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Traces of potentials arising from translation invariant operators


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TRACES OF POTENTIALS ARISING
FROM TRANSLATION INVARIANT OPERATORS

by D. R. ADAMS

For a function $u$ in the space $W^{\alpha,p}(E_n)$ (the usual Sobolev space of $L_p$ functions on Euclidean $n$-space $E_n$ with distribution derivatives of orders $\leq \alpha$ in $L_p$), it is possible to characterize the « restriction » or trace of $u$ (call it $u^*$) to certain lower dimensional manifolds $M$ provided their dimension $d$, satisfies $d > n - \alpha p$.

In this paper the characterization is given in terms of the Lebesgue or $L_p$ class of $u^*$, where the norm of $u^*$ is taken with respect to an appropriate measure $\mu$ concentrated on $M$. In particular, it is known that if $n - d < \alpha p < n$ and $p > 1$, then $u^* \in L_r(M)$, $1 \leq r \leq dp/(n - \alpha p)$, when $M$ is «smooth» and $\mu$ is the surface area measure on $M$. The usual procedure for proving this is to first obtain the result for a subset of a $d$-dimensional hyperplane in $E_n$ and then extend via a change of variables to manifolds which are diffeomorphic images of $d$-dimensional coordinate patches. In such a method, the essence is to work coordinate wise, from $E_n$ down to the hyperplane. This paper presents a new method for achieving this, which in addition allows an extension of the trace result to sets $M$ of fractional Hausdorff dimension $d$, $0 < d \leq n$.

Since every $u \in W^{\alpha,p}$ can be represented as a Bessel potential of an $L_p$ function (see [1]), we will consider functions $u$ in the form of potentials $T(f)$, where $f \in L_p$ and $T \in S_\alpha$, $S_\alpha$ being the class of translation invariant operators of smoothness $\alpha$, $\alpha > 0$ (see section 1 for the definitions). Theorem 1 then states that for each class $S_\alpha$ there is a corresponding class of «appropriate» measures $\mu_1^\perp(d)$ for which $T(f) \in L^p_\perp(\mu_1)$, $p_\perp = dp/(n - \alpha p)$,
n - d < \alpha p < n. Here the set \( M \) is the support of \( \mu \). When \( T \) is the Riesz potential operator, Theorem 1 can be improved (Theorem 2). In this case, the condition \( \mu \in \mathcal{L}^+_{\gamma_1,d} \) is both necessary and sufficient for the map \( T: L_p \to L_p^* (\mu) \) to be continuous.

The class of measures \( \mathcal{L}^+_{\gamma_1,d} \) is closely related to the Hausdorff \( d \)-dimensional measure \( H_d \) by a well known theorem of O. Frostman (see [3]). In particular if \( H_d (M) > 0 \), then there is a measure \( \mu \) concentrated on \( M \) such that \( \mu \in \mathcal{L}^+_{\gamma_1,d} \) and \( \mu \neq 0 \).

Theorem 1 can also be viewed another way: if \( \mu_0 \) is given then the condition --- \( \mu_0 \) restricted to \( M \) belongs to \( \mathcal{L}^+_{\gamma_1,d} \) --- will be a sufficient condition on \( M \) to insure that \( T (f) \) has a \( L_p^* (\mu_0) \) trace on \( M \). For example, if \( \mu_0 = H_d \), \( d \) a positive integer, then any smooth compact manifold in \( E_n \) satisfies this condition.

I am grateful to G. Stampacchia for pointing out the result appearing in the appendix of [9]. It is the forerunner of lemmas 1 and 2.

Section 1 contains the preliminaries and section 2, the statements and proofs of the main results. In section 3, I have attempted to point out relationships between Theorem 1 and various results appearing in the literature.

1. Preliminaries.

1.1. Let \( A_\gamma \) denote the usual Banach space of all bounded H"older continuous functions of exponent \( \gamma > 0 \) defined on \( E_n \) (see for example [8] for a precise definition). A linear transformation \( T \) which maps \( A_\gamma \) into \( A_{\gamma+\alpha} \), \( \alpha > 0 \), boundedly and which commutes with translations will be termed a linear translation invariant operator of smoothness \( \alpha \) and the class of all such \( T \) will be denoted by \( S_\alpha \). Here we will be content to list the various properties of \( T \in S_\alpha \) needed for this paper.

If \( T \in S_\alpha \), then it is known (see [8]) that for \( 0 < \alpha < 1 \), \( T \) applied to any smooth function \( f \) is given by \( T (f) (x) = \int k (x - y) f (y) \, dy \) where \( k \), the kernel of \( T \), satisfies

\[
\int |k(x)| \, dx < \infty, \tag{1}
\]

\[
\int |k(x - y) - k(x)| \, dx \lesssim Q |y|^{\alpha}, \tag{2}
\]

\( Q \) a constant independent of \( y \). Here the symbol \( \int_{E_n} \, dx \) denotes integration over \( E_n \) with respect to \( n \)-dimensional Lebesgue measure \( m_n \).
Our main interest lies with the Riesz potential operator whose kernel is \( h_\alpha(x) = |x|^{n-\alpha} \) and the Bessel potential operator \( J^{\alpha} \). In general, \( J^{\alpha} \) is defined for all real \( \alpha \) by: \( J^{\alpha} \) is the mapping given by the convolution with a tempered distribution and whose Fourier Transform is \((2\pi)^{-n/2} \cdot (1 + |x|^2)^{-\alpha/2} \). When \( \alpha > 0 \), the kernel of \( J^{\alpha} \) is denoted by \( g_\alpha \) and satisfies, in addition to (1) and (2)

\[
0 < g_\alpha(x) \leq Q h_\alpha(x), \quad \text{for all } x \in E_n.
\]

Here \( Q \) is a constant independent of \( x \). For additional properties of \( g_\alpha \) see [1].

A basic feature of the map \( J^{\beta} \) is the fact that it is a bicontinuous isomorphism of \( A_\gamma \) to \( A_{\gamma+\beta} \) as long as \( \gamma + \beta > 0 \). From this, it easily follows that for any \( T \in S_\alpha, \ T = J^{\beta} T J^{-\beta}, \) i.e. \( T \) commutes with \( J^{\beta} \).

1.2. By \( \mathcal{M} \) we will understand the collection of all completions of Borel measures on \( E_n \) and by \( \mathcal{L}_1 \) those \( \mu \in \mathcal{M} \) for which \( \|\mu\|_1 = \text{total variation of } \mu < \infty \). We will use the Morrey space notation \( \mathcal{L}^{1;\alpha} \) to denote those \( \mu \in \mathcal{M} \) for which

\[
V_x (\mu, r) = |\mu| (\{ y : |x - y| < r \}) \leq Ar^d,
\]

for all \( x \in E_n \) and all \( r \geq 0 \). Here \( 0 < d \leq n \) and \( |\mu| \) denotes the sum of the positive and negative parts of \( \mu \). \( A \) is a constant independent of \( x \) and \( r \).

The notation \( \|\cdot\|_p \) will represent the usual Lebesgue \( p \)-norm, \( 1 \leq p \leq \infty \), with respect to \( m \). For any other measure \( \mu \in \mathcal{M}^+, \) the symbol \( \|\cdot\|_{p,\mu} \) will be used. \( L_p \) and \( L_p(\mu) \) will denote the corresponding Lebesgue function spaces. \( \mathcal{M}_0 \) denotes the measures with compact support.

The superscript \( « + » \) is used to indicate the subclass of non-negative elements. The letter \( Q \) will denote various constants, possibly not the same constant in any one proof, whereas \( A, A_1, \) etc. will denote specific constants.

2. The main results.

2.1. The results of principal interest are Theorems 1 and 2 below

**Theorem 1**: For \( T \in S_\alpha \) and \( \mu \in \mathcal{L}^{1;\alpha}_+, \) there exists a constant \( Q \) such that for all \( f \) in \( L_p \),

\[
\| T(f) \|_{p',\mu} \leq Q \| f \|_p
\]

provided \( n - d < ap < n, \ 0 < d \leq n, \ p > 1 \). Here \( p^* = dp/(n - ap) \) and \( Q \) is independent of \( f \).
REMARK 1: Special cases of Theorem 1 are known, e.g. when \( \mu = m_n \), it is the Theorem of Stein-Zygmund (see Section 2.4); when \( \mu = m_d \), \( d \) an integer, and \( T \) the Riesz potential operator, it becomes the imbedding result of II' in [5].

The program for proving Theorem 1 will be: (a) to establish necessary and sufficient conditions on a non-negative Borel measure \( \mu \) in order that the above inequality holds for the Riesz potential operator, (b) to show then that Theorem 1 holds for the Bessel potential operator (using (3)), and finally (c), to establish Theorem 1 for general \( T \) (using (b) and the Theorem of Stein-Zygmund). Thus the main burden of the proof of Theorem 1 is in establishing (a). This can be stated as follows:

**Theorem 2:** The necessary and sufficient condition for

\[
\| h_\alpha \ast f \|_{p', \mu} \leq Q \| f \|_p
\]

to hold for all \( f \in L_p, Q \) a constant independent of \( f \), with \( \mu \in \mathcal{M}^+ (p' = \frac{dp}{n - \alpha p}, 0 < d \leq n, n - d < \alpha p < n, p > 1) \) is that \( \mu \in L^+_d \).

2.2. The proof of the sufficiency for Theorem 2 involves an estimate on the number \( \| h_\alpha \ast \mu^k \|_{p'} \), where \( p' = \frac{p}{p - 1} \) and \( \mu^k \) denotes \( \mu \) restricted to \( K \), \( K \) a compact set in \( E_n \) of positive \( \mu \) measure. To obtain the desired estimate, two main cases are considered, namely \( 1 < p \leq 2 \) and \( p > 2 \). In the first case, \( h_\alpha \ast \mu^K \) is estimated in the \( L_\infty \) norm and then \( h_\alpha \ast \mu^K \) in the \( L_p \) norm (lemmas 1 and 2).

For the second case we note that

\[
\| h_\alpha \ast \mu^K \|_{p'} = \int h_\alpha \ast (h_\alpha \ast \mu^K)^{1/(p-1)} \, d\mu^K = \int u^K \, d\mu^K
\]

where \( u^K(x) = h_\alpha \ast f^K(x), f^K(y) = [h_\alpha \ast \mu^K(y)]^{1/(p-1)} \). Hence it suffices to estimate \( u^K \) in the \( L_\infty \) norm. To do this, observe that

\[
u^K(x) = \int h_\alpha (x - y) f^K(y) \, dy = \int_0^\infty h_\alpha (r) \, dV_x (f^K, r).
\]

Here \( V_x (f^K, r) \) is given by (4), for a measure with density \( f^K \). In lemmas 3-5, estimates for the functions \( f^K(y) \) and \( V_x (f^K, r) \) are obtained. Finally, lemma 6 is the desired estimate on \( u^K \).
Lemma 1: \( h_{ap} \ast \mu^K (x) \leq A_1 \mu (K)^{\omega p - n + d} \), for all \( x \in E_n \), \( A_1 = 1 + A (n - ap)/(\omega p - n + d) \).

Proof: \( h_{ap} \ast \mu^K (x) = \int_0^\infty h_{ap}(r') d V_x(\mu^K, r) \) and altho the function \( V_x(\mu^K, r) \) is not in general continuous in \( r \) (for each fixed \( x \)), it is non-decreasing and left continuous. This formula follows from the definitions of the integrals involved.

Integrating by parts we get

\[
\int_0^\infty h_{ap}(r') d V_x(\mu^K, r) = - \int_0^\infty V_x(\mu^K, r) d h_{ap}(r) = (n - ap) \int_0^\infty V_x(\mu^K, r) r^{ap - n - 1} \, dr,
\]

since as \( r \to 0 \), \( h_{ap}(r) \cdot V_x(\mu^K, r) \leq A r^{ap - n + d}, \ ap - n + d > 0 \) whereas \( h_{ap}(r) \cdot V_x(\mu^K, r) \leq \mu (K)^{\omega p - n} \), as \( r \to \infty \), \( ap - n < 0 \). Thus

\[
h_{ap} \ast \mu^K (x) \leq (n - ap) \left( \int_0^\sigma + \int_\sigma^\infty \right) V_x(\mu^K, r) r^{ap - n - 1} \, dr = (n - ap) (I_1 + I_2).
\]

\[
I_1 \leq A \int_0^\sigma r^{d + ap - n - 1} \, dr = \frac{A}{ap - n + d}, \omega p - n + d,
\]

\[
I_2 \leq \mu (K) \int_\sigma^\infty r^{ap - n - 1} \, dr = \frac{1}{n - ap} \cdot \mu (K), \omega p - n.
\]

The result now follows by choosing \( \sigma = \mu (K)^{1/d} \).

Lemma 2: For \( 1 < p \leq 2 \),

\[
\| h_a \ast \mu^K \|_{p'} \leq A_2 \cdot \mu (K)^{\omega p - n + dp/dp},
\]

where \( A_2 \cdot C \left( \frac{\omega p}{2}, \frac{\omega p}{2} \right) \cdot A_1^{1/p} \cdot C (\alpha, \beta) \) is the Riesz convolution constant, i.e. \( h_a \ast h_\beta = C (\alpha, \beta), h_a \ast h_\beta \) for \( \alpha + \beta < n \).
Proof: For $1 < p < 2$ (the case $p = 2$ can be handled by a trivial modification) choose $\theta : 0 < \theta < 1$ and $1 = \theta p / 2 + (1 - \theta) p$, then

$$h_a (x) = h_{ap/2} (x)^\theta \cdot h_{ap} (x)^{1-\theta}.$$  

Thus from Hölder's inequality, we have

$$h_a \ast \mu^K (y) \leq [h_{ap/2} \ast \mu^K (y)]^p \cdot [h_{ap} \ast \mu^K (y)]^{1-p},$$

and

$$(7) \quad \| h_a \ast \mu^K \|_{p'} \leq \| h_{ap} \ast \mu^K \|_{\infty}^{(1-2p')/p'} \| h_{ap/2} \ast \mu^K \|_{2/p'}$$

by the choice of $\theta$, i.e. $\theta p' = 2$. But

$$(8) \quad \| h_{ap/2} \ast \mu^K \|_{2} = C \left( \frac{ap}{2} \cdot \frac{ap}{2} \right) \int h_{ap} \ast \mu^K d\mu^K$$

$$\leq C \left( \frac{ap}{2} \cdot \frac{ap}{2} \right) \| h_{ap} \ast \mu^K \|_{\infty} \cdot \mu (K).$$

(7) and (8), together with lemma 1, now give the desired result.

Lemma 3: For $p > 2$,

$$V_x (f^K, r) \leq A_3 \mu (K)^{1/p-1} \cdot r^{n-(n-a)/(p-1)}$$

for all $x \in E_n$ and all $r \geq 0$. $A_3 = \omega_n (1 + 3^a/a)^{1/(p-1)}$, $\omega_n$ = area of the unit sphere in $E_n$.

Proof: Since $p > 2$, Hölder's inequality gives

$$(9) \quad V_x (f^K, r) \leq (\omega_n r^n)^{1-1/(p-1)} \cdot [V_x (h_a \ast \mu^K, r)]^{1/(p-1)}$$

$$= (\omega_n r^n)^{1-1/(p-1)} \cdot [I_3 + I_4]^{1/(p-1)},$$

where

$$I_3 = \int_{K \setminus \{ |x - z| > 2r \}} d\mu (z) \int_{|x-y| < r} h_a (y - z) dy$$

which never exceeds

$$\int_{K \setminus \{ |x - z| > 2r \}} d\mu (z) \int_{|x-y| < r} h_a (r) dy$$

since $|y - z| \geq r$. Thus $I_3 \leq r^{n-a} \cdot \omega_n \cdot r^n \cdot \mu (K)$. 


And
\[ I_4 = \int_{K \cap \{|x-z| \leq 2r\}} d \mu(z) \int_{|y-z| < r} h_a(y-z) \, dy \]
\[ \leq \int_{K \cap \{|x-z| \leq 2r\}} d \mu(z) \int_{|y-z| \leq 3r} h_a(y-z) \, dy, \]
Since now $|y-z| \leq 3r$. Thus $I_4 \leq o_n \mu(K)(3r)^n/\alpha$.

**Lemma 4:** For $p > 2$ and $0 < \alpha < n - d$,
\[ V^*_{\alpha} (f^K, r) \leq A_{4r} r^{n-(n-a-d)/(p-1)} \]
for all $x \in E_n$ and all $r \geq 0$. $A_4 = o_n A^{1/(p-1)} \left( \frac{n-a}{n-a-d} + \frac{d^*}{\alpha} \right)^{1/(p-1)}$.

**Proof:** Equivalently this lemma asserts that $f^K$ belongs to the Morrey class $L_{(1:b)}$, with $b = n - (n - \alpha - d)/(p - 1)$ when $\alpha < n - d$ (compare this to lemma 5).

For fixed $x$, we consider $y$ such that $|x-y| < r$, then
\[ h_a \ast f^K(y) \leq \left( \int_{|x-z| \geq 2r} + \int_{|x-z| < 2r} \right) h_a(y-z) \, d \mu(z) = I_5 + I_6. \]

Let
\[ \varphi_y(q) = \int_{|x-z| \geq 2r \cap \{|y-z| < q\}} \, d \mu(z) \]
and note:

(i) $\varphi_y(q) = 0$, when $0 \leq q \leq r$;

(ii) $\varphi_y(q) \leq V^*_\alpha(\mu, q) \leq Aq^\alpha$, for all $y \in E_n$ and $q \geq 0$;

(iii) For each fixed $y$, $\varphi_y(q)$ is non-decreasing in $q$ and left continuous.
\[ I_5 = \int_{r}^{\infty} h_a(y) \, d \varphi_y(q) = (n - \alpha) \int_{r}^{\infty} \varphi_y(q) \, q^{n-\alpha-1} \, dq \]
using (i) and then integrating by parts.

Also note that $h_a(q) q y (q) \to 0$, as $q \to \infty$ by (ii). Thus

$$I_5 \leq (n - \alpha) A \int_r^\infty q^{n-\alpha-\alpha-1} d_0 = \frac{(n - \alpha)}{(n - \alpha - d)} A r^{n-\alpha+\alpha}.$$  

$$\int_{|z| < r} I_6 dy \leq \int_{|z| < 2r} d \mu (z) \int_{|y-z| < 3r} h_a (y-z) dy = \omega_n (3r)^{n/\alpha} \cdot V_{n} (\mu, 2r).$$

With these estimates and (9) of lemma 3, the result follows.

**Lemma 5:** For $p > 2$ and $0 \leq n-d < \alpha$,

$$f^K (y) \leq A_5 \cdot \mu (K)^{(d+a-n)/d (p-1)}$$

for all $y \in E_n$. $A_5 = [A (1 + (n-\alpha)/((\alpha - n + d)) + 1]^{1/(p-1)}$.

**Proof:** In contrast to lemma 4, $f^K$ is no longer in a Morrey class, but in a Hölder class with exponent $(d + \alpha - n)/(p - 1)$.

$$[f^K (y)]^{p-1} = \int h_a (y-z) d \mu^K (z)$$

$$= \left( \int_{|z| > r} + \int_{|z| \leq r} \right) h_a (y-z) d \mu^K (z) = I_7 + I_8.$$

Again with $\sigma = \mu (K)^{1/d}$,

$$I_7 \leq \sigma^{n-\alpha} \mu (K),$$

$$I_8 \leq \sigma \int_0^\sigma h_a (q) d V_y (\mu, q)$$

$$\leq h_a (\sigma) V_y (\mu, \sigma) + (n-\alpha) \int_0^\sigma V_y (\mu, q) q^{n-\alpha-1} d q$$

$$\leq \left( A + \frac{(n-\alpha)}{(\alpha - n + d)} A \right) \sigma^{n-\alpha+\alpha},$$

the result now follows easily.
LEMMA 6: For $p > 2$, there is a constant $A_6$ independent of the set $K$ such that

$$u^K(x) \leq A_6 \mu(K)^{(ap-n+d)/(p-1)}.$$ 

Hence by (5), $\|h_\alpha \ast \mu^K\|_{p'} \leq A_6^{1/p'} \mu(K)^{(ap-n+dp)/dp}$.

PROOF: case (1) $0 < \alpha < n - d$: Integrating by parts in (6),

$$u^K(x) = (n - \alpha) \int_0^\infty V_x(f^K, r) r^{n-\alpha-1} dr$$

since by lemma 4, $V_x(f^K, r) \cdot h_\alpha(r)$ is $O(r^{ap-n+d}/(p-1))$ as $r \to 0$, and is $O(r^{ap-n}/(p-1))$ as $r \to \infty$, by lemma 3. Thus

$$u^K(x) = (n - \alpha) \left( \int_0^\sigma + \int_0^\infty \right) V_x(f^K, r) r^{n-\alpha-1} dr = (n - \alpha) (I_9 + I_{10}).$$

Applying lemma 4 to $I_9$ and lemma 3 to $I_{10}$, we have

$$I_9 \leq \frac{A_4 (p - 1)}{(ap - n + d)} \cdot o'^{ap-n+d}/(p-1),$$

and

$$I_{10} \leq \frac{A_3 (p - 1)}{(n - \alpha p)} \cdot \mu(K)^{1/(p-1)} \cdot o'(ap-n)/(p-1).$$

The result follows taking $\sigma = \mu(K)^{1/d}$.

case (2) $0 \leq n - d < \alpha$:

$$u^K(x) = \left( \int_{|x-y| \leq \sigma} + \int_{|x-y| > \sigma} \right) h_\alpha(x - y) f^K(y) dy = I_{11} + I_{12}.$$

Applying lemma 5 to $I_{11}$ and lemma 3 to $I_{12}$, we have

$$I_{11} \leq \frac{A_5 \alpha o_\alpha}{\alpha} \mu(K)^{(d+a-n)/d/(p-1)} \cdot o^n,$$

$$I_{12} \leq \frac{(n - \alpha)(p - 1)}{(n - \alpha p)} A_3 \mu(K)^{1/(p-1)} \cdot o'(ap-n)/(p-1),$$

with the same choice of $\sigma$. 

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case (3) $0 < \alpha = n - d$: This case is resolved by interpolating between cases (1) and (2) as follows: choose pairs $(\alpha_i, p)$ with $n - d \leq \alpha_i p < n$, $i = 0, 1$ but $0 < \alpha_0 < n - d < \alpha_1 < n$. Let $\alpha = \theta \alpha_0 + (1 - \theta) \alpha_1$, $0 < \theta < 1$.

As before $h_a(x) = h_{a_0}(x^0) \cdot h_{a_1}(x^1)$ and upon applying Hölder’s inequality, we have

$$f^K(y) \leq [f^K_0(y)]^{\theta} \cdot [f^K_1(y)]^{1-\theta}$$

where $f^K_i(y) = [h_{a_i} \cdot \mu^K(y)]^{1/(p-1)}$, $i = 0, 1$. Thus

$$u^K(x) \leq [h_{a_0} \cdot f^K_0(x)]^{\theta} \cdot [h_{a_1} \cdot f^K_1(x)]^{1-\theta} = [u^K_0(x)]^{\theta} \cdot [u^K_1(x)]^{1-\theta}.$$  

Case (1) gives $u^K_0(x) \leq A_0 \mu(K)^{\alpha_0 p/(n-p)}$ and case (2) gives $u^K_1(x) \leq A_0 \mu(K)^{\alpha_1 p/(n-p)}$. Hence it is now clear that a finite constant $A_6$ may be chosen with the required properties.

2.3. PROOF OF THEOREM 2: For the sufficiency, lemmas 2 and 6 are used to show that for fixed $\alpha$, the Riesz potential operator is of weak type $(L_p, L_{p^*}((\mu)))$, when $n - d < \alpha p < n$.

Let $E_t = \{x : h_a \cdot f(x) > t\}$, $t > 0$ and $f \in L_p$.

$$t \mu(E_t) \leq \int h_a \cdot f |(x) \, d\mu(x) = \int h_a \cdot \mu(E_t(x) \cdot f(x) \, dx \leq \|h_a \cdot \mu(E_t)\|_p \|f\|_p.$$

Now since lemmas 2 and 6 hold for all compact sets $K$, and the constants $A_2$ and $A_6$ are independent to $K$, these estimates must also hold for $K$ replaced by $E_t$, a $G_\delta$-set, since $\mu$ is a Borel measure. Thus

$$t \mu(E_t) \leq Q^\prime \|f\|_p \mu(E_t)^{\alpha p - n + dp} \|dp \| dp$$

or

$$\mu(E_t) \leq \left(\frac{Q^\prime \|f\|_p}{t}\right)^{p^*}, \quad p^* = dp/(n - \alpha p).$$

We now apply the well known interpolation theorem of Marcinkiewicz to deduce the strong type estimate required.

To prove the necessity, we choose a particular $L_p$ function, namely the characteristic function of the ball $B_r(x_0) = \{x : |x - x_0| < r\}$, $r > 0$ and $x_0 \in E_n$ arbitrary. Denote this function by $\chi_r(x)$. $\|\chi_r\|_p = (\omega_n r^{n})^{1/p}$. On the other hand

$$\|h_a \cdot \chi_r\|_{p^*}^{1/p^*} \geq \left\{ \int \left(\int_{|x-y| < r} h_a |x-y| \, dy \right)^{p^*} \, d\mu(x) \right\}^{1/p^*}$$

$$\geq h_a(2r) \omega_n r^n |V_{x_0}(\mu, r)|^{1/p^*}.$$
Thus if $h_a : L_p \to L_{p^*}(\mu)$ is continuous, we immediately get
\[ V_{\omega_0}(\mu, r) \leq A' r^d. \]

The proof of Theorem 2 is now complete.

REMARK 2: It is interesting to note that the region in the $(p, \alpha)$ plane $1 < p < \infty, 0 < \alpha < n$, for which the above result holds is the region between the two hyperbolas $\alpha p = n - d$ and $\alpha p = n$. It is possible to «shift» this region to obtain a result of additional interest (see Remark 6).

Making the changes: $\mu \to \mu_2$, $m_n \to \mu_1$ and $h_a \to h_{a+d_1}$, where $0 \leq d_1 < d_2 \leq n$, $\mu_1 \in \mathcal{L}_{1; n-d_1}$ and $\mu_2 \in \mathcal{L}_{1; n-d_2}$, we easily get
\[ \| h_{a+d_1} * f_{\mu_1} \|_{p^*, \mu_1} \leq Q \| f \|_{p, \mu_1} \]
where $p^* = (d_2 - d_1) p/(n - \alpha p - d_1)$, $n - d_2 < \alpha p < n - d_1$, $p > 1$. Here $f_{\mu_1}$ denotes a measure with density $f$.

2.4. To see that Theorem 1 holds for $h_a$ replaced by $g_a$, it is only necessary to combine (3) with Theorem 2.

Necessary and sufficient conditions on $\mu$ for $g_a$ are possible only if the variation of $\mu$ is allowed to grow more rapidly at infinity.

PROOF OF THEOREM 1: This extension of Theorem 2 can now be established by applying the theorem of Stein-Zygmund [8]. This result may be stated as follows:

THEOREM: If $T \in S_a$, then there is a constant $Q$ such that
\[ \| T(f) \|_q \leq Q \| f \|_p, \quad q = np/(n - \alpha p) \]
for all $f \in L_p$; $Q$ is independent of $f$. $1 < p < \infty, 0 < \alpha p < n$.

For $T \in S_a$, choose $\beta$ satisfying $\left(\frac{n-d}{dp}\right)_{n-\alpha p} < \beta < \alpha$. Note that this is always possible since $\alpha p > n - d$. From 1.1, $T \cdot J^{-\beta} \in S_{a-\beta}$, thus the theorem of Stein-Zygmund yields
\[ \| T \cdot J^{-\beta}(f) \|_q \leq Q_1 \| f \|_p \]
where $q = np/[n - (\alpha - \beta) p]$. Using Theorem 2 (which we now know is true for the Bessel potential operators),
\[ \| T(f) \|_{p^*, \mu} = \| J^\beta(T \cdot J^{-\beta}(f)) \|_{q^*, \mu} \leq Q_2 \| T \cdot J^{-\beta}(f) \|_q \]
where \( q^* = \frac{dq}{n - \beta q} = \frac{dp}{n - \alpha p} = p^* \). Hence
\[
\| T(f) \|_{p^*, \mu} \leq Q_2 \cdot Q_1 \| f \|_p.
\]
Note that the conditions \( 1 < q < q^* < \infty \) are satisfied when \( 1 < p < \infty \), \( n - d < \alpha p < n \), and by the choice of \( \beta \).

**Remark 3:** It might be noted that if the more general interpolation theorem of R. Hunt [4] had been used in place of the theorem of Marcinkiewicz, it would be possible to deduce that any \( T \in S_a \) maps the Lorentz space \( L(p, q) \) continuously into \( L(p^*, s)(\mu) \) with \( q \leq s \), the usual restrictions on \( \alpha, p \) and \( d \). In particular when \( q = s = \infty \), \( T \) maps weak \( L_p \) continuously into weak \( L_{p^*}(\mu) \).

### 3. Related results.

3.1. We begin by giving potential versions of two classical trace theorems.

**Theorem 3:** Let \( T \in S_a \) and \( \mu \in \mathcal{E}^+_{1,d} \), then there is a constant \( Q \) such that for all \( f \in L_p \)
\[
\| \Lambda_t T(f) \|_{r, \mu} \leq Q \left| t \right|^{-n[p + d/r]} \| f \|_p
\]
where \( \max [dp/[n - (\alpha - 1)p], p] < r < p^* \), \( n - d < \alpha p < n \). Here \( \Lambda_t \) denotes the first difference; \( Q \) is to be independent of \( f \) and \( t \).

**Proof:** The restrictions on \( r \) insure that the exponent of \( \left| t \right| \) is always positive and less than 1. It is easy to see that \( \Lambda_t T(f)(x) = (\Lambda_t k) \ast f(x) \), where \( k \) is the kernel of \( T \).

We write \( T = (J^{-\theta} \cdot T) \cdot J^\theta = k_{a-\theta} \ast g_{\theta} \) where \( k_{a-\theta} \) is the kernel of \( J^{-\theta} \). \( T, \theta \) chosen to satisfy initially \( (n - d)/p < \theta < \alpha \). Then
\[
\Lambda_t T(f)(x) = \int (k_{a-\theta}(x) \ast g_{\theta} \ast f_z(x)) \, dz
\]
since \( k = k_{a-\theta} \ast g_{\theta} \). Here \( f_z(y) \) denotes \( f(y - z) \). By the inequality of Minkowski and Theorem 1, we have
\[
\| \Lambda_t T(f) \|_{r, \mu} \leq \int |\Lambda_t k_{a-\theta}(z)| \cdot |g_{\theta} \ast f_z|\, dz
\]
\[
\leq \int |\Lambda_t k_{a-\theta}(z)| \cdot Q \| f_z \|_p \, dz
\]
where \( r = dp/(n - \theta p) \). Since \( \|f_i\|_p = \|f\|_p \), we have, using (2)
\[
\|A_\mu T(f)\|_{r, \mu} \leq Q \|f\|_p |t|^{\alpha - \theta}, \quad 0 < \alpha - \theta < 1.
\]
But \( \theta = n/p - d/r \), hence the theorem follows.

**Theorem 4**: For \( T \in S_n \), \( f \mapsto T(f) \) is a compact mapping of \( L_p \) into \( L_r(\mu) \) for any \( \mu \in \mathcal{M}_+^+ \cap \mathcal{L}_{1-d}^+ \), \( 1 \leq r < p^* \), \( n - d < \alpha p < n \).

**Proof**: Let \( \{f_k\} \) be a bounded sequence in \( L_p \), then there exists a weakly convergent subsequence \( f_k \rightarrow f, f \in L_p \). By (1) and (2) \( k_{n-\theta} \ast f_k \rightarrow k_{n-\theta} \ast f \) strongly in \( L_p \) locally, by the familiar Riesz compactness criterion. But since \( \mu \in \mathcal{M}_q \), \( T(f_k) \rightarrow T(f) \) in \( \mu \) measure and thus using Theorem 1 the result follows by a standard argument.

3.2. We now consider a «dual» to Theorem 1 and then apply it to obtain an extension of a theorem of Campanato [2].

**Theorem 5**: Let \( T \in S_n \) and \( \mu \in \mathcal{L}_{1-d}^+ \), then there is a constant \( Q \) such that for all \( g \in L_q(\mu) \),
\[
\|T(g, \mu)\|_{q, \mu} \leq \|g\|_{q, \mu} Q
\]
where \( q = np/(d + q(n - \alpha - d)) \) and \( 1 < q < \bar{q} < \infty \). Here \( Q \) is independent of \( g \).

**Proof**: Let \( g \in L_{p^*}(\mu) \) and \( f \in L_p \), \( p^* = dp/(n - \alpha p) \), then
\[
\int T(g, \mu) f \, dx = \int T(f) g \, d\mu \leq \|T(f)\|_{p^*, \mu} \|g\|_{p^*, \mu} \leq Q \|f\|_{p^*} \|g\|_{p^*, \mu},
\]
the last inequality following from Theorem 1. The result now follows by taking \( q = p^* \) and \( \bar{q} = p^* \).

Let \( \mathcal{L}_{1-n}^\lambda \), \( 1 \leq t < \infty \), \( 0 < \lambda < n \), denote the class of measures \( \mu \in \mathcal{L}_{1-n}^\lambda \) which are absolutely continuous with respect to \( m_n \) with density \( f \) satisfying \( |f|^t \cdot m_n \in \mathcal{L}_{1-n}^\lambda \).

**Theorem 6**: If \( \mu \in \mathcal{L}_{1-n}^\lambda \), then \( h_{n-\theta} \ast \mu \in \mathcal{L}_{1-n-\theta}^\lambda(\lambda - \alpha) \) where \( 1 \leq t < \lambda/(\lambda - \alpha) \), \( 0 < \alpha < \lambda \).
PROOF: In Theorem 5, take \( g(x) = \chi_{2r}(x) \), the characteristic function of the ball \( B_{2r}(x_0) \), and \( t = \tilde{q}, \tilde{q}/q = (n - t(\lambda - \alpha))/(n - \lambda), \tilde{d} = n - \tilde{\lambda} \), then

\[
\int_{|x-x_0|<\tilde{r}} \left| h_a \ast \mu_{2r} \right|^t \, dx \leq Q' r^{n-t(\lambda-a)}
\]

where \( \mu_{2r} \) is \( \mu \) restricted to \( B_{2r}(x_0) \). The condition \( 1 < q < \tilde{q} < \infty \) is equivalent to \( n/(n-\alpha) < t < \lambda/(\lambda - \alpha) \).

It now remains to estimate the \( t \)-power of the variation over \( B_r(x_0) \) of \( h_a \ast (\mu - \mu_{2r}) \). However, this quality is just the variation of \( |I_5|^t \) over \( B_r(x_0) \) (\( I_5 \) as in lemma 4) with \( x_0 \) playing the role of \( x \) and \( \alpha < \lambda \).

Next, if \( 1 \leq t < n/(n-\alpha) \), then

\[
\left\{ \int_{|x-x_0|<r} \left| h_a \ast \mu_{2r} \right|^t \, dx \right\}^{1/t} \leq \int_{|y-x_0|<2r} \left[ \int_{|x-x_0|<r} h_a(x-y)^t \, dx \right]^{1/t} \, d|\mu|(y) \leq Qr^{n-t(\lambda-a)/t}.
\]

The remaining integral is handled as before.

Finally, for any \( t \) in the interval \([1, \lambda/(\lambda - \alpha)]\), a simple interpolation argument gives the result.

REMARK 4: The result of Campanato [2] can be stated as follows: If \( f \in L_p; n-\alpha \), then \( h_a \ast f \in L_t; n-\sigma \) where \( \alpha p < \lambda, 1 \leq t < \lambda p/(\lambda - \alpha p) \), and \( \lambda > \sigma > t(\lambda - \alpha p)/p \), \( 1 < p < \infty \).

His proof fails when \( p = 1 \), which Theorem 6 now treats. Note, when \( p = 1 \), it is possible to take \( \sigma = t(\lambda - \alpha) \).

REMARK 5: It might be interesting to find conditions on \( \mu \) for which \( t = \lambda/(\lambda - \alpha) \) is allowed in Theorem 6, for in general, it is known that \( \mu \in L_{1, -1} \) is not sufficient. Indeed, if \( \lambda = \alpha p \), then \( \lambda/(\lambda - \alpha) = p' \) and such a \( \mu \) does not even insure that \( h_a \ast \mu \in L_{p'} \) locally (for this see [6]). From the proof of Theorem 6, it appears that any condition on \( \mu \) which insures \( \| h_a \ast \mu_{2r} \|^t \leq Qr^{n-t}, t = \lambda/(\lambda - \alpha) \) will be sufficient. Two such conditions are:

(i) \( h_a : L_{\lambda/a} \rightarrow L_{\lambda/a}(\mu) \) continuously, \( 0 < \alpha < \lambda \), and

(ii) \( h_a \ast (h_a \ast \mu)^{n/(\lambda - \alpha)} \) is bounded on \( E_n \).

The proofs require no new ideas.
REMARK 6: Finally, we observe that (10) is an extension of a theorem of Stein and Weiss [7] — see in particular their Theorem B* in the case $p < q$, which corresponds to our case $p < p^\ast$. The example referred to in the above remark easily shows that no extension of this generality is possible when $p = p^\ast$.

REFERENCES