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Properties of hypersurfaces which are characteristic for spaces of constant curvature

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PROPERTIES OF HYPER\textit{SURFACES}
WHICH ARE CHARACTERISTIC FOR SPACES
OF CONSTANT CURVATURE

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If is well-known that the spaces of constant curvature are characterized among all Riemannian spaces by the following property: \textit{for any} \((n - 1)\)-dimensional linear element \(E_{n-1}\) of a Riemann space \(N\) (dim \(N = n \geq 3\)) there is a totally geodesic hypersurface \(M \subset N\) which is tangent to \(E_{n-1}\). (Cf. [1]). The purpose of this Note is to present a number of theorems of the above type; only the requirement that our hypersurfaces should be totally geodesic will be replaced be another geometrical or analytical postulates. Umbilical points of hypersurfaces and so called \textit{normal Bianchi identity} play the leading part here.

Throughout the paper we shall keep all the notations and conventions of the famous book by Kobayashi and Nomizu ([2], [3]).

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Let \(N\) be a Riemannian manifold of dimension \(n \geq 3\), \(g\) the corresponding Riemann metric on \(N\) and \(M \subset N\) a hypersurface. Because all the postulates put on hypersurfaces will be purely local, we can suppose (if not otherwise stated) that \(M\) is a \textit{small} hypersurface, diffeomorphic with an open region in \(\mathbb{R}^{n-1}\).

Denote by \(\nabla, \nabla'\) the covariant differentiation on \(N, M\) respectively. Let \(\xi\) be a field of unit normal vectors and \(X, Y\) tangent vector fields on \(M\). The formulas of \textit{Gauss and Weingarten} are given by

\begin{align}
(1) \quad & \nabla_X Y = \nabla'_X Y + h(X, Y) \xi, \\
(2) \quad & \nabla_X \xi = - A X.
\end{align}

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Here \( h(X, Y) \) is the second fundamental form and \( A \) is a symmetric transformation on each tangent space \( T_x(M) \), \( x \in M \). Moreover, we have
\[
g(AX, Y) = g(X, AY) = h(X, Y).
\]

A point \( x \in M \) is called an umbilical point if \( A = \lambda \cdot I \) on \( T_x(M) \), where \( \lambda \) is a scalar and \( I \) is an identity transformation.

**Definition 1.** A hypersurface \( M \subset N \) is called a \( U \)-sphere if \( A = \lambda \cdot I \) on the tangent bundle \( T(M) \), where \( \lambda \) is a constant, \( \lambda \neq 0 \).

Denote by \( R, R' \) the Riemann curvature tensors of \( N, M \) respectively. At any point \( x \in M \) we have
\[
R(X, Y) \xi = (V_Y A)(X) - (V_X A)(Y),
\]
\[
\text{proj}_{T(M)} R(X, Y) Z = R'(X, Y) Z + g(AZ, Z) AY - g(AZ, Z) AX,
\]
where \( X, Y, Z \in T_x(M) \) and \( \xi \) is a normal vector to \( M \) at \( x \). (Cf. [3]).

We introduce new tensors \( A(X, Y), B(X, Y) Z \) on \( M \) by
\[
A(X, Y) = (V_Y A)(X) - (V_X A)(Y),
\]
\[
B(X, Y) Z = g(AZ, Z) AY - g(AZ, Z) AX.
\]

Let us remind the first Bianchi identity \( \sigma (R'(X, Y) Z) = 0 \) and the second Bianchi identity \( \sigma ((V_X R')(Y, Z)) = 0 \), where \( X, Y, Z \) are tangent vector fields on \( M \) and \( \sigma \) denotes the cyclic sum with respect to \( X, Y, Z \).

The tensor \( B(X, Y) Z \) defined by (6) also satisfies the first Bianchi identity but not the second one, in general.

**Definition 2.** We say that a hypersurface \( M \subset N \) satisfies the normal Bianchi identity if \( \sigma ((V_X B)(Y, Z)) = 0 \) for any vector fields \( X, Y, Z \) on \( M \).

**Remark.** This definition is independent of the choice of a normal unit vector field \( \xi \).

A routine calculation leads to the following

**Proposition 1.** The normal Bianchi identity holds on a hypersurface \( M \subset N \) if and only if
\[
A(X, Y) \wedge AZ + A(Y, Z) \wedge AX + A(Z, X) \wedge AY = 0
\]
in the vector bundle \( \Lambda^2 T(M) \).
If $N$ is a Riemann manifold with the constant curvature $C$, then

$$R(X, Y)Z = C[g(Y, Z)X - g(X, Z)Y].$$

Hence and taking into account (3) - (6) we obtain

$$\Delta(X, Y) = (V^2_XA)(Y) - (V^2_YX) = 0 \quad (Equation \ of \ Codazzi),$$

$$R(X, Y)Z = C[g(Y, Z)X - g(X, Z)Y] - K'(X, Y)Z \quad (Equation \ of \ Gauss).$$

Finally, (7) and (9) imply

PROPOSITION 2. If $N$ is a space of constant curvature then any hypersurface $M \subset N$ satisfies the Codazzi equation $\Delta(X, Y) = 0$ and also the normal Bianchi identity.

In order to prove a converse we shall remind some well-known definitions.

For each plane $p$ in the tangent space $T_x(N)$, i.e., for any 2-dimensional subspace of $T_x(N)$, the sectional curvature $K(p)$ is defined by $K(p) = R(e_1, e_2, e_1, e_2) = g(R(e_1, e_2)e_2, e_1)$, where $\{e_1, e_2\}$ is an orthonormal basis of $p$. (In the following we shall put $K(e_1, e_2) = R(e_1, e_2, e_1, e_2)$ for abbreviation).

A point $x \in N$ is called isotropic if the sectional curvature $K(p)$ is the same for any plane $p \subset T_x(N)$.

Now we shall present a number of lemmas the statements of which are well-known.

**LEMMA 1.** Suppose that there is an orthonormal basis $\{e_1, \ldots, e_n\}$ in $T_x(N)$ and a constant $C$ such that $K(e_i, e_j) = C$ for any $i, j = 1, \ldots, n$, $i \neq j$. Then $x$ is an isotropic point of $N$.

**LEMMA 2.** Suppose that $g(R(e_i, e_j)e_k, e_j) = 0$ for any orthonormal triplet $\{e_i, e_j, e_k\}$ of vectors of $T_x(N)$. Then $x$ is an isotropic point of $N$.

**PROOF.** Consider an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_x(N)$, and for any triplet of indices $[i, j, k] \subset [1, \ldots, n]$ put $f_i = \frac{e_i + e_k}{\sqrt{2}}, f_j = e_j, f_k = \frac{e_i - e_k}{\sqrt{2}}$. The equality $g(R(f_i, f_j)f_k, f_j) = 0$ implies $K(e_i, e_j) = K(e_k, e_j)$ and we can use Lemma 1.
LEMMA 3. (Schur's lemma). Let $N$ be a Riemannian manifold of dimension $n \geq 3$. If all points of $N$ are isotropic, then $N$ is a space of constant curvature. (Cf [2]).

Now we can derive

PROPOSITION 3. Let $N$ be a Riemannian manifold of dimension $n \geq 3$ with the following property: to any point $x \in N$ and any hyperplane $E_{n-1} \subset T_x(N)$ there is a hypersurface $M \subset N$ such that

a) $M \cap x, T_x(M) = E_{n-1}$, b) $M$ satisfies the Codazzi equation at $x$.

Then $N$ is a space of constant curvature.

PROOF. Let a point $x \in N$ and an orthonormal triplet $\{e_i, e_j, e_k\}$ of vectors of $T_x(N)$ be given. Denote $E_{n-1} = \{X \in T_x(N) \mid g(X, e_k) = 0\}$. Let $M \subset N$ be a hypersurface satisfying the conditions $a)$, $b)$ of the Proposition with respect to $E_{n-1}$. According to (3) we obtain $R(e_i, e_j) e_k = 0$. Now we apply Lemma 3 to complete the proof.

Propositions 2 and 3 give us

THEOREM 1. Let $N$ be a Riemannian manifold of dimension $n \geq 3$. Then the following two statements are equivalent:

(i) Any hypersurface $M \subset N$ satisfies the Codazzi equation $A(X, Y) = 0$.

(ii) $N$ is a space of constant curvature.

We are in a position to prove also the following

THEOREM 2. Let $N$ be a Riemannian manifold of dimension $n \geq 4$. Then the following two statements are equivalent:

(i) Any hypersurface $M \subset N$ satisfies the normal Bianchi identity.

(ii) $N$ is a space of constant curvature.

PROOF. The implication (ii) $\implies$ (i) was stated in Proposition 2. Let us prove the converse. Let $x \in N$ be a fixed point and consider a system of normal coordinates $[x^1, \ldots, x^n]$ in a neighbourhood $V_x$ of $x$. In this way the neighbourhood $V_x$ of $x$ can be represented as an open region $U_x$ of a coordinate space $\mathbb{R}^n(x^1, \ldots, x^n)$; the point $x$ is mapped onto the origin $0 \in \mathbb{R}^n$. The space $\mathbb{R}^n$ provided with its canonical euclidean metric is called the osculating euclidean space of $N$ at the point $x$. (Cf. [1]). We have a canonical isomorphism of $T_x(N)$ onto $T_0(\mathbb{R}^n)$. Suppose that $\{e_i, e_j, e_k\}$ is an orthonormal triplet of vectors of $T_x(N)$ and $\{e'_i, e'_j, e'_k\}$ the corresponding orthonormal triplet of $T_0(\mathbb{R}^n)$. 

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Let us construct a small piece $M'$ of a cylinder $S^1 \times \mathbb{R}^{n-2} \subset \mathbb{R}^n$ with the following properties:

- $M'$ passes through the origin $0 \in \mathbb{R}^n$ and $M' \subset U_x$,
- $M'$ is normal to $e_k$ at $0$,
- $M'$ with respect to the canonical identification

$$T_0(M') = T_0(S^1) \oplus T_0(\mathbb{R}^{n-2})$$

(orthogonal decomposition !)

we have $e_i', \ e_j' \in T_0(\mathbb{R}^{n-2})$.

Let $Z$ be a vector of $T_x(N)$ corresponding to a vector $Z' \in T_0(S^1)$.

If $M$ is a hypersurface of $N$ corresponding to $M'$, then it is well-known that the second fundamental form of $M \subset N$ at $x$ is the same as the second fundamental form of $M' \subset \mathbb{R}^n$ at $0$. Hence $Ae_i = A\epsilon_j = 0, AZ = \lambda \cdot Z$. Formula (7) implies $A(e_i, e_j) = \mu \cdot Z$ and from (3) we obtain $R(e_i, e_j)e_k = \mu \cdot Z$. Hence $g(R(e_i, e_j)e_k, e_l) = 0$ and $x$ is an isotropic point of $N$ according to Lemma 2. Now the Schur's lemma completes our proof.

**Remark.** If $N$ is a Riemannian manifold of dimension 3, then the normal Bianchi identity is trivially satisfied on each surface $M \subset N$.

The rest of this paper is devoted mainly to the study of umbilical points and $U$ spheres. The following theorem of the linear algebra is well-known and it will be useful for our further calculations:

**Lemma 4.** Let $g, h$ be two quadratic forms in a real vector space $\mathbb{R}^n$ and let $g$ be positively definite. Then there is a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ such that

$$g = \sum_{i=1}^{n} (\xi^i)^2, h = \sum_{i=1}^{n} \lambda_i (\xi^i)^2$$

with respect to the dual basis $\{\xi^1, \ldots, \xi^n\}$ of $(\mathbb{R}^n)^\ast$.

**Convention:** $N(C)$ will denote a Riemannian manifold with the constant curvature $C$.

**Proposition 4.** Let $N(C)$ be a space of dimension $n \geq 4$ and $M \subset N(C)$ a hypersurface. Then the following two statements are equivalent:

- (i) $M$ is a $U$-sphere.
- (ii) $M$ is a space $M(C')$, where $C' > C$.

**Proof.** Let us multiply (6) and (10) by the vector $X$ to the right and put $Z = Y$. We obtain $g(AX, Y)^2 - g(AX, X)g(AY, Y) = C [g(X, X) g(Y, Y) - g(X, Y)^2] - g(R'(X, Y) Y, X)$. where $g(AX, Y) = h(X, Y)$ is the second fundamental form of the hypersurface $M \subset N$. Denote by $K_M$ the
sectional curvature of $M$. If $X, Y$ are orthonormal vectors of $T_x(M)$, it follows

$$ g(AX, Y)^2 - g(AX, X)g(AY, Y) = C - K_M(X, Y). $$

The proof of (i) $\Rightarrow$ (ii): If $M$ is a $U$-sphere, $A = \lambda \cdot I$, $\lambda \neq 0$, we obtain $K_M(X, Y) = C + \lambda^2 = \text{const.}$, q.e.d.

The proof of (ii)$\Rightarrow$ (i): Let us re-write (11) in the form $h(X, Y)h(Y, Y) - h(X, Y)^2 = K_M(X, Y) - C$. If (ii) is satisfied, we obtain

$$ h(X, X)h(Y, Y) - h(X, Y)^2 = C' - C > 0 $$

for any orthonormal pair $\{X, Y\}$ of $T_x(M)$.

Let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal basis of $T_x(M)$ such that $h(X, Y)$ assumes a diagonal form with respect to this basis (Lemma 4). Put $h(e_i, e_i) = \lambda_i$ for $i = 1, \ldots, n - 1$; then (12) implies $\lambda_i\lambda_j = C' - C > 0$ for $i, j = 1, \ldots, n - 1 \geq 3$ as required, we obtain $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-1} = \pm \sqrt{C' - C}$ and $h = \pm \sqrt{C' - C} \cdot g$ on $M$. Hence follows $A = \pm \sqrt{C' - C} \cdot I$ on $M$; the sign depends on the orientation of $M$.

REMARK. If $\dim N(C) = 3$, the implication (ii)$\Rightarrow$ (i) is false as the following example shows: let $M$ be a small piece of a sphere $S^2$ in the euclidean space $E^3$; consider a non-trivial isometric deformation of $M$ in $E^3$. The deformed surface is a space of constant curvature but not a $U$-sphere.

Let $N$ be a Riemannian manifold. To any point $x \in N$ and any sufficiently small number $r > 0$ the locus of all points at the distance $r$ from $x$ is a regular submanifold $S(x; r)$ of dimension $n - 1$, called a metric sphere with center $x$ and radius $r$.

PROPOSITION 5. Let $N(C)$ be a space of dimension $n \geq 3$. Then any sufficiently small metric sphere in $N(C)$ is a space form $N(C')$ with $C' > C$.

PROOF. It suffices to cite some classical results only (cf. [5]).

For $C = 0$ the assertion is trivial.

For $C > 0$ (elliptic geometry) we put $k = 1/\sqrt{C}$. The following result it well-known: The metric of the space $N(1/k^2)$ induces on a sphere of radius $r$ the metric of a euclidean sphere of radius $k \cdot \sin(r/k)$.

For $C < 0$ (hyperbolic geometry) we put $k = 1/\sqrt{-C}$. Let us remember the following theorem: The metric of the space $N(-1/k^2)$ induces on a sphere of radius $r$ the metric of a euclidean sphere of radius $k \cdot \sinh(r/k)$. Hence our Proposition follows.
PROPOSITION 6. Let \( N(C) \) be a space of dimension \( n \geq 3 \). Then any sufficiently small metric sphere in \( N(C) \) is a \( U \)-sphere.

PROOF. If the dimension \( n \geq 4 \), the result follows from Propositions 4 and 5. For \( n = 3 \), we can identify \( N(C) \) with a totally geodesic hypersurface of a space form \( N'(C) \) of dimension 4. Any metric sphere in \( N(C) \) can be represented as an intersection of a metric sphere in \( N'(C) \) by \( N(C) \). Hence our result follows.

REMARK. The converse of Proposition 6 is true in the euclidean and the elliptic case only. As for a hyperbolic space \( N(-1/k^2) \), any \( U \)-sphere in \( N \) is locally either a metric sphere (for \( |\lambda| > 1/k \)), or a limit hypersurface bearing a euclidean metric (for \( |\lambda| = 1/k \)), or finally an equidistant hypersurface (for \( 0 < |\lambda| < 1/k \)).

PROPOSITION 7. Let \( N \) be a Riemannian space of dimension \( n \geq 3 \). Suppose that to any linear \((n - 1)\)-dimensional element \( E_{n-1} \) of \( N \) there is a \( U \)-sphere tangent to \( E_{n-1} \). Then \( N \) is a space of constant curvature.

PROOF. Let \( M \) be a \( U \)-sphere tangent to \( E_{n-1} \). We have \( A = \lambda \cdot I \) along \( M \), \( \lambda = \text{const} \). Hence we see that the Codazzi equation \((V' \cdot A)(X) = (V' \cdot A)(Y) = 0 \) holds at the base point \( x \) of \( E_{n-1} \). Now we can apply Proposition 3.

PROPOSITION 8. Let \( N \) be a Riemannian space of dimension \( n \). Then to any linear \((n - 1)\)-dimensional element \( E_{n-1} \) of \( N \) there is a metric sphere \( M \) tangent to \( E_{n-1} \).

PROOF. Let \( x_0 \) be the base point of \( E_{n-1} \). According to [2], Theorem 8.7, there is a spherical normal coordinate neighbourhood \( U(x_0; \theta) \) such that each point of \( U(x_0; \theta) \) has a normal coordinate neighbourhood containing \( U(x_0; \theta) \). Let \( \gamma \) be a geodesic emanating from \( x_0 \) and orthogonal to \( E_{n-1} \) at \( x_0 \). Choose a point \( y \) on \( \gamma \) at the distance \( d < \frac{\theta}{2} \) from \( x_0 \). Then the metric sphere \( S(y; d) \) satisfies our Proposition.

THEOREM 3. Let \( N \) be a Riemannian manifold of dimension \( n \geq 3 \). Then the following statements (i), (ii), (iii) are equivalent:

(i) To any \((n - 1)\)-dimensional linear element \( E_{n-1} \) of \( N \) there is a \( U \)-sphere tangent to \( E_{n-1} \).

(ii) Any sufficiently small metric sphere of \( N \) is a \( U \)-sphere.

(iii) \( N \) is a space of constant curvature.
PROOF: (iii) $\Rightarrow$ (ii) - Proposition 6,
(ii) $\Rightarrow$ (iii) - Proposition 7,
(i) $\Rightarrow$ (ii) - Proposition 8.

Now, let us remind some further definitions. The Ricci field is a covariant tensor field of degree 2 on $N$ defined as follows: for any vectors $X, Y \in T_x(N)$ and any orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_x(N)$ we put $S(X, Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i)$. A Riemannian manifold $N$ is called an Einstein manifold if $S = \lambda \cdot g$ on $N$, where $\lambda$ is a constant.

REMARK. From a more general point of view: whenever we speak about an Einstein manifold, we mean an Einstein manifold the signature of which in zero (cf. [4]).

It is well-known that an Einstein manifold of dimension $n \leq 3$ is a space of constant curvature. We can characterize Einstein manifolds of dimension 4 as follows:

**Lemma 5.** A 4-dimensional Riemannian manifold $N$ is Einsteinian if and only if the following property is satisfied: For any point $x \in N$ and any orthogonal decomposition of $T_x(N)$ into two planes $p, p'$ we have $K(p) = K(p')$.

**Proof.** 1) Let $N$ be Einsteinian, $S = \lambda \cdot g$, and let us have an orthogonal decomposition $T_x(N) = p \perp p'$. Choose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ such that $p = \{e_1, e_2\}$, $p' = \{e_3, e_4\}$. Then $S(e_i, e_i) = \lambda \cdot g(e_i, e_i) = \lambda$ for $i = 1, \ldots, 4$; on the other hand $S(e_i, e_j) = \sum_{j=1}^{4} g(R(e_i, e_i)e_j, e_j) = \sum_{j=1}^{4} K(e_i, e_j)$ ($i \neq j$). We obtain easily $K(p) = K(p') = K(e_1, e_2) - K(e_3, e_4) = (1/2) [S(e_1, e_1) + S(e_2, e_2) - S(e_3, e_3) - S(e_4, e_4)] = (1/2)(2\lambda - 2\lambda) = 0$, q. e. d.

2) Let us accept the implication $T_x(N) = p \perp p' \Rightarrow K(p) = K(p')$. On a fixed space $T_x(N)$ the Ricci tensor $S$ is a symmetric bilinear form and $g$ is positively definite. Thus, according to Lemma 4, there is an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $T_x(N)$ in which $S$ takes on a diagonal form. It means that $S(e_i, e_j) = 0$ for $i \neq j$. Further, $S(e_i, e_i) = \sum_{j=1}^{4} K(e_i, e_j)$ for $i = 1, \ldots, 4$, and because $K(e_1, e_2) = K(e_3, e_4)$, $K(e_1, e_3) = K(e_2, e_4)$, $K(e_1, e_4) = K(e_2, e_3)$, we obtain easily $S(e_1, e_1) = \ldots = S(e_4, e_4)$. Hence we
get \( S = \lambda \cdot g \) on \( T_x(N) \). According to [2], p. 292, the coefficient \( \lambda \) is independent of the point \( x \in N \). Consequently, \( N \) is an Einstein space.

The following Theorem is added to complete our theory of \( U \)-spheres:

**Theorem 4.** Let \( N \) be an Einstein manifold of dimension \( n \geq 3 \) and \( M \subset N \) a hypersurface all points of which are umbilical. Then \( M \) is a \( U \)-sphere, or a totally geodesic submanifold.

**Proof.** We suppose that \( A = \mu I \) on \( M \) where \( \mu \) is a real function. Then

\[
\Delta (X, Y) = (V_x A)(X) - (V_Y A)(Y) = (Y \mu) X - (X \mu) Y.
\]

Let \( \{e_1, ..., e_{n-1}\} \) be an orthonormal basis of \( Z_x(M) \) and \( e_n \) a unit vector of \( T_x(N) \) which is normal to \( T_x(M) \). According to (3) and (13) \( R(e_i, e_j) e_n = \lambda (e_i \mu) e_j - (e_j \mu) e_i \) for \( i, j = 1, ..., n - 1 \) and \( S(e_j, e_n) = \sum_{i=1}^{n-1} g(R(e_i, e_j) e_n, e_i) = (n - 2) (e_j \mu) \). On the other hand, \( S(e_j, e_n) = \lambda \cdot g(e_j, e_n) = 0 \) because \( N \) is Einsteinian. Hence we obtain \( e_j \mu = 0 \) for any \( j = 1, ..., n - 1 \) or, what is the same, the differential \( (d\mu)_x = 0 \). Consequently, \( d\mu = 0 \) on \( M \) and \( \mu = \text{const.} \)

In the end we shall study the normal Bianchi identity at the umbilical points of a hypersurface. We start with

**Lemma 6.** Let \( N \) be a Riemannian manifold of dimension \( n \geq 3 \). Then to any \( (n - 1) \)-dimensional linear element \( E_{n-1} \) of \( N \) there is a hypersurface \( M \subset N \) tangent to \( E_{n-1} \) at its base point \( x \) such that \( A = I \) at \( x \).

**Proof.** We employ the concept of an osculating euclidean space as in the proof of Theorem 2.

**Proposition 9.** Let \( N \) be a Riemannian manifold of dimension \( n \geq 4 \) and \( x \in N \) a fixed point. Then the following two statements are equivalent:

(i) Any hypersurface \( M \subset N \) for which \( x \) is an umbilical point satisfies the normal Bianchi identity at \( x \).

(ii) For any two totally orthogonal planes \( p, p' \) of \( T_x(N) \) we have \( K(p) = K(p') \).

**Proof.** (i) \( \Rightarrow \) (ii). Let \( e_1, e_2, e_3, e_4 \) be orthonormal vectors of \( T_x(N) \) such that \( p = (e_1 e_2), p' = (e_3 e_4) \). Put \( E_{n-1} = \{ X \in T_x(N) \mid g(X, e_3) = 0 \} \). Ac-
According to Lemma 6 there is a hypersurface $M \subset N$ such that $T_x(M) = E_{n-1}$ and $x$ is a «unit» umbilic: $AX = X$ for any $X \in E_{n-1}$. Formula (7) takes on the form

$$(14) \quad A(Y) \cdot Z + A(Z) \cdot X + A(X) \cdot Y = 0 \quad (X, Y, Z \in E_{n-1}).$$

Put $X = e_1$, $Y = e_2$, $Z = e_3$, $\xi = e_4$, then (3) and (14) imply

$$(15) \quad R(e_1, e_2) \cdot e_4 + R(e_2, e_3) \cdot e_4 + R(e_3, e_4) \cdot e_4 \cdot e_2 = 0.$$

According to the Cartan's lemma there are numbers $a, b, c, d, f, g$ such that

$$R(e_1, e_2) e_4 = a \cdot e_4 + b \cdot e_4 + c \cdot e_2$$

$$R(e_2, e_3) e_4 = b \cdot e_3 + d \cdot e_1 + f \cdot e_2$$

$$R(e_3, e_4) e_4 = c \cdot e_3 + f \cdot e_1 + g \cdot e_2$$

Particularly, we have $g(R(e_1, e_2) e_4, e_1) = g(R(e_2, e_3) e_4, e_3) = b$. Now, the orthonormal quadruple $e_3, e_4$ was arbitrary and it can be replaced by the quadruple $[f_1, f_2, f_3, f_4]$, where

$$(17) \quad f_1 = e_1, f_2 = \frac{e_2 + e_4}{\sqrt{2}}, f_3 = e_3, f_4 = \frac{e_2 - e_4}{\sqrt{2}}.$$

We get $g(R(f_1, f_2) f_4, f_3) + g(R(f_2, f_3) f_4, f_3) = 0$, whence

$$K(e_1, e_2) = K(e_3, e_4) = K(e_1, e_3) = K(e_4, e_4).$$

Similarly, we obtain

$$K(e_2, e_3) + K(e_4, e_4) = K(e_2, e_1) + K(e_4, e_1).$$

By the subtraction we get finally

$$K(p) = K(e_1, e_2) = K(e_3, e_4) = K(p').$$

(ii) $\Rightarrow$ (i): Suppose first dim $N = 4$. Let $M \subset N$ be a hypersurface having an umbilic at $x$. Denote by $e_4$ a unit vector of $T_x(N)$ normal to $E_3 = T_x(M)$. It suffices to prove (15), or equivalently, (16) for any orthonormal triplet $[e_1, e_2, e_3]$ of $E_4$. 

Now, each $R(e_i, e_j) e_k(i, j = 1, 2, 3)$ is normal to $e_4$ and hence it is a linear combination of $e_1, e_2, e_3$. We have only to show the relation

$$g(R(e_1, e_2) e_4, e_1) + g(R(e_2, e_3) e_4, e_3) = 0$$

for any orthonormal triple $\{e_1, e_2, e_3\}$ of $E_6$. For this purpose, take the quadruple $\{f_1, f_2, f_3, f_4\}$ given by (17) and write up the identity $K(f_1, f_2) = K(f_2, f_4)$. Taking into account $K(e_1, e_2) = K(e_2, e_3), K(e_1, e_3) = K(e_1, e_4)$, we obtain the wanted relation.

Consider now the case $\dim N \geq 5$. Then, according to the assumption (ii) and Lemma 1, $x$ is an isotropic point of $N$. Formula (8) shows that (15) is satisfied trivially for any orthonormal quadruple $\{e_1, e_2, e_3, e_4\}$ of $T_x(N)$. This completes the proof.

Now we can formulate two theorems, which are consequences of Lemma 5, Proposition 9, Lemma 1 and Lemma 3.

**Theorem 5.** Let $N$ be a Riemannian manifold of dimension $n = 4$. Then the following two statements are equivalent:

(i) Any hypersurface $M \subset N$ satisfies the normal Bianchi identity at its umbilical points.

(ii) $N$ is an Einstein space.

**Theorem 6.** Let $N$ be a Riemannian manifold of dimension $n \geq 5$. Then the following two statements are equivalent:

(i) Any hypersurface $M \subset N$ satisfies the normal Bianchi identity at its umbilical points.

(ii) $N$ is a space of constant curvature.

**Remark.** Theorem 6 is closely related to a theorem by T. Y. Thomas: If $M \subset E^n, n \geq 5$, is a hypersurface the type number of which is $\geq 4$ at each point, then the Codazzi equations are algebraic consequences of the Gauss equations. (Cf. [6]).

**Appendix.**

In Theorem 4 we have characterized space of constant curvature as Riemannian manifolds containing «sufficiently many» $U$-spheres. A direct generalization of a $U$-sphere is an umbilical hypersurface, i.e., a hypersurface $M \subset N$ all points of which are umbilical. One can ask about the nature of spaces containing «sufficiently many» umbilical hypersurfaces. The answer is given by the following Theorem:
THEOREM 7. Let $N$ be a Riemannian manifold of dimension $n \geq 4$. Then the following two statements are equivalent:

(i) To any $(n - 1)$ dimensional linear element $E_{n-1}$ of $N$ there is an umbilical hypersurface tangent to $E_{n-1}$.

(ii) $N$ is locally conformally flat.

PROOF. (ii) $\Longrightarrow$ (i). Remind the well-known fact that all umbilical points of a submanifold $M \subset N$ remain umbilical under conformal transformations of the metric $g$ on $N$ (See e.g. [7]). Thus, we can always use an auxiliary euclidean neighbourhood to realize a geometrical construction required by (i).

(i) $\Longrightarrow$ (ii). Let $x \in N$, $E_{n-1} \subset T_x(N)$ be given, and let $M \subset N$ be a hypersurface such that $T_x(M) = E_{n-1}$, $A = \mu \cdot I$ on $M$. Similarly as in the proof of Theorem 4 we obtain the relation $R(e_i, e_j) e_n = (e_j \mu) e_i - (e_i \mu) e_j$ for any $e_i, e_j \in E_{n-1}$ and a unit $e_n$ normal to $E_{n-1}$. Hence $e_j \mu = g(R(e_i, e_j)e_n, e_k) = g(R(e_k, e_j)e_n, e_k)$ for any orthonormal quadruple $[e_i, e_j, e_k, e_n]$ of $T_x(N)$, $e_n \perp E_{n-1}$. Because the element $E_{n-1} \subset T_x(M)$ can be arbitrary, we obtain $g(R(e_i, e_j)e_k, e_l) = g(R(e_k, e_j)e_i, e_l)$ for any orthonormal quadruple $[e_i, e_j, e_k, e_l]$ of $T_x(N)$. Using a similar argument as in the proof of Proposition 9 (Formula (17), we obtain finally:

$$K(e_1, e_2) + K(e_3, e_4) = K(e_1, e_4) + K(e_2, e_3)$$

for any orthonormal quadruple $[e_1, e_2, e_3, e_4]$ of $T_x(N)$.

As was shown by R. S. KULKARNI, [8], this last condition is equivalent to the requirement that the conformal curvature tensor $C$ vanishes at the point $x$. Consequently, the Riemannian space $N$ is locally conformally flat.

As a consequence of Theorem 7 we can state

THEOREM 8. Let $N$ be a Riemannian manifold of dimension $n \geq 4$. Suppose that any sufficiently small metric sphere of $N$ consists of umbilical points. Then $N$ is locally conformally flat.

PROBLEM. Does a converse of Theorem 8 hold, too?

Added in Proof, March 15, 1972: It has been pointed to me by R. S. Kulkarni that the conclusion of Theorem 8 may be strengthened to the effect that $N$ be a space of constant curvature.
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