ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

R. A. HAGER
J. ROSS

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 26, nº 2 (1972), p. 283-290
<http://www.numdam.org/item?id=ASNSP_1972_3_26_2_283_0>

A REGULARITY THEOREM FOR LINEAR SECOND ORDER ELLIPTIC DIVERGENCE EQUATIONS

by R. A. HAGER and J. ROSS

0. Introduction.

In this paper we obtain a local regularity theorem for solutions of the equation

$$D_i (a_{ij}(x) \; D_j u(x)) = 0, \tag{1}$$

where the a_{ij} are Holder continuous functions satisfying $a_{ij} \xi_i \xi_j \geq \lambda \xi^2$ and $x = (x_1, \dots, x_n)$, $n \geq 2$. A function u which is locally of class $W^{1,p}(\Omega)$, $p \geq 1$, is called a solution of (1) in an open set Ω of E^n if

$$\int_\Omega a_{ij}(x) \, D_j u \; D_i \Phi \; dx = 0 \tag{2}$$

for all smooth Φ with compact support in Ω.

Serrin [7] shows that when the a_{ij} are assumed to be only bounded and measurable one must necessarily require solutions of (1) to be of class $W^{1,2}$ in order to retain the general framework of elliptic equations. For each p, $1 < p < 2$, and all $n \geq 2$ he displays an equation of the form (1) which has a solution in $W^{1,p}$ which is not continuous. Ladyzhenskaya and Ural'tseva [3] give similar examples which show that the Dirichlet problem for (1) is not uniquely solvable in the small when solutions are assumed to be of class $W^{1,p}$, $1 < p < 2$.

These examples contrast with the well-known results of De Georgi and Nash which state that solutions of class $W^{1,2}$ are necessarily Holder continuous provided only that the a_{ij} are bounded and measurable.

Pervennto alla Redazione il 2 Marzo 1971.

The purpose of this paper is to show that if the a_{ij} are Holder continuous then any solution of class $W^{1,p}$, for some p, with $1 < p < 2$, is necessarily of class $W^{1,2}$. It then follows from the results of Campanato [2] that u has first derivatives which are Holder continuous with the same Holder exponent as the coefficients.

Other authors, notably Morrey [5, theorem 5.5.3], have similar results when u is a solution of a certain Dirichlet problem. We also mention a paper of Meyers [4] in which it is shown that a solution of (1) of class $W^{1,2}$ is also of class $W^{1,p}$ for some $p > 2$ where p depends on the modulus of ellipticity λ. He assumes only that the a_{ij} are bounded and measurable.

1. Notation.

Throughout this paper we assume that Ω is a bounded open set in E^n and that F and F_i are compact subsets of Ω. The a_{ij} are assumed to be uniformly Holder continuous in Ω, i. e., there exists constants $C_0 > 0$ and α, $0 < \alpha < 1$ such that

$$| a_{ij}(x) - a_{ij}(y) | < C_0 \, | x - y |^{\alpha}$$

for all x, y in Ω.

We use the notation $D_i = \dfrac{\partial}{\partial x_i}$, $D_i^k = \dfrac{\partial^k}{\partial x_i^k}$, and if $k = (k_1, \dots, k_n)$ with $|k| = k_1 + \dots + k_n$ then $D^k = D_1^{k_1} \dots D_n^{k_n}$. w_x is used to denote $(D_1 w, \dots, D_n w)$. $W^{m,p}(\Omega)$ is the completion of $C^m(\Omega)$ with respect to the norm

$$\| w \|_{m,p,\Omega} = \{ \sum_{|k| \leq m} \| D^k w \|^p \}^{1/p} ,$$

the norms on the right being the ordinary L^p norms. A function w is locally of class $W^{m,p}(\Omega)$ if $\| w \|_{m,p,F} < \infty$ for every F contained in Ω.

Since the results in this paper are local in nature it is sufficient to restrict our attention to small spheres. If F is a compact subset of Ω and P is a fixed point of F we let $S(R)$ be the sphere of radius R centered at P. For simplicity we use $\| w \|_{m,p,R}$ to denote the norm of w in $W^{m,p}(S(R))$. We also denote $a_{ij}(P)$ by a_{ij}^0.

Let $K(x)$ be a non negative smooth averaging kernel defined on E^n with support in $|x| < h$ and such that $\int K(x)\, dx = 1$, $K(x) <$ const. h^{-n}. For x in F and h sufficiently small we define the integral average w_h of w by

$$w_h(x) = \int K(x - y)\, w(y)\, dy.$$

For the standard properties of the integral average we refer to Serrin [8]. We mention only that if w is locally of class $W^{m,p}(\Omega)$ then $D^k w_h = (D^k w)_h$ is a smooth function which converges to $D^k w$ in $L^p(F)$ for $|k| \leq m$.

Primes always denote Holder conjugates, $\frac{1}{p} + \frac{1}{p'} = 1$. All constants except those specifically denoted by subscripts are denoted by C.

2. Preliminary Lemmas.

LEMMA 1 (Sobolev). Let w be of class $W^{1,p}(\Omega)$ and suppose $\frac{1}{r} > \frac{1}{p} - \frac{1}{n}$. Then

$$\| w \|_{o,r,\Omega} \leq C_1 \| w \|_{1,p,\Omega}.$$

Here Ω must satisfy a cone condition.

LEMMA 2. Let f be in $C_0(S(R))$ and let A_{ij}, $1 \leq i, j \leq n$, be constants, which satisfy $|A_{ij}| \leq M_1$ and $A_{ij}\xi_i\xi_j \geq \lambda_1 \xi^2$. Then there is a solution of

$$A_{ij} D_i D_j v = f \qquad \text{in} \qquad S(R)$$

such that

$$\| v \|_{2,p,R} \leq C_2 \| f \|_{o,p,R}$$

for all p satisfying $1 < p < \infty$. The constant C_2 depends only on n, M_1, λ_1, p, and R.

This result follows from lemma 4.1 of Agmon [1] and standard limiting procedures.

LEMMA 3. Suppose w is of class $L^p(S(R + h_0))$, $q > p$, and $h < h_0$. Then

$$\| w_h \|_{q,R} \leq C_3 h^{-n(1/p - 1/q)} \| w_h \|_{p,R}^{p/q} \| w \|_{p,R+h}^{1-p/q}$$

where $\| w \|_{p,R}$ denotes the ordinary $L_p(S(R))$ norm.

PROOF. Let $B(h)$ be the ball of radius h about x where x is in $S(R)$. Then

$$\int | w_h |^q \, dx = \int_{S(R)} \left| \int_{B(h)} K(x - y)\, w(y)\, dy \right|^{q-p} | w_h(x) |^p \, dx$$

$$\leq \int_{S(R)} \left| \left(\int_{B(h)} K^{p'} dy \right)^{1/p'} \left(\int_{B(h)} | w |^p \, dy \right)^{1/p} \right|^{q-p} | w_h |^p \, dx$$

$$\leq C\,(h^{-n}\,h^{n/p'})^{q-p}\int_{S(R)}\|\,w\,\|^{q-p}_{p,\,B(h)}\,|\,w_h\,|^p\,dx$$

$$\leq Ch^{-n(q-p)/p}\,\|\,w\,\|^{q-p}_{p,\,R+h}\int_{S(R)}|\,w_h\,|^p\,dx$$

$$= Ch^{-n(q-p)/p}\,\|\,w\,\|^{q-p}_{p,\,R+h}\,\|\,w_h\,\|^p_{p\;R}\,.$$

The result now follows by taking *qth* roots.

3. Main result.

Theorem. Let $u \in W^{1,p}(\Omega)$, $p > 1$, be a solution of (1) in Ω. If F is any compact subset of Ω then $u \in W^{1,2}(F)$. Moreover,

$$\|\,u\,\|_{1,\,2,\,F} \leq C\,\|\,u\,\|_{1,\,p,\,\Omega}$$

where C depends on α, n, p, λ, C_0 and dist. $(F, \partial\Omega)$.

Proof. The proof is by means of a finite iteration. We interpose between Ω and F a finite sequence of nested, closed sets,

$$\Omega \supset F_1 \supset \ldots \supset F_N \equiv F.$$

Here dist $(F_i, F_{i+1}) > 0$ for all i and the number of closed sets will depend only on α, p and n. The main portion of the proof consists in showing that if $u \in W^{1,\zeta}(F_i)$ then $u \in W^{1,z}(F_{i+1})$ where $z \equiv n\zeta/(n-\alpha\zeta) > \zeta$.

We assume that both h and R are less than $\dfrac{1}{4}\min\limits_i \text{dist}\,(F_i, F_{i+1})$. By choosing the function Φ in (2) to be the averaging kernel K, equation (2) takes the form

$$D_i\,(a_{ij}\,D_j\,u)_h = 0.$$

Thus if ψ is any smooth function with compact support in $S(2R)$ it follows that

$$\int (a_{ij}\,D_j\,u)_h\,D_i\,\psi\,dx = 0. \tag{3}$$

Let $\eta(x)$ be a non-negative smooth function which has compact support in $S(2R)$, $\equiv 1$ in $S(R)$ and $0 \leq \eta(x) \leq 1$ in $S(2R)$. We can do this, moreover, in such a way that $|\eta_x| \leq$ const. R^{-1}. Let v^k be a solution of the problem

$$a^o_{ij} D_i D_j v^k = \text{sign}\, D_k u_h \{\eta \mid D_k u_h \mid\}^{z-1} \quad \text{in} \quad S(2R) \tag{4}$$

which satisfies the conclusion of lemma 2. We now set $\psi = \eta D_k v^k$ in (3). The fact that this choice is admissible easily follows from standard limiting procedures. Equation (3) then takes the form

$$\int (a_{ij} D_j u)_h D_i (\eta D_k v^k)\, dx = 0. \tag{5}$$

Using y to denote the integration variable in the averaging process and letting $\delta a_{ij} \equiv a_{ij}(y) - a_{ij}(x)$, (5) can be written in the form

$$\int (\delta a_{ij} D_j u)_h D_i (\eta D_k v^k)\, dx$$
$$+ \int [a_{ij}(x) - a^o_{ij}] D_j u_h D_i (\eta D_k v^k)\, dx$$
$$+ \int a^o_{ij} D_j u_h D_i (\eta D_k v^k)\, dx = 0.$$

Performing the indicated differentiations and several integrations by parts we obtain

$$\int (\delta a_{ij} D_j u)_h D_i \eta D_k v^k\, dx + \tag{6}$$
$$+ \int (\delta a_{ij} D_j u)_h \eta D_i D_k v^k\, dx$$
$$+ \int [a_{ij}(x) - a^o_{ij}] D_j u_h D_i \eta D_k v^k\, dx$$
$$+ \int [a_{ij}(x) - a^o_{ij}] \eta D_j u_h D_i D_k v^k\, dx$$
$$+ \int a^o_{ij} D_j u_h D_i \eta D_k v^k\, dx$$
$$- \int a^o_{ij} D_j u_h D_k \eta D_i v^k\, dx$$
$$+ \int a^o_{ij} D_k u_h D_j \eta D_i v^k\, dx$$
$$+ \int \eta D_k u_h a^o_{ij} D_i D_j v^k\, dx$$
$$= 0.$$

From (4) it follows that the last term on the left hand side of (6) can be written

$$\int \eta D_k u_h \overset{o}{a}_{ij} D_i D_j v^k \, dx = \| \eta D_k u_h \|^z_{o,z,2R}. \tag{7}$$

We now estimate, in order, the first seven terms of the left hand side of (6).

$$\left| \int (\delta a_{ij} D_j u)_h D_i \eta D_k v^k \, dx \right|$$

$$\leq C R^{-1} \| (\delta a_{ij} D_j u)_h \|_{o,z,2R} \| D_k v^k \|_{o,z',2R}$$

$$\leq C C_0 C_3 R^{-1} h^{\alpha - n(1/\zeta - 1/z)} \| | u_x |_h \|^{\zeta/z}_{0,\zeta,2R} \| u_x \|^{1-\zeta/z}_{0,\zeta,2R+h} \| v^k \|_{2,z',2R}$$

$$\leq C C_0 C_2 C_3 R^{-1} \| | u_x |_h \|^{\zeta/z}_{o,\zeta,2R} \| u_x \|^{1-\zeta/z}_{o,\zeta,2R+h} \| \eta D_k u_h \|^{z-1}_{o,z,2R}.$$

$$\left| \int (\delta a_{ij} D_j u)_h \eta D_i D_k v^k \, dx \right|$$

$$\leq C \| (\delta a_{ij} D_j u)_h \|_{o,z,2R} \| D_i D_k v^k \|_{o,z',2R}$$

$$\leq C C_0 C_2 C_3 \| u_{hx} \|^{\zeta/z}_{o,\zeta,2R} \| u_x \|^{1-\zeta/z}_{o,\zeta,2R+h} \| \eta D_k u_h \|^{z-1}_{o,z,2R}.$$

$$\left| \int (a_{ij}(x) - \overset{o}{a}_{ij}) D_j u_h D_i \eta D_k v^k \, dx \right|$$

$$\leq C C_0 R^{\alpha-1} \| u_{hx} \|_{o,\zeta,2R} \| D_k v^k \|_{o,\zeta',2R}$$

$$\leq C C_0 C_1 R^{\alpha-1} \| u_{hx} \|_{o,\zeta,2R} \| v^k \|_{2,z',2R}$$

$$\leq C C_0 C_1 C_2 R^{\alpha-1} \| u_{hx} \|_{o,\zeta,2R} \| \eta D_k u_h \|^{z-1}_{o,z,2R}.$$

Here the use of lemma 1 requires that $1/\zeta' > 1/z' - 1/n$, the validity of which is a consequence of the definition of z.

$$\left| \int (a_{ij}(x) - \overset{o}{a}_{ij}) \eta D_j u_h D_i D_k v^k \, dx \right|$$

$$\leq C C_0 R^{\alpha} \| \eta u_{hx} \|_{o,z,2R} \| D_i D_k v^k \|_{o,z',2R}$$

$$\leq C C_0 C_2 R^{\alpha} \| \eta u_{hx} \|_{o,z,2R} \| \eta D_k u_h \|^{z-1}_{o,z,2R}.$$

The remaining three terms of (6) are quite similar and so the estimations are nearly identical. For example

$$\left| \int a^o_{ij} D_j u_h D_i \eta D_k v^k \, dx \right|$$

$$\leq CR^{-1} \| u_{hx} \|_{o,\zeta,2R} \| D_k v^k \|_{o,\zeta',2R}$$

$$\leq CC_1 R^{-1} \| u_{hx} \|_{o,\zeta,2R} \| v^k \|_{2,z',2R}$$

$$\leq CC_1 C_2 R^{-1} \| u_{hx} \|_{o,\zeta,2R} \| \eta D_k u_h \|^{z-1}_{o,z,2R} .$$

Inserting these estimates and (7) into (6), there results

(8) $$\| \eta D_k u_h \|^z_{o,z,2R}$$

$$\leq CC_0 C_2 C_3 (1 + R^{-1}) \| \, | u_x |_h \|^{\zeta/z}_{o,\zeta,2R} \| u_x \|^{1-\zeta/z}_{o,\zeta,2R+h} \| \eta D_k u_h \|^{z-1}_{o,z,2R}$$

$$+ CC_1 C_2 (R^{\alpha-1} + 3R^{-1}) \| u_{hx} \|_{o,\zeta,2R} \| \eta D_k u_h \|^{z-1}_{o,z,2R}$$

$$+ CC_0 C_2 R^{\alpha} \| \eta u_{hx} \|_{o,z,2R} \| \eta D_k u_h \|^{z-1}_{o,z,2R} .$$

Summing over k in (8) and choosing R sufficiently small it follows that

(9) $$\| \eta u_{hx} \|^z_{o,z,2R}$$

$$\leq C \{ \| \, | u_x |_h \|^{\zeta/z}_{o,\zeta,2R} \| u_x \|^{1-\zeta/z}_{o,\zeta,2R+h}$$

$$+ \| u_{hx} \|_{o,\zeta,2R} \} \| \eta u_{hx} \|^{z-1}_{o,z,2R} .$$

Dividing both sides of (9) by $\| u_{hx} \|^{z-1}_{o,z,2R}$ and letting h tend to zero there results

(10) $$\| u_x \|_{o,z,R} \leq C \| u_x \|_{o,\zeta,2R} .$$

The result of the theorem now easily follows. To perform the finite iteration, let $z_1 = p$, $z_i = nz_{i-1}/(n - \alpha z_{i-1})$ for $i = 2, \ldots, N$ where $N > [n(2-p)/2\alpha p] + 1$. Using standard covering arguments it follows that

$$\| u_x \|_{o,z_{i+1},F_{i+1}} \leq C \| u_x \|_{o,z_i,F_i} , \qquad i = 1, \ldots, N-1.$$

Thus

$$\| u_x \|_{o,z_N,F} \leq C \| u_x \|_{o,p,F_1} .$$

A simple computation shows that $1/z_i = 1/p - i\alpha/n$ so that $z_N > 2$. This completes the proof.

4. Concluding remarks.

The proof of our theorem shows that u is not only of class $W^{1,2}(F)$ but in fact of class $W^{1,q}(F)$ for all finite $q > p$, that is, u is demiregular, [6].

In using the results of Campanato [2] to deduce that a solution of (1) is of class $C^{1,\alpha}(F)$ one must observe that the condition of symmetry which Campanato requires is not needed. It enters only in lemma 5.II and can be avoided with additional straightforward computations.

The restriction to a homogeneous equation with no lower order terms was made for simplicity. Appropriate additional terms could easily be handled.

It is an open question whether the conclusion of our theorem remains valid if some weaker modulus of continuity of the a_{ij} is assumed.

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San Diego State College
San Diego, California 92115