} = 0$ are obvious. As Lemma 1.3 shows, it is sufficient to prove the theorem for $u \in C^\infty_c (R^n)$. According to a theorem by Peetre ([8], Lemma 4)

$$\| F_x u (\cdot, \xi^0) \|_{H^{s} (\cdot, \xi^0)} \leq \| A (\cdot, \xi^0) \| F_x u (\cdot, \xi^0) \|_{H^{s} (\cdot, \xi^0)}$$

if $|\xi'| \geq M$ and if $u \in C^\infty_0 (R^n)$. The proof is based on the Paley-Wiener theorem. We multiply (3.2) by $A_1^s$ and integrate in $\xi'$ (cf. (1.3), (1.4)) getting

$$
\| u \|_{B^{+,s}_{1,-1}}^+ \leq \| A u \|_{B^{+,s}_{1,-1}}^+ + \| 1 + M_1^s \| u \|_{B^{+,s-1}_{1,-1}}^+ .
$$

It follows that

$$
(3.3) \quad \| u \|_{B^{+,s}_{1,-1}}^+ \leq \left( \left( A + \sum_1^m a_j Q_j \right) u \right)_{B^{+,s-1}_{1,-1}}^+ + \sum_1^m \| a_j Q_j u \|_{B^{+,s-1}_{1,-1}}^+ + C \| u \|_{B^{+,s-1}_{1,-1}}^+ .
$$

But

$$
\| a_j Q_j u \|_{B^{+,s-1}_{1,-1}}^+ \leq \| B_- a_j Q_j B_-^{-1} u \|_{B^{+,s-1}_{1,-1}}^+ + \| (a_j B_- - B_- a_j) Q_j B_-^{-1} u \|_{B^{+,s-1}_{1,-1}}^+ .
$$

The last term can be estimated by use of Theorem 2.1, and in view of Lemma 1.1, we can estimate the first term in the following way;

$$
\sup a_j \| Q_j B_-^{-1} u \|_{B^{+,s-1}_{1,-1}}^+ + K \| Q_j B_-^{-1} u \|_{B^{+,s-1}_{1,-1}}^+ \leq C_j (\sup a_j \| u \|_{B^{+,s-1}_{1,-1}}^+ + K \| u \|_{B^{+,s-1}_{1,-1}}^+ ).
$$

Here $C_j$ is independent of $u$ and $a_j$. We have now proved that

$$
\| a_j Q_j u \|_{B^{+,s-1}_{1,-1}}^+ \leq (C_j \sup_1 a_j | | + \epsilon) \| u \|_{B^{+,s-1}_{1,-1}}^+ + C \| u \|_{B^{+,s-1}_{1,-1}}^+ ,
$$

which together with (3.3) gives the desired estimate (3.1), if we assume for instance, that $\epsilon > \sum_{j=1}^m C_j \sup_1 a_j | | < 1/2$.

4. Regularity.

In this section

$$
\mathcal{A} = A + \sum a_j Q_j
$$

is formally hypoelliptic. Before the main regularity theorem we formulate a result on regularity in the $x'$-directions.
THEOREM 4.1. Let \( u \in H_{\mathbb{R}^+}^+ \), for some \( r \) and let \( u \) satisfy (1.5). Define \( \mathcal{A}(x,D) \) as in Theorem 3.1. Then

\[
\mathcal{A} \in H_{\mathbb{R}^+}^+.
\]

PROOF. It is always possible to choose \( r \), so that \( r = s - v \) for some integer \( v \). If \( r \leq s - 1 \) then the quotient

\[
\frac{u(x_1, x' + h) - u(x_1, x')}{|h|}
\]

is bounded in \( H_{\mathbb{R}^+}^+ \), by Theorem 3.1. Then by Lemma 1.4, \( u \in H_{\mathbb{R}^+}^+ \). By iteration, this proves the theorem.

THEOREM 4.2. Let \( u \in D'(\mathbb{R}^+_-) \) and satisfy (1.5) Then

\[
u \in (H_{\mathbb{R}^+}^+)_{\text{loc}} \quad \text{if} \quad \mathcal{A}u \in (H_{\mathbb{R}^+}^+)_{\text{loc}}.
\]

PROOF. The theorem means that \( \psi u \in H_{\mathbb{R}^+}^+ \) if \( \psi \in C_0^\infty (\mathbb{R}^+_-) \). It is no restriction to take all \( Q_j \) hypoelliptic and \( \psi \) with « small » support. For each such function \( \psi \), we take another \( \Phi \) of the same type with \( \Phi = 1 \) in a neighbourhood of \( \text{supp} \psi \). We first show that \( \Phi \) is hypoelliptic when \( \text{supp} \Phi \) is small enough for \( \mathcal{A} \) to fulfil the conditions of Theorem 3.1 in some open set \( \omega \supset \text{supp} \Phi \). From the fact that

\[
|B_-| \leq K A_{m_0}^{-m_0} A_{1}^{m_0}
\]

for some \( m_0 \), it follows

\[
\mathcal{A}u \in (H_{\mathbb{R}^+}^+)_{\text{loc}}
\]

if

\[
\mathcal{A}u \in (H_{\mathbb{R}^+}^+)_{\text{loc}}.
\]

As the \( Q_j \)'s are hypoelliptic, there is a \( d > 0 \) such that for large \( \xi' \)

\[
|Q_j/Q_j| \leq |\xi'|^{-d}|a|, \quad |A^a|/A \leq |\xi'|^{-d}|a|.
\]

Take

\[
\Phi_0 \in C_0^\infty (\mathbb{R}^+_-)
\]
with \( \Phi_0 = 1 \) in a neighbourhood of \( \text{supp } \Phi \) and \( \text{supp } \Phi_0 \subset \omega \). As \( u \in D' \), we have

\[
\Phi_0 u \in H^+_{\sigma + 1, \tau}
\]

for some integer \( \sigma \) and real \( \tau \). If \( \sigma < m_+ \), we construct a sequence of \( C_0^\infty (\mathbb{R}^n) \) functions

\[
\Phi_0, \Phi_1, \ldots, \Phi_\mu = \Phi, \mu = m_+ - \sigma
\]

with \( \Phi_{j-1} = 1 \) in a neighbourhood of \( \text{supp } \Phi_j \). Let \( m_j = m_+ + m_- \) be the order of the derivative \( D_1 \) in \( \mathcal{A} \) and \( m' \) the total order. As

\[
(D^\alpha \Phi_1) (A^n + \sum a_j Q_j) \Phi_0 u \in H^+_{\sigma - m_1 + 1, \tau - m'_j + 1} \text{ when } \alpha = 0
\]

and

\[
\Phi_1 (A + \sum a_j Q_j) u \in H^+_{\sigma - m_1, \tau - m_0},
\]

Leibniz' formula shows that

\[
\Phi_1 (A + \sum a_j Q_j) u + \sum \left| a \right| \neq 0 \Phi_1 (A^n + \sum a_j Q_j) \Phi_0 u =
\]

\[
= (A + \sum a_j Q_j) \Phi_1 u \in H^+_{\sigma - m_1 + 1, \min (\tau + 1, \sigma - m_0)}.
\]

Then by partial regularity (see e.g. [5]), for some \( \tau' \)

\[
\Phi_1 u \in H^+_{\sigma + 1, \tau'}
\]

and so, by iteration, for some \( \tau' \)

\[
\Phi u \in H^+_{m_1, \tau'}.
\]

For some \( \tau' \) this will give

\[
(4.1)
\]

\[
\Phi u \in H^+_{m_1, \tau'}. \tag{4.1}
\]

Take \( \tau \) so that with \( q = c/b \)

\[
\nu = \frac{(s - \tau) q}{d}
\]

is an integer. Let \( (\psi_j)_\nu \) be a sequence analogous to \( (\Phi_j)_\nu \), and with

\[
\psi_0 = \Phi, \psi_\nu = \psi.
\]
The terms in

\[(A + \sum a_j Q_j) \psi_1 u = \psi_1 (A + \sum a_j Q_j) u + \sum_{|a| \neq 0} D^a \psi_1 (A^a + a_j Q_j^a) \psi_0 u\]

can be estimated as in the proof of Theorem 2.1 and Theorem 3.1. With our choice of \(d\) this gives

\[\| a_j D^a \psi_1 Q_j^a \psi_0 u \|_{B^{-1}, r+\frac{d}{q}} \leq K \| \psi_0 u \|_{B^1, r}, \]

and so

\[\| D^a \psi_1 A^a \psi_0 u \|_{B^{-1}, r+\frac{d}{q}} \leq K \| \psi_0 u \|_{B^1, r}, \]

and so

\[(A + \sum a_j Q_j) \psi_1 u \in H^{+}_{B^{-1}, r+\frac{d}{q}}.\]

Then by Theorem 4.1

\[\psi_1 u \in H^{+}_{B^1, r}.\]

Repeating this \(v\) times gives

\[\Phi u \in H^{+}_{B^1, r},\]

and so

\[u \in (H^{+}_{B^1, r})^{\text{loc}}.\]

**Corr. 4.1.** If \(u \in C^{\infty}(R^n_+)\) and \(u \in D' (R^n_+)\) satisfies (1.5), then \(u \in C^{\infty}(R^n_+).\)

**Proof.** This follows by partial regularity from Theorem 4.2.
REFERENCES