ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

L. FUCHS

On quasi-injective modules

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 23, nº 4 (1969), p. 541-546

<http://www.numdam.org/item?id=ASNSP_1969_3_23_4_541_0>

ON QUASI-INJECTIVE MODULES

By L. Fuchs

Dedicated to B. H. Neumann on his 60 th birthday

The purpose of this note is to point out certain analogies between injective and quasi-injective left modules over an arbitrary ring R with identity.

We shall show that quasi-injective modules can be characterized in the same way as injective modules M by the extensibility of homomorphisms $L \longrightarrow M$ (where L is a left ideal of R) to $R \longrightarrow M$, but in the quasi-injective case only homomorphisms are admitted whose kernels contain the annihilator left ideal of some $a \in M$.

The notion of K-bounded module (with K an ideal of R) is introduced as a module M which is annihilated by K and which contains an element whose annihilator is exactly K. For K-bounded modules quasi-injectivity turns out to be equivalent to R/K-injectivity. A K-bounded module is a direct summand of every module containing it as a pure submodule where purity can be taken in two, inequivalent ways.

Finally, the so-called exchange property will be proved for quasi-injective modules.

1. By a ring we mean an associative ring with 1 and by a module a unital left module over a ring R.

An R module M is said to be *quasi-injective* (1) if every R-homomorphism of every R submodule of M into M is induced by an R-endomorphism of M. A module is quasi-injective exactly if it is a fully invariant submodule of its injective hull.

For an R-module M, we denote by $\Omega(M)$ the set of all left ideals L of R such that L contains $\operatorname{Ann} a = \{r \in R \mid ra = 0\}$ for some $a \in M$.

Pervenuto alla Redazione il 24 Dicembre 1968.

(1) For properties of quasi-injective modules we refer e.g. to Faith [5].

LEMMA 1. *The following conditions on an R-module M are equivalent:*

(i) *M is quasi-injective;*

(ii) *if B is a submodule of a cyclic submodule $A = Ra$ of M and if $\beta: B \to M$ is any R-homomorphism, then there is an extension $\alpha: A \to M$ of β;*

(iii) *if B is a submodule of any R-module A with $\Omega(A) \subseteq \Omega(M)$, then every R-homomorphism $\beta: B \to M$ can be extended to an R-homomorphism $\alpha: A \to M$.*

The implication (i) $\Longrightarrow$ (ii) is trivial. To prove (ii) $\Longrightarrow$ (iii), assume (ii) and let A, B, β be given as in (iii). We use the standard argument and consider submodules C of A and homomorphisms $\gamma: C \to M$ such that $B \subseteq C \subseteq A$ and $\gamma \mid B = \beta$. If the pairs (C, γ) are ordered in the obvious way, then we can pick out a maximal pair (C_0, γ_0) in the set of pairs (C, γ). By way of contradiction, suppose there is an $a \in A$ not in C_0.

Clearly, $L = \{r \in R \mid ra \in C_0\}$ is a left ideal of R contained in $\Omega(A)$, and hence in $\Omega(M)$. Choose some $x \in M$ such that $L \supseteq \operatorname{Ann} a \supseteq \operatorname{Ann} x$, and consider the submodule $N = Lx$ of M. The correspondence $rx \longmapsto \gamma_0(ra)$ with $r \in L$ defines a homomorphism $\varphi': N \to M$ which can be extended, in view of our hypothesis (ii), to a homorphism $\varphi: Rx \to M$. Now let $C' = C_0 + Ra$ and let $\gamma': C' \to M$ be defined as $\gamma': c + ra \longmapsto \gamma_0(c) + \varphi(rx)$ for $c \in C_0$, $r \in R$. It is easy to check that γ' is a well-defined homomorphism, so $(C_0, \gamma_0) < (C', \gamma')$ contradicts the maximal choice of (C_0, γ_0). Hence $C_0 = A$ and $\gamma_0 = \alpha$ is an extension of β.

The choice $A = M$ in (iii) yields (i). This completes the proof.

Condition (ii) may be reformulated to give a characterization of quasi-injectivity which is similar to a well-known characterization of injectivity [1].

LEMMA 2. *An R-module M is quasi-injective if and only if for every left ideal L of R and for every R-homomorphism $\eta: L \to M$ with* $\operatorname{Ker} \eta \in \Omega(M)$ *there exists an R-homomorphism $\psi: R \to M$ that extends η* [2].

If $a \in M$ is such that $\operatorname{Ann} a \subseteq \operatorname{Ker} \eta$, then η induces an R-homomorphism $\beta: La \to M$, and the equivalence with (ii) becomes evident.

In connection with Lemma 2 let us notice that $\Omega(M)$ can be replaced by the filter (i.e. the dual ideal) $\bar{\Omega}(M)$ generated by $\Omega(M)$ in the lattice of all submodules of M. In fact, if M is quasi-injective, then so is $M \oplus \ldots \oplus M$ with a finite number n of summands and $\Omega(M \oplus \ldots \oplus M)$ contains all $L_1 \cap \ldots \cap L_n$ with $L_j \in \Omega(M)$. Together with $M \oplus \ldots \oplus M$ also M must have the property stated in Lemma 2 for the finite intersections $L_1 \cap \ldots \cap L_n$.

---

(2) Notice that the stated condition makes sense only for $L \in \Omega(M)$.

2. Next we introduce the notion of bounded modules.

Let K be a left ideal of R. An R-module M will be called *K-bounded* if $\Omega(M)$ consists exactly of the left ideals of R which contain K. That is, $K \in \Omega(M)$ is the unique minimal element of $\Omega(M)$, or, in other words, $\Omega(M)$ is the filter generated by K (3).

It follows at once that K must be two-sided, since it is the annihilator of M. We can thus form the factor ring R/K and may consider M as an R/K-module in the obvious way. If we do so, then we are led to

THEOREM 1. *A K-bounded R-module M is quasi injective if and only if it is injective as an R/K-module.*

In the K-bounded case, the condition in Lemma 2 amounts to R/K-injectivity. Hence Theorem 1 holds.

For $K = 0$, we have: a 0-bounded quasi-injective is injective.

If we drop the hypothesis of K-boundedness, then — under rather restrictive conditions — a similar result can be established with R/K replaced by a topological ring which is constructed as an inverse limit [4].

3. Following Cohn [2], we call a submodule N of the R-module M *pure* if for all right R modules U, the homomorphism $U \otimes_R N \longrightarrow U \otimes_R M$ [induced by the inclusion $N \longrightarrow M$] is monic. This is equivalent to the following condition which is more suitable for our purposes: if

$$\sum_{j=1}^{n} r_{ij}\, x_j = a_i \in N \qquad (i = 1, 2, \dots, m) \tag{1}$$

is a finite set of equations in the unknowns $x_1, \dots, x_n$ where $r_{ij} \in R$, and if this system has a solution in M, then it has a solution in N too.

An R-module A is called *algebraically compact* (see [6], [8]) if it is a direct summand in every R-module in which it is a pure submodule. Or, equivalently, if

$$\sum_{j} r_{ij}\, x_j = a_i \in A \qquad (i \in I) \tag{2}$$

is an arbitrary set of equations in the unknowns x_j ($j \in J$) [where I and J are arbitrary index sets and each equation contains but a finite number of

(3) For Z-modules, i.e. for abelian groups, K-boundedness means that the group is a direct sum of cyclic groups of order n and orders m dividing n or it contains an element of infinite order, according as $K = (n)$ or $K = (0)$.

non-zero $r_{ij} \in R$], and if every finite set of equations in (2) has a solution in A, then the entire system (2) is solvable in A.

Theorem 2. *A K-bounded quasi-injective R-module is algebraically compact.*

Let A be a K-bounded quasi-injective R-module and (2) a system of equations which is finitely solvable in A. If we consider A as an R/K-module and replace r_{ij} by $r_{ij} + K = \overline{r_{ij}}$, then (2) may be viewed as a system of equations over the R/K-module A. Finite solvability implies that this system is compatible in the sense of Kertész [7], thus it has a solution in the injective R/K-module A. This is evidently a solution of the original form (2), hence the algebraic compactness of A follows.

Algebraic compactness is preserved under direct products and direct summands, hence

Corollary 1. *Let M_i $(i \in I)$ be K_i-bounded quasi-injective R-modules and M a direct summand of their direct product ΠM_i. Then M is algebraically compact* (4).

4. There are various definitions of purity for modules which all reduce to ordinary purity for abelian groups. We are going to show that Theorem 2 holds if we replace purity in the sense of P. M. Cohn by the following one.

A submodule N of the R-module M is now called *pure* if

$$LN = N \cap LM$$

for all two-sided ideals L of R. Algebraic compactness can be defined in the same way as in 3 by using this definition of purity.

Next we prove Theorem 2 for this algebraic compactness. Let A be a K-bounded quasi injective R-module and let M contain A as a pure submodule. The module M/KM is annihilated by K, thus $\Omega(M/KM)$ contains only left ideals containing K. In view of $0 = KA = A \cap KM$, the natural homomorphism $\varphi : M \longrightarrow M/KM$ maps A isomorphically upon φA which is thus quasi-injective. The two last sentences imply, by Lemma 1, that the identity map of φA extends to a homomorphism $M/KM \longrightarrow \varphi A$ showing that $M/KM = \varphi A \oplus N/KM$ for a submodule N of M. Hence $M = A \oplus N$, and A is algebraically compact.

---

(4) Notice that for abelian groups the converse also holds: every algebraically compact group is a summand of a direct product of K-bounded quasi-injectives.

5. Next we turn our attention to the so-called exchange property which was systematically discussed by Crawley and Jónsson [3].

Recall that an R-module M is said to have the *exchange property* if for every R-module A containing M and for submodules N and $A_i\,(i \in I)$ of A, the direct decomposition

$$A = M \oplus N = \bigoplus_{i \in I} A_i \qquad (I = \text{arbitrary index set}) \tag{3}$$

implies the existence of R-submodules B_i of $A_i\,(i \in I)$ satisfying

$$A = M \oplus (\bigoplus_{i \in I} B_i). \tag{4}$$

It is known [3] that M has the exchange property if it has the stated property with the A_i subject to the condition that each A_i is isomorphic to a submodule of M. The following result generalizes a theorem of Warfield [9] from injectives to quasi injectives.

THEOREM 3. *A quasi-injective module has the exchange property.*

Let M be a quasi-injective R-module and assume (3) holds for R-modules N, $A_i\,(i \in I)$ with A_i isomorphic to submodules of M. Select a submodule B of A which is maximal with respect to the properties: (i) $B = \bigoplus_i B_i$ with $B_i \subseteq A_i$, and (ii) $M \cap B = 0$. We claim (4) holds with these B_i.

Denote by φ the natural homomorphism $A \to A/B$. Because of (ii), $\varphi \mid M$ is monic, so $\varphi(M)$ is a quasi-injective submodule of the R-module $A/B = \bigoplus_i A_i/B_i$ where A_i/B_i has been identified with $(A_i + B)/B$ under the canonical isomorphism. The maximal choice of B guarantees that no A_i/B_i has a non-zero submodule with 0 intersection with $\varphi(M)$, that is, $\varphi(M) \cap \cap (A_i/B_i)$ is essential in A_i/B_i, and so $\bigoplus_i [\varphi(M) \cap (A_i/B_i)]$ and a fortiori $\varphi(M)$ is essential in A/B. Now φ maps A_i into A/B, but since A_i is isomorphic to a submodule of $\varphi(M)$ and $\varphi(M)$ is fully invariant in its injective hull (which contains A/B), we see that $\varphi(A_i)$ must be contained in $\varphi(M)$. Consequently, φ maps the whole A into $\varphi(M)$, i. e. $\varphi(M) = A/B$, so M and B generate A. This proves $A = M \oplus B$.

An immediate consequence is:

COROLLARY 2. *Assume that*

$$A = M_1 \oplus \dots \oplus M_m = \bigoplus_{i \in I} N_i$$

are two direct decompositions of an R-module A where every M_j and every

N_i is a quasi-injective R-module, and I is an arbitrary index set. Then they have isomorphic refinements, i. e. there exist R-modules A_{ji} $(j = 1, \ldots, m;\ i \in I)$ such that

$$M_j \cong \bigoplus_{i \in I} A_{ji} \quad \text{and} \quad N_i \cong A_{1i} \oplus \ldots \oplus A_{mi}$$

for every j and i, respectively.

An application of Theorem 3 yields $A = M_1 \oplus (\bigoplus_i N_i')$ for submodules N_i' of N_i. Write $N_i = N_i' \oplus A_{1i}$ to get $M_1 \cong \bigoplus_{i \in I} A_{1i}$ and $M_2 \oplus \ldots \oplus M_m \cong$ $\cong \bigoplus_i N_i'$. A simple induction completes the proof.

It is an open problem whether or not two infinite decompositions have isomorphic refinements [5].

---

(5) This holds for injectives as was shown by Warfield [9].

REFERENCES

[1] H. Cartan and S. Eilenberg, *Homological algebra* (Princeton, 1956).

[2] P. M. Cohn, *On the free product of associative rings*, Math. Z. 71 (1959), 380-398.

[3] P. Crawley and B. Jónsson, *Refinements for infinite direct decompositions of algebraic system*, Pac. Journ. Math. 14 (1964), 797-855.

[4] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology* (Princeton, 1952).

[5] C. Faith, *Lectures on injective modules and quotient rings*, Lecture Notes in Mathematics, vol. 49 (1967).

[6] L. Fuchs, *Algebraically compact modules over Noetherian rings*, Indian Journ. Math. 9 (1967), 357-374.

[7] A. Kertész, *Systems of equations over modules*, Acta Sci. Math. Szeged 18 (1957), 207-234.

[8] R. B. Warfield, Jr., *Purity and algebraic compactness for modules*. Pac. Journ. Math. 28 (1969), 699-720.

[9] R. B. Warfield, Jr., *Decompositions of injective modules*.