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DIFFERENTIABLE DISTRIBUTION SEMI-GROUPS

by VIOREL BARBU

Introduction.

Distribution semi-groups of operators in a Banach space were introduced and studied by Lions [1] (cf. also Foiaş [2], Yoshinaga [10], [11], Peetre [3]). J. L. Lions has obtained the characterization of the infinitesimal generator of an exponential distribution semi-group and recently his result has been generalized by Chazarain [4], [5] (cf. also Foiaş [2], Larsson [9]), for regular distribution and hyper-distribution semi-groups. In their works, Da Prato-Mosco [6], [7] and Fujiwara [8] have generalized the notion of holomorphic semi-group (cf. Yosida [14]) to that of holomorphic distribution semi-groups and have given a characterization of the infinitesimal generator of such a distribution semi-group.

In this paper we extend some of their results for differentiable distribution semi-groups.

§. 1. General results on distribution semi-groups.

We use the notations and the terminologies of L. Schwartz [12], [13] for infinitely differentiable functions and for distributions. We set $R = \{t\,;\, -\infty < t < \infty\}$ and denote: $\mathcal{D}$ the space of all infinitely differentiable functions with compact suppport in R, $\mathcal{E}$ the space of infinitely differentiable function on R; $\mathcal{D}^+$ the space of all $\varphi \in \mathcal{D}$ such that $\operatorname{supp} \varphi \subset [0, \infty)$ topologized as in Schwartz [12]; $\mathcal{S}$ the space of rapidly decreasing $\mathcal{E}$ functions and $\mathcal{E}'$ the space of scalar distributions with compact support. We denote also by $\mathcal{D}_-$ the strict inductive limit of the spaces $\mathcal{E}_a = \{\varphi \in \varepsilon\,;\, \operatorname{supp} \varphi(-\infty, a]\}$. Let X be a Banach space and $L(X, X)$ the space of all continuous linear

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operator on X topologized with the operator norm. We denote also by $\mathcal{D}'(L(X, X))$, $\mathcal{D}'_+(L(X, X)$ and $\mathcal{S}'(L(X, X))$ the vector-valued distribution spaces : $L(\mathcal{D}, L(X, X)$, $L(\mathcal{D}_-, L(X, X))$ and $L(\mathcal{S}, L(X, X))$ respectively.

A vector-valued distribution $T \in \mathcal{D}'_+(L(X, X))$ is called a distribution semi-group (D. S. G. in short) if it satisfies the following conditions :

i) $T(\varphi * \psi) = T(\varphi)\, T(\psi)$ for any $\varphi, \psi \in \mathcal{D}^+$.

ii) The support of T is contained in $[0, \infty)$.

iii) The linear subspace $[T(\mathcal{D}^+)X]$ generated by $T(\mathcal{D}^+)X$ is dense in X.

iv) If $x \in X$ and $T(\varphi)x = 0$ for any $\varphi \in \mathcal{D}^+$, then $x = 0$.

Let $R_+\{t\,;\, t > 0\}$ and $\bar{R}_+\{t\,;\, t \geq 0\}$. If $\mu \in \mathcal{E}'(\bar{R}_+)$ then we de define a closable and densely defined operator $T(\mu)$ on $[T(\mathcal{D}^+)X]$, by the formula

$$(1.1) \qquad T(\mu)\, x = \sum_{i=1}^{n} T(\varphi_i^* \mu)\, x_i, \quad \text{for} \quad x = \sum_{i=1}^{n} T(\varphi_i)\, x_i\,;\ x_i \in X,\ \varphi_i \in \mathcal{D}^+.$$

Let us denote the closure of $T(\mu)$ again by $T(\mu)$. The linear operator $A = T(-D\,\delta_0)$ is called the infinitesimal generator of T. Here δ_t is the Dirac measure concentrated at $\mathcal{T} = t$ and D is the derivation symbol.

For any $\varphi(t)$ defined on R we denote by $\varphi_+(t)$ the function

$$\varphi_+(t) = \begin{cases} \varphi(t) & \text{for} \quad t \geq 0, \\ 0 & \text{for} \quad t < 0. \end{cases}$$

We say that a D. S. G. T is regular if $T(\varphi_+) = T(\varphi)$ for any $\varphi \in \mathcal{D}_-$. A regular D. S. G., T is called of exponential growth (E. D. S. G. in short) if there exists a number α such that $e^{-\alpha t}\, T \in \mathcal{S}'(L(X, X))$.

THEOREM 1 (Lions). A closed linear operator A in X with domain $D(A)$ dense in X, generates an E. D. S. G. if and only if there is a number $\alpha \geq 0$ such that

i) for any λ with $Re\,\lambda > \alpha$, $\lambda I + A$ defines an isomorphism of $D(A)$ onto X.

ii) $\|(\lambda I - A)^{-1}\| \leq \text{pol}(|\lambda|)$ for $\text{Re}\,\lambda > \alpha$ where pol $(|\lambda|)$ denote a polynomial with non-negative coefficients.

For the proof see [1] and [10]. The following theorem is due to Chazarain [4] (cf. also Foiaş [2]).

THEOREM 2. Let A be a closed and dense operator on X. Then A is the infinitesimal generator of a regular S. G. D. if and only if the following conditions hold :

i) There exist the constants α, β, γ; $\alpha, \gamma \geq 0$ such that $(\lambda I - A)^{-1} \in L(X, X)$ fon any λ in the domain

$$\Lambda = \{\lambda \in C;\ \operatorname{Re}\lambda \geq \alpha \log |\operatorname{Im}\lambda| + \beta;\ \operatorname{Re}\lambda \geq \gamma\} \tag{1.2}$$

ii) $\| (\lambda I - A)^{-1} \|_{L(X,X)} \leq \operatorname{pol}(|\lambda|)$, for $\lambda \in \Lambda$.

We shall give a sketch of the proof for this theorem.

Necessity. Since $T \in \mathcal{D}'_{+}(L(X, X))$ is regular it follows (cf. Yoshinaga [10]) that $T \in \mathcal{D}'_{+}(L(X, D_A))$ and

$$\left(\frac{d}{dt} - A\right) * T = \delta_0 \otimes I_X; \quad T * \left(\frac{d}{dt} - A\right) = \delta_0 \otimes I_{D_A} \tag{1.3}$$

where D_A is the domain of A topologized by the norm $\| x \| = \| x \| + \| Ax \|$ and I_X (resp. I_{D_A}) is the identical application on X (resp. D_A). Let $\varrho(t)$ be a $\mathcal{D}$-function such that $\varrho(t) = 1$ on $\{t;\ |t| < 1\}$ and $\varrho(t) = 0$ for $|t| > 2$. We denote by E (resp. Φ) the distribution ϱT (resp. $\varrho' T$) and set $\widehat{E}(\varrho) = E(e^{-\lambda t})$; $\widehat{\Phi}(\lambda) = \Phi(e^{-\lambda t})$ for any complex λ. From (1.3) we have

$$(\lambda I - A)\widehat{E}(\lambda) = I_X - \widehat{\Phi}(\lambda);\ \widehat{E}(\lambda)(\lambda I - A) = I_{D_A} - \widehat{\Phi}(\lambda). \tag{1.4}$$

Since $\Phi \in \mathcal{E}'(L(X, X))$ and $\operatorname{supp}\Phi \geq 1$, by a well known argument it follows

$$\| \widehat{\Phi}(\lambda) \|_{L(X,X)} \leq C(1 + |\lambda|)^N \exp(-\operatorname{Re}\lambda), \quad \text{for any } \lambda \in C. \tag{1.5}$$

From (1.4) this implies that $(\lambda I - A)^{-1} \in L(X, X)$ for $\lambda \in \Lambda = \{\lambda;\ \operatorname{Re}\lambda \geq \alpha \log |\operatorname{Im}\lambda| + \beta;\ \operatorname{Re}\lambda \geq \gamma\}$, with $\alpha, \gamma > 0$ convenable chosen. Moreover we get

$$\| (\lambda I - A)^{-1} \|_{L(X,X)} \leq c \| \widehat{E}(\lambda) \|_{L(X,X)}, \quad \text{for } \lambda \in \Lambda. \tag{1.6}$$

But $\operatorname{supp} E \subset [0, 1]$ and by a Paley-Wiener theorem argument it follows

$$\| \widehat{E}(\lambda) \|_{L(X,X)} \leq \operatorname{pol}(|\lambda|) \quad \text{for any } \lambda \in \Lambda.$$

This inequality together (1.6) proves (ii).

Sufficiency. Define $T \in \mathcal{D}'(L(X, X))$ by the formula

$$T(\varphi) = (2\pi i)^{-1} \int_{\Gamma} (\lambda I - A)^{-1} \widehat{\varphi}(-\lambda)\, d\lambda, \tag{1.7}$$

where Γ is the frontier of Λ and

$$\widehat{\varphi}(\lambda) = \int e^{-\lambda t} \varphi(t)\, dt.$$

From (i), (ii) it follows that T is a regular D. S. G.

§. 2. Differentiable distribution semi-groups.

DEFINITION. A regular D. S. G. T. is called differentiable if for every $t > 0$, $T(\delta_t) \in L(X, X)$ and the application $t \to T(\delta_t)$ from R^+ in $L(X, X)$ is differentiable.

REMARKS. 1º If T is differentiable, then the distribution $T \in \mathcal{D}'(L(X, X))$ is given on R^+ by a differentiable $L(X, X)$-valued function. In fact for any $x \in [T(\mathcal{D}^+) X]$, we have

$$T(\varphi)\, x = \int_0^\infty T(\delta_t)\, x\, \varphi(t)\, dt \qquad \forall\, \varphi \in \mathcal{D}(R^+). \tag{2.1}$$

Since the space $[T(\mathcal{D}^+) X]$ is dense in X, this implies that $T = T(\delta_t)$ on R^+.

2º Let T be a differentiable D. S. G. and $A = T(-\delta_0')$ its infinitesimal generator. Then for every $t > 0$, $T(\delta_t) X \subset D_A$ and

$$\frac{d}{dt} T(\delta_t)\, x = A\, T(\delta_t)\, x, \quad \text{for any } x \in X \text{ and } t > 0. \tag{2.2}$$

To prove this, we consider x an arbitrary element of X and set $y(t) = T(\delta_t) x$ for $t > 0$. Let x_n be a sequence of $[T(\mathcal{D}^+) X]$ such that $x_n \to x$. It is obvious that $A T(\delta_t)\, x_n = d/dt\, T(\delta_t)\, x_n \to d/dt\, T(\delta_t)\, x$ for $n \to \infty$. Since A is closed, this implies that $y(t) \in D(A)$ and $y'(t) = Ay(t)$ for any $t > 0$.

The following theorem gives a characterization for the generator of a differentiable D. S. G.

THEOREM 3. Let A be a closed operator on X with domain D_A dense in X. A necessary and sufficient condition for A generate a differentiable regular D. S. G. is : for every $\delta > 0$ there exist positive constants C_δ and M_δ such that $(\lambda I - A)^{-1} \in L(X, X)$ for any complex λ in the domain

$$\Lambda_\delta = \{\lambda\, ;\ \operatorname{Re} \lambda \geq -\delta \log |\operatorname{Im} \lambda| + C_\delta\} \cup \{\lambda\, ;\ \operatorname{Re} \lambda \geq \gamma\}$$

and for $\lambda \in \Lambda_\delta$,

$$\| (\lambda I - A)^{-1} \|_{L(X,X)} \leq M_\delta \operatorname{pol}(|\lambda|), \tag{2.3}$$

where γ is a non-negative constant independent of δ.

Proof. *Necessity.* Let $\varphi(t)$ be a $\mathcal{D}$-function so that $\operatorname{supp} \varphi \subset \{t; |t| \leq 1\}$ and $\varphi(t) = 1$ in $|t| < 2^{-1}$. Denote by $\varphi_\varepsilon(t)$, $\varepsilon > 0$, the function $\varphi(t/\varepsilon)$ and by E_ε, Φ_ε the vector-valued distribution $\varphi_\varepsilon T$ and $\varphi'_\varepsilon T$ respectively. It is obvious that Φ_ε is differentiable and $\operatorname{supp} \Phi_\varepsilon \subset \{t; 2^{-1}\varepsilon \leq t \leq \varepsilon\}$. Put

$$M_\varepsilon = \sup_{2^{-1}\varepsilon < t < \varepsilon} \| D^1 (\varphi'_\varepsilon(t)\, T(t)) \|.$$

It is easy to see that

$$\| \widehat{\Phi}_\varepsilon(\lambda) \| \leq 2^{-1} \varepsilon M_\varepsilon |\operatorname{Im} \lambda|^{-1} \sup_{\varepsilon/2 \leq t \leq \varepsilon} \exp(-t \operatorname{Re} \lambda), \qquad \lambda \in C.$$

Hence $\| \widehat{\Phi}_\varepsilon(\lambda) \| \leq 2^{-1}$ for any complex λ in the domain

$$\Sigma_\varepsilon = \{\lambda ; \log |\operatorname{Im} \lambda| \geq -\varepsilon \operatorname{Re} \lambda + \log M_\varepsilon ; \operatorname{Re} \lambda \leq 0\} \cup$$
$$\cup \{\lambda ; \log |\operatorname{Im} \lambda| \geq -\varepsilon/2 \operatorname{Re} \lambda + \varepsilon/2 \log M_\varepsilon ; \operatorname{Re} \lambda \geq 0\}.$$

Remembering (1.4) this implies that

$$\| (\lambda I - A)^{-1} \|_{L(X,X)} \leq 2^{-1} \| \widehat{E}_\varepsilon(\lambda) \|_{L(X,X)}, \qquad \text{for } \lambda \in \Sigma_\varepsilon. \tag{2.4}$$

On the other hand, since $\operatorname{supp} E \subset [0, \varepsilon]$ we have (see L. Schwartz 12, Th.)

$$\| \widehat{E}_\varepsilon(\lambda) \|_{L(X,X)} \leq \sup_{t \in [0,\varepsilon]} \sum_{j=0}^{m} | D^j (e^{-\lambda t} \varphi_\varepsilon(t)) |, \qquad \lambda \in C.$$

Hence

$$\| \widehat{E}_\varepsilon(\lambda) \|_{L(X,X)} \leq M_\varepsilon \operatorname{pol}(|\lambda|) |\operatorname{Im} \lambda|, \qquad \text{for } \lambda \in \Sigma_\varepsilon \tag{2.5}$$

where the degree of the polynomial $\operatorname{pol}(|\lambda|)$ is equal to the order of the distribution T in a neighbourhood of the origin. Therefore $(\lambda I - A)^{-1} \in L(X, X)$ and satisfies (2.3) for any $\lambda \in \Sigma_\varepsilon$. From theorem 2 it follows then, that there exists a non-negative constant γ such that $\| (\lambda I - A)^{-1} \| \leq \operatorname{pol}(|\lambda|)$ for $\operatorname{Re} \lambda \geq \gamma$. If we choose N_ε so that $\log N_\varepsilon / M_\varepsilon \geq \gamma$, we deduce that the estimate (2.3) is verified for any λ in the domain

$$\{\lambda\,;\ \operatorname{Re}\lambda \geq -\varepsilon^{-1}\log|\operatorname{Im}\lambda| + \varepsilon^{-1}\log N_\varepsilon\} \cup \{\lambda\,;\ \operatorname{Re}\lambda \geq \gamma\}.$$

Choosing $\delta = \varepsilon^{-1}$ this implies that $(\lambda I - A)^{-1}$ satisfies (2.3) in any domain Λ_δ.

Sufficiency. It is obvious that T is an E. D. S. G. Hence we may write

$$T(\varphi) = (2\pi i)^{-1}\int_{\operatorname{Re}\lambda=\gamma} (\lambda I - A)^{-1}\,\widehat{\varphi}(-\lambda)\,d\lambda, \qquad \varphi \in \mathcal{D}. \tag{2.6}$$

As $(\lambda I - A)^{-1}$ is holomorphic in every Λ_δ and $\|(\lambda I - A)^{-1}\,\widehat{\varphi}(-\lambda)\|$ rapidly tends to zero at infinity, we can change the path of integration and obtain

$$T(\varphi) = (2\pi i)^{-1}\int_{\Gamma_\delta} (\lambda I - A)^{-1}\,\widehat{\varphi}(-\lambda)\,d\lambda,$$

where Γ_δ is the boundary of the domain $\{\lambda\,;\ \operatorname{Re}\lambda \geq -\delta\log|\operatorname{Im}\lambda| + C_\delta\,;$ $\operatorname{Re}\lambda \leq \gamma$. Let $\{\varrho_k\}_{k=0}^{\infty} \subset \mathcal{D}^{+}$ be a sequence of regularization for Dirac distribution, i. e. $\varrho_n(t) \geq 0$, $\int \varrho_n(t)\,dt = 1$ and $\operatorname{supp}\varrho_n \to 0$. We have

$$D_t^k\, T(\delta_t * \varrho_n) = (2\pi i)^{-1}\int_{\Gamma_\delta} e^{\lambda t}\,(\lambda I - A)^{-1}\,\lambda^k\,\widehat{\varrho}_n(-\lambda)\,d\lambda \tag{2.7}$$

for any non-negative integer k. We set $\Gamma_\delta = \Gamma_\delta^1 \cup \Gamma_\delta^2$, where Γ_δ^1 is given by $\{\operatorname{Re}\lambda = -\delta\log|\operatorname{Im}\lambda| + C_\delta\,;\ |\operatorname{Im}\lambda| \geq A_\delta = \exp(\delta^{-1}(C_\delta - \gamma))\}$ and Γ_δ^2 by $\{\operatorname{Re}\lambda = \gamma\,;\ |\operatorname{Im}\lambda| \leq \exp(\delta^{-1}(C_\delta - \gamma))\}$. We write

$$T_j^{(k)}(t) = (2\pi i)^{-1}\int_{\Gamma_\delta^j} e^{\lambda t}\,(\lambda I - A)^{-1}\,\lambda^k\,\widehat{\varrho}_n(-\lambda)\,d\lambda\,; \qquad j = 1, 2, \dots$$

It is obvious that $T_2^{(k)}(t)$ is defined for every $t \geq 0$ and

$$\|T_2^{(k)}(t)\|_{L(X,X)} \leq M_\delta^{m+k+1}\exp(\gamma t) \qquad \text{for} \quad t \geq 0. \tag{2.8}$$

where M_δ is another non-negative constant. Let $\lambda = \sigma + i\eta$; then since on Γ_δ^1, $\sigma = -\delta \log|\eta| + C_\delta$ we have

$$\| T_1^{(k)}(t) \|_{L(X,X)} \leq M_\delta \exp(C_\delta t) \int |\eta|^{m+k-t\delta} d\eta .$$

Hence

$$\| T_1^{(k)}(t) \| \leq M_\delta \exp(C_\delta t), \qquad \text{for} \quad t > (m+k+1)\delta^{-1} . \tag{2.9}$$

Therefore we find a constant $M_{k,\delta}$ such that

$$\| D_t^k T(\delta_t * \varrho_n) \|_{L(X,X)} \leq M_{k,\delta} \exp(C_\delta t) \tag{2.10}$$

for $t > (m+k+1)\delta^{-1}$. But for any $x \in [T(\mathcal{D}^+)X]$ we have

$$D_t^k T(\delta_t * \varrho_n)x \rightarrow D_k^t T(\delta_t)x, \qquad t > 0, \quad k = 0, 1, \ldots$$

uniformly en every compact. Since the space $[T(\mathcal{D}')X]$ is dense in X this implies that $D_t^k T(\delta_t) \in L(X,X)$ for $t > (m+k+1)\delta^{-1}$. Since δ is arbitrary this proves the differentiability of T. Moreover we have proved that $D_t^k T(\delta_t) \in L(X,X)$ for any $t > 0$ and $k = 0, 1, \ldots$. Combining with the first part of the proof it follows that if a regular D. S. G., T is differentiable then the application $t \rightarrow T(\delta_t)$ from R^+ in $L(X,X)$ is infinitely differentiable.

COROLLARY. Let A be a closed and densely defined operator on the Banach space X. If the conditions of Theorem 3 are satisfied, then the abstract Cauchy problem $(A\,C\,P)_0$:

$$\frac{du(t)}{dt} - Au(t) = 0, \qquad \text{for} \quad t > 0, \tag{2.11}$$

$$u(0) = 0,$$

has a solution $u \in C^\infty(R^+, X)$ for every $x \in X$.

PROOF. Let T be the D. S. G. generates by A. Then from remark 2 it follows that $T(\delta_t)x$ solves $(ACP)_0$ for any $x \in X$.

REMARKS 1° If T is a differentiable regular D. S. G., then

$$\| D^k T(\delta_t) \|_{L(X,X)} = 0(\exp(\gamma_0 t)), \qquad \text{for} \quad t \rightarrow \infty \tag{2.1'}$$

and $k = 0, 1, \ldots$, where γ_0 is a non-negative constant.

2^0 In particular if T is a strongly continuous semi-group of bounded linear operators on X, then according to formula (2.5) it follows that T is differentiable if and only if for $\lambda \in \Lambda_\delta$

$$\| (\lambda I - A)^{-1} \|_{L(X, X)} \leq M_\delta \, | \operatorname{Im} \lambda \, | . \tag{2.12}$$

Thus we find a result proved by Pazzy [16].

§ 3. Analytic and non-quasianalytic D. S. G.

Let Y be a Banach space and $L = \{L_k\}_{k=0}^{\infty}$ and increasing sequence of non-negative numbers such that

$$L_k^{2k} \leq L_{k-1}^{k-1} \, L_{k+1}^{k+1} \, ; \qquad L_{mk+n} \leq q\,(m)\, L_k \, , \qquad k = 0, 1, \ldots \tag{3.1}$$

where m and n are non-negative integres and $r \longrightarrow q\,(r)$ is a positive and monotone increasing function. If Ω is an open set of R we denote by $C^L(\Omega, Y)$ the space of infinitely differentiable Y-valued functions $u\,(t)$ in Ω, such that for any compact subset K there exists $M > 0$ such that

$$\sup_{t \in K} \| D^j u\,(t) \| \leq M^{j+1} \, L_j^j \, ; \qquad j = 0, 1, \ldots \tag{3.2}$$

The space $C^L(\Omega, Y)$ is topologized as projective limit of all $\{C^L(K, Y)\,; K \subset \Omega\}$. The function class $C^L(\Omega, Y)$ is called non-quasi-analytic if it contains a non-trivial regular function with compact support contained in Ω. The Carleman-Denjoy criterion states that C^L is on-quasianalytic if and only if

$$\Sigma \, L_j^{-1} < \infty .$$

If $Y = R$ we often omit R and write $C^L(Y)$. In particular, if $L_j = (j\,!)^{\varrho/j}$, C^L is the classical Gevrey class G which is non-quasianalytic for $1 < \varrho < \infty$. For $\varrho = 1$ we obtain the class of real analytic functions. If L is a non-quasianalytic sequence we denote by $C_0^L(\Omega, Y)$ the space $C^L(\Omega, Y) \cap C_0^\infty(\Omega, Y)$.

DEFINITION. A D. S. G., $T \in \mathcal{D}'_+(L(X, X))$ is said to be of class C^L if the mapping $t \longrightarrow T(\delta_t)$ is of class C^L on R^+.

In particular, fort $L_j = (j\,!)^{\varrho/j}$ the semi-group T is called ϱ-hypoanalytic; $1 \leq \varrho < \infty$. As above we remark that if the semi-group T is of class C^L then the distribution $T \in \mathcal{D}'(L(X, X))$ is defined on $R^+ = \{t\,;\, t > 0\}$ by a $L(X, X)$-valued function of class C^L.

Let $L = \{L_j\}_{j=0}^{\infty}$ be a non-quasianalytic sequence and $\omega_L(t)$ be a scalar function defined by

$$\omega_L(t) = \sum_{j=0}^{\infty} t^j / L_j^j\,; \qquad t \geq 0.$$

Then we have

THEOREM 4. Let T be a regular D.S.G. and $A = T(-\delta_0')$ be its infinitesimal generator. T is of class C^L if and only if for every $0 < \varepsilon < 1$ there exist C_ε and $M_\varepsilon > 0$ such that

i) $(\lambda I - A)^{-1} \in L(X, X)$ for any λ in the domain

$$\Sigma_\varepsilon = \{\lambda\,;\, \mathrm{Re}\,\lambda \geq -\log \omega_L(C_\varepsilon\,|\,\mathrm{Im}\,\lambda\,|) + \gamma\} \tag{3.3}$$

and

ii) $\|(\lambda I - A)^{-1}\|_{L(X, X)} \leq M_\varepsilon\,\mathrm{pol}(|\lambda|)\exp(\varepsilon\,|\,\mathrm{Re}\,\lambda\,|)$, for $\lambda \in \Sigma_\varepsilon$ where γ is a positive constant independent of ε.

PROOF. *Necessity.* Assume that for every $0 < \varepsilon < 1$.

$$\| D^k T(t)\|_{L(X, X)} \leq M_\varepsilon^{k+1} L_k^k\,, \quad t \in [\varepsilon/2, \varepsilon],\ k = 0, 1, \ldots \tag{3.4}$$

We choose $\varphi \in C_0^L$ such that $\mathrm{supp}\,\varphi \subset \{t\,;\, |t| \in 1\}$, $\varphi(t) = 1$ in $|t| \leq 2$, and denote: $\Phi_\varepsilon = \varphi_\varepsilon' T$; $E_\varepsilon = \varphi_\varepsilon T$ where $\varphi_\varepsilon(t) = \varphi(t/\varepsilon)$. From (3.4) we obtain

$$\| \Phi_\varepsilon^{(k)}(t)\| \leq M M_\varepsilon \varepsilon^{-1}(2 N_\varepsilon)^{-k} L_k^k\,, \qquad k = 0, 1, \ldots \tag{3.5}$$

where $M > 0$ and $N_\varepsilon^{-1} = 2\max(M\varepsilon^{-1}, M\varepsilon)$. Or,

$$\| \widehat{\Phi}_\varepsilon(\lambda)\| = M(L_k / 2N_\varepsilon\,|\,\mathrm{Im}\,\lambda\,|)^k \int_{\varepsilon/2}^{\varepsilon} \exp(-t\,\mathrm{Re}\,\lambda)\,dt, \quad k = 0, 1, \ldots$$

Thus for any λ complex in the domain

$$\Lambda_\varepsilon = \{\lambda\,;\, \mathrm{Re}\,\lambda \geq \varepsilon^{-1}\log\omega_L(N_\varepsilon\,|\,\mathrm{Im}\,\lambda\,|) + M_\varepsilon^1\,;\ \mathrm{Re}\,\lambda \leq \gamma_\varepsilon\}$$

we have $\|\widehat{\Phi}_\varepsilon(\lambda)\| \leq 2^{-1}$. Here M_ε^1 and γ_ε are another non-negative constants. Since the semi group T is regular we may assume that $\gamma_\varepsilon = \infty$. As in the proof of theorem 3, this implies that $(\lambda I - A)^{-1} \in L(X, X)$, and

$$\|(\lambda I - A)^{-1}\| \leq p_\varepsilon(|\lambda|)\exp(\varepsilon\,|\,\mathrm{Re}\,\lambda\,|), \text{ for } \lambda \in \Lambda_\varepsilon\,. \tag{3.6}$$

where p_ε is a polinomial with non-negative coefficients. Since the sequence $\{L_k\}$ satisfies (3.1), the function $r \longrightarrow \log \omega_L(r)$ is sub-additive. Hence we may find another constant $C_\varepsilon > 0$ such that (3.6) to be satisfied for any λ in the domain

$$\{\lambda ;\ \mathrm{Re}\,\lambda \geq -2 \log \omega_L (C_\varepsilon \,|\, \mathrm{Im}\,\lambda \,|) + M_\varepsilon^1\}.$$

Without loss of the generality we may assume that $\varepsilon \longrightarrow C_\varepsilon$ is bounded and $\lim_{\varepsilon \to 0} M_\varepsilon^1 = \infty$. Let a be a non-negative constant such that $C_\varepsilon \leq a$ for $0 < \varepsilon < 1$. Using the above argument it follows that there exist $b > 0$ such that

$$\| (\lambda I - A)^{-1} \| \leq \mathrm{pol}(\,|\,\lambda\,|\,) \exp (N \,|\, \mathrm{Re}\,\lambda \,|) \tag{3.8}$$

for $\mathrm{Re}\,\lambda \geq -\log \omega_L (a \,|\, \mathrm{Im}\,\lambda \,|) + b$. For ε enough small we may suppose $b < M_\varepsilon^1$. Hence

$$\| (\lambda I - A)^{-1} \| \leq p_\varepsilon (\,|\,\lambda\,|\,) \exp (\varepsilon \,|\, \mathrm{Re}\,\lambda \,|) \tag{3.9}$$

for $\mathrm{Re}\,\lambda \geq -\log \omega_L (C_\varepsilon \,|\, \mathrm{Im}\,\lambda \,|) + b$; and, $|\,\mathrm{Im}\,\lambda \,| \geq C_\varepsilon^{-1} \omega_L^{-1} (\exp (M_\varepsilon^{-1} - b))$. Using (3.8) we get that the estimate (3.3) satisfied in the whole domain Σ_ε with $\gamma = b$.

Sufficiency. From (3.3) it follows that $\| (\lambda I - A)^{-1} \| = 0 (\mathrm{pol}(\,|\,\lambda\,|\,))$ for $\mathrm{Re}\,\lambda > \gamma$. Hence, as in the proof of theorem 3, we get

$$D_t^k T (\delta_t * \varrho_n) = (2\pi i)^{-1} \int_{\Gamma_\varepsilon} e^{\lambda t} (\lambda I - A)^{-1} \lambda^k \widehat{\varrho_n} (-\lambda)\, d\lambda \tag{3.10}$$

where $\{\varrho_n\}$ is a sequence of regularization of $\mathcal{D}^+$ and Γ_ε is the frontier of the domain

$$\{\mathrm{Re}\,\lambda \geq -\log \omega_L (C_\varepsilon \,|\, \mathrm{Im}\,\lambda \,|) + \gamma ;\ \mathrm{Re}\,\lambda \leq \gamma\}.$$

It is easily verified that

$$\| D_t^k T (\delta_t * \varrho_n) \|_{L(X, X)} \leq M_\varepsilon^{m+k+1} \exp (\gamma\, t) \int_{\overline{R}_+} \eta^{m+k} \omega_L^{\varepsilon - t} (\eta)\, d\eta \tag{3.11}$$

where m is the degree of the polynomial $\mathrm{pol}(\,|\,\lambda\,|\,)$.

Since, for any non-negative integer k, $\omega_L(\eta) \geq \eta^j L_j^{-j}$, it follows that. the right side of (3.11) is bounded by $M^{m+k+1} \exp (\gamma\, t)\, L_{(m+k+1)p+1}^{m+k+2}$

where p is the largest integer smaller then $(t-\varepsilon)^{-1}$. Because of the properties of the sequence $\{L_k\}$ we find another constant M_ε we such that

$$\| D_t^k\, T(\delta_t * \varrho_n) \|_{L(X,X)} \leq M_\varepsilon^{k+1} L_k^k (\exp(\gamma t))$$

for $t > \varepsilon$ and $k = 0, 1, \dots$ According to an argument used in the proof of theorem 3, this implies that

$$\| D^k\, T(\delta_t) \|_{L(X,X)} \leq M_\varepsilon^{k+1} + L_k^k \exp(\gamma t),\ k = 0, 1, \dots;\ t > \varepsilon > 0. \tag{3.13}$$

Since ε is arbitrary, the proof is complete.

COROLLARY. Let T be a regular D.S.G. and A its infinitesimal generator. The semi-group T is ϱ-hypoanalytic; $1 \leq \varrho < \infty$, if and only if for every $\varepsilon > 0$ there exist constants C_ε and M_ε such that $(\lambda I - A)^{-1} \in L(X, X)$ and satisfies

$$\| (\lambda I - A)^{-1} \| \leq M_\varepsilon \operatorname{pol}(|\lambda|) \exp(\varepsilon |\operatorname{Re} \lambda|) \tag{3.14}$$

for

$$\operatorname{Re} \lambda \geq - C_\varepsilon |\operatorname{Im} \lambda|^{1/\varrho} + \gamma \tag{3.15}$$

where γ is a non-negative constant independent of ε.

PROOF. The non-quasianalytic case $\varrho > 1$ is a consequence of theorem 4. We assume that $\varrho = 1$. It is easily proved that there exists a sequence $\varphi_k \in \mathcal{D}$, $k = 0, 1, \dots$ such that

$$\operatorname{supp} \varphi_k \subset \{t;\ |t| \leq 1\};\ \varphi_k(t) \equiv 1 \quad \text{for } |t| \leq 2^{-1}$$

and

$$|\varphi_k^{(j)}(t)| \leq M^{j+1} k^j, \quad \text{for } j \leq k. \tag{3.16}$$

Put

$$\varphi_{\varepsilon,k}(t) = \varphi_k(t/\varepsilon);\quad E_{\varepsilon,k} = \varphi_{\varepsilon,k} T;\quad \Phi_{\varepsilon,k} = \varphi'_{\varepsilon,k} T.$$

If $T(\delta_t) \in G^1(R^+, L(X,X))$, then as in the proof of theorem 4 we find a constant $M_\varepsilon > 0$ such that

$$\| \widehat{\Phi}_{\varepsilon,k}(\lambda) \|_{L(X,X)} \leq M_\varepsilon (k/M_\varepsilon |\operatorname{Im} \lambda|)^k \int_{\varepsilon/2}^{\varepsilon} \exp(-t \operatorname{Re} \lambda)\, dt \tag{3.17}$$

for any non-negative integer k. Take k equal to the largest integer smaller than $M_\varepsilon \mid \mathrm{Im}\, \lambda \mid e^{-1}$. Thus from (3.17) we obtain

$$\| \widehat{\Phi}_{\varepsilon,\, k}(\lambda) \|_{L(X,\, X)} \leq M_\varepsilon^1 \exp(-M_\varepsilon e^{-1} \mid \mathrm{Im}\, \lambda \mid) \int_{\varepsilon/2}^{\varepsilon} \exp(-\mathrm{Re}\, \lambda t)\, dt.$$

As above this implies that, $(\lambda I - A)^{-1} \in L(X, X)$ and satisfies

$$\| (\lambda I - A)^{-1} \| \leq \underset{\varepsilon}{\mathrm{pol}} (\mid \lambda \mid) \exp(\varepsilon \mid \mathrm{Re}\, \lambda \mid)$$

for

$$\mathrm{Re}\, \lambda \geq 2\varepsilon^{-1} \log(2M_\varepsilon) - M_\varepsilon (e\varepsilon)^{-1} \mid \mathrm{Im}\, \lambda \mid ; \quad \text{and} \quad \mid \mathrm{Im}\, \lambda \mid < M_\varepsilon^{-1} e (k+1).$$

Since k is arbitrary, this implies that $(\lambda I - A)^{-1}$ satisfies the estimate (3.14) in a domain of the form

$$\mathrm{Re}\, \lambda \geq - C_\varepsilon \mid \mathrm{Im}\, \lambda \mid + D_\varepsilon ; \ \mathrm{Re}\, \lambda \geq \gamma$$

Sufficiency of (3.14) follows just in the proof of theorem 4.

§ 4. Distribution semi-groups of class A^ϱ.

If a D.S.G. T is differentiable, then for any integer $k \geq 0$, $\| D_t^k T(\delta_t) \|_{L(X,\, X)}$ is of exponential growth for $t \to \infty$. In this section we also impose a restriction of the origin for $D_t^k T(\delta_t)$.

DEFINITION. Let $1 \leq \varrho < \infty$. A regular D.S.G., T is said to be of class A^ϱ, if for $t > 0$,

$$\| D_t^k T(\delta_t) \|_{L(X,\, X)} \leq p(t^{-\varrho}) (Mt)^{-\varrho k} (k!)^\varrho \exp(\gamma t) ; \ k = 0, 1, \ldots \tag{4.1}$$

where M, γ are non-negative constants and $p(r)$ is a polynomial with non negative coefficients.

The semi-groups of class A^ϱ can be characterized in the following way (see theorem 4).

THEOREM 5. Let A be a closed operator on X with the domain D_A dense in X. Then A is the infinitesimal generator for a D.S.G. of class A^ϱ if and only if there exist positive constants α and β such that

(i) $(\lambda I - A)^{-1} \in L(X, X)$ for

$$\lambda \in \Lambda = \{\lambda \mid \mathrm{Re}\, \lambda > - \alpha \mid \mathrm{Im}\, \lambda \mid^{1/\varrho} + \beta\}. \tag{4.2}$$

(ii) $\| (\lambda I - A)^{-1} \| \leq \mathrm{pol}(\mid \lambda \mid)$ for any $\lambda \in \Lambda$.

PROOF. *Necessity.* From (4.1) it is obvious that $e^{-\gamma t}\, T \in \mathcal{S}'\,(L\,(X, X))$ where γ is a non-negative constant. Therefore the semigroup T is of exponential growth and from Lions's theorem it follows that $(\lambda I - A)^{-1}$ exists and satisfies the estimate (ii) for $\mathrm{Re}\ \lambda > \gamma$. Moreover by a well known result (cf. Schwartz [12], Yoshinaga [11]) there exists a function $f \in \mathcal{E}^0\,(L\,(X, X))$ and an integer $m \geq 0$ such that

(4.3) $$\| f(t) \|_{L(X,\,X)} = 0\,((1 + t^2)^m \quad \text{for } t \to \infty$$

and $e^{-\gamma t}\, T$ may be expressed as

(4.4) $$e^{-\gamma t}\, T = D^m f.$$

The function $f(t)$ is regular on R^+ and from (4.1) we have

(4.5) $$\| D^k f(t) \|_{L(X,\,X)} \leq Mp\,(t^{-e})\,(Mt^{-e})^k\,(k\,!)^e\,, \qquad k \geq m,\ t > 0$$

and

(4.6) $$\| D^k f(t) \|_{L(X,\,X)} \leq M_1\, p\,(t^{-e})\, t^{m-k} \qquad \text{for} \quad 0 \leq k \leq m$$

Let j and k be two non-negative integers such that $\varrho\,(k+p) < j \leq \varrho\,(k+p) + 1$, where p is the degree of $p\,(r)$. Since $\mathrm{supp}\ f \subset [0, \infty)$, for j and k taken as. above we have

$$\lambda^k\, D^j\, \widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t}\, D^k\,(t^j\, f(t))\, dt, \qquad \mathrm{Re}\ \lambda > \varepsilon$$

where ε is an arbitrary positive number. Then (4.5) and (4.6) imply that

$$\| D^j\, \widehat{f}(\lambda) \|_{L(X,\,X)} \leq M^{j+1}\, j\,!\, |\, \lambda\, |^{-k}$$

for $\mathrm{Re}\ \lambda > \varepsilon$ and $\varrho\,(k + p) < j \leq \varrho\,(k + p) + 1$. Hence

(4.7) $$\| D^j\, \widehat{f}(\lambda) \|_{L(X,\,X)} \leq M^{j+1}\, j\,!\, |\, \lambda\, |^{-(j-1)\varrho^{-1}} + p, \ \text{for } \mathrm{Re}\ \lambda > \varepsilon > 0.$$

Then the analyticity of $\widehat{f}(\lambda)$ in the domain $\{\lambda : \mathrm{Re}\ \lambda > 0\}$ and the estimate (4.7) imply that $\widehat{f}(\lambda)$ can be extended holomorphically in a domain of the form

$$\Sigma = \{\lambda \in C\,;\, |\ \mathrm{Re}\ \lambda - \varepsilon\, | < M^{-1}\, |\, \mathrm{Im}\ \lambda\, |^{1/\varrho}\}$$

and $\|\widehat{f}(\lambda)\|_{L(X, X)} \leq M |\lambda|^{p+\varrho^{-1}}$ for $\lambda \in \Sigma$. We observe that

$$T(e^{-\lambda t}) = \widehat{T}(\lambda) = (\lambda - \gamma)^m \widehat{f}(\lambda - \gamma), \qquad \text{for Re } \lambda > \gamma.$$

Hence we have proved that $\widehat{T}(\lambda)$ exists and satisfies the estimate

$$\|\widehat{T}(\lambda)\|_{L(X, X)} \leq \text{pol}(|\lambda|) \tag{4.8}$$

for $|\text{Re } \lambda - \gamma - \varepsilon| < M^{-1} |\text{Im } \lambda|^{1/\varrho}$. Because $\widehat{T}(\lambda) = (\lambda I - A)^{-1} \in L(X, X)$ for Re $\lambda > \gamma$, the analyticity of $(\lambda I - A)^{-1}$ implies that it satisfies the estimate (ii) for

$$\text{Re } \lambda > - M^{-1} |\text{Im } \lambda|^{1/\varrho} + \gamma.$$

Sufficiency. From (i) and (ii) it follows that the operator A generates en E. D. S. G. and

$$T(\varphi) = (2\pi i)^{-1} \int_{\text{Re} = \beta + \varepsilon} (\lambda I - A)^{-1} \widehat{\varphi}(-\lambda)\, d\lambda, \quad \varphi \in \mathcal{D}$$

where ε is an arbitrary positive number. As in the proof of theorem 5 we have

$$D_t^k\, T(\delta_t * \varrho_n) = (2\pi i)^{-1} \int_{\Gamma_\varepsilon} e^{\lambda t} (\lambda I - A)^{-1} \lambda^k\, \widehat{\varrho_n}(-\lambda)\, d\lambda \tag{4.9}$$

where Γ_ε is the curve given by

$$\Gamma_\varepsilon = \{\lambda = \sigma + i\eta\, ; \sigma = -\alpha |\eta|^{1/\varrho} + \beta + \varepsilon; \quad -\infty < \sigma \leq \beta + \varepsilon\}.$$

Then our estimates of $(\lambda I - A)^{-1}$ and $\widehat{\varrho_n}(-\lambda)$ imply that

$$\|D_t^k\, T(\delta_t * \varrho_n)\|_{L(X, X)} \leq C^{k+1} \exp(\beta + \varepsilon)\, t \int_0^\infty \eta^{p+k} \exp(-t\, \alpha \eta^{1/\varrho})\, d\eta$$

for any $t > 0$ and $k = 0, 1, \dots$. Hence

$$\|D_t^k\, T(\delta_t * \varrho_n)\|_{L(X, X)} \leq \text{pol}(t^{-\varrho})\, (Ct^{-\varrho})^k\, \Gamma(\varrho k) \exp(\beta + \varepsilon)\, t. \tag{4.10}$$

Here $\Gamma(r)$ is Euler's function and C is a positive constant independent of ε. Consequently the semi- group T is of class A^ϱ and the proof is complete.

For $\varrho \geq 1$ and $\gamma \geq 0$ we denote by A_γ^ϱ the class of regular D. S. G., T such that for any $\varepsilon > 0$

$$(4.11) \qquad \| D_t^k\, T(\delta_t) \|_{L(X,X)} \leq \underset{\varepsilon}{\mathrm{pol}}\, (t^{-\varrho})\, (Mt)^{-\varrho k}\, (k\,!)^\varrho \exp(\gamma + \varepsilon)\, t.$$

As a consequence of theorem 5 and its proof we obtain (see also Da Prato-Mosco [7]).

COROLLARY. A closed and densely defined operator A on X generates a D. S. G. of class A_γ^ϱ if and only if there exists $\alpha > 0$ such that $(\lambda I - A)^{-1} \in L(X, X)$ and satisfies

$$\| (\lambda\, I - A)^{-1} \| \leq \underset{\varepsilon}{\mathrm{pol}}\, (|\,\lambda\,|)$$

for Re $\lambda > -\alpha\, |\, \mathrm{Im}\, \lambda\, |^{1/\varrho} + \gamma + \varepsilon$, where ε is an arbitrary non-negative number.

Let $T \in \mathcal{D}'_+(L(X, X))$ be a regular D. S. G. T is said holomorphic (cf. Fujiwara [8], Da Prato-Mosco [6]) in the sector $\Sigma = \mu\,;\, |\arg \mu\,| < \alpha\,;\, 0 < \alpha < \pi/2\}$ if $t \longrightarrow T(\delta_t)$ can be extended at an holomorphic function T_μ in this sector. It is obvious that a D. S. G. of class A^ϱ with $\varrho = 1$ is holomorphic in a sector of the complex plane. Conversely from Cauchy's formula it follows that any holomorphic D. S. G. in a sector Σ is of class A^1. We can now formulate the following result (cf [8]).

THEOREM 6. A closed and dense linear operator A generates a D. S. G. which is holomorphic in the sector $\Sigma = \{\mu\,;\, |\arg \mu\,| < \alpha < \pi/2\}$ if and only if there exists a real γ such that for any $\varepsilon > 0$ and any λ in the sector

$$\Lambda = \{\lambda \mid |\arg(\lambda - \gamma)| < \pi/2 + \alpha - \varepsilon\}$$

we have $(\lambda I - A)^{-1} \in L(X, X)$ with the estimate

$$(4.12) \qquad \| (\lambda I - A)^{-1} \|_{L(X,X)} \leq \mathrm{pol}\, (|\,\lambda\,|).$$

PROOF. The sufficiency of condition (4.12) is a consequence of theorem 5. Also the necessity can be obtained by an adaptation of the proof of theorem 5, but we shall give a direct proof. Il the semi-group T is holomorphic in the sector $\Sigma = \{\mu\,;\, |\arg \mu\,| < \alpha < \pi/2\}$, then according to theorem 3, there exists a real γ such that $e^{-\gamma t}\, T = D^m f$ where $f(t)$ is a $\mathcal{E}^0$

$(L\,(X,\,X))$-function satisfying (4.3). Put

$$\widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt, \quad \text{for Re } \lambda > 0.$$

Because $f(t)$ is analytic in Σ and $\| e^{-\lambda t} f(t) \|$ rapidly tends to zero at infinity, we may write

$$\widehat{f}(\lambda) = \int_\Gamma e^{-\lambda\mu} f(\mu)\, d\mu, \quad \text{for Re } \lambda > 0. \tag{4.13}$$

where $\qquad \Gamma = \{\mu\,;\, \mu = te^{-i(\alpha-\varepsilon)}\,;\, t > 0\}$ for $\operatorname{Im} \lambda \geq 0$

and $\qquad \Gamma = \{\mu\,;\, \mu = te^{i(\alpha-\varepsilon)}\,;\, t > 0\}$ for $\operatorname{Im} < 0$.

This implies that $\widehat{f}(\lambda)$ can be extended at an holomorphic function $\widehat{f}(\lambda)$ in the domain

$$\{\lambda\,;\, \text{Re } \lambda > -(\alpha - \varepsilon)\, |\operatorname{Im} \lambda|\}.$$

Again following the proof of theorem 5 we obtain that $(\lambda I - A)^{-1} \in (L\,(X,\,X)$ and satisfies (4.12) for

$$\text{Re } \lambda > -(\alpha - \varepsilon)\, |\operatorname{Im} \lambda| + \gamma.$$

Thus theorem 6 is proved.

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