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EXISTENCE AND APPROXIMATION OF WEAK SOLUTIONS OF ABSTRACT DIFFERENTIAL EQUATIONS

by M. DJEDOUR

Let (,) and $\| \|$ denote the scalar product and the norm in the Hilbert space H.

In the following we will be concerned with the differential equation:

$$u^{(n)}(t) + A_1 u^{(n-1)}(t) + A_2 u^{(n-2)}(t) + \dots + A_n u(t) = f(t) \tag{1}$$

where $A_1, A_2, \dots, A_n$ are linear operators defined on the Hilbert space H. The operators $A_1, \dots, A_n$ are supposed to be continuous on H except for one of them, A_{k_0} which is generally (an unbounded operator) a closed operator defined on a dense subset $\mathcal{D}_{A_{k_0}}$ of H.

The function $f(t)$ belongs to $L^2_{\text{loc}}(R; H)$, the space of all H-valued strongly measurable fonctions such that the norm $\| g(t) \|$ is square integrable on every compact subset of R.

In Lemma 2, we show that (1) has a local solution then, with Lemma 6 (Density) and 7 (approximation) we are able to prove the existence of a solution of (1) in the sense of Definition I below.

For convenience, let:

$$K^*(a,b) = \{\Phi : \Phi(t) \in C_0^n(a, b); \mathcal{D}_{A^*_{k_0}}), A_j^* \Phi \in C^{n-j}((a, b); H), \quad j = 1, \dots, n\}$$

$$K^*(a,b) = \{\Phi : \Phi(t) \in C_0^n(a, b); \mathcal{D}_{A_{k_0}}), A_j \Phi \in C^{n-j}((a, b); H), \quad j = 1, \dots, n\}$$

$$K^* \quad = \{\Phi : \Phi(t) \in C_0^n(R; \mathcal{D}_{A^*_{k_0}}); A_j^* \Phi \in C^{n-j}(R; H), \quad j = 1, \dots, n\}$$

$$K \quad = \{\Phi : \Phi(t) \in C_0^n(R, D_{A_{k_0}}); A_j \Phi \in C^{n-j}(R; H); \; j = 1, \dots, n\}.$$

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DEFINITION I

a) For a given function $f(t) \in L^2((a,b); H)$ we say that $u(t) \in L^2((a,b); H)$ is a weak solution of (1) on (a, b) if the following hold:

$$(2) \qquad \int_a^b \Big(u(t), (-1)^n \Phi^{(n)} + \sum_{j=1}^{n} (-1)^{n-j} (A_j^* \Phi)^{(n-j)}\Big)\, dt = \int_a^b (f(t), \Phi(t))\, dt$$

for all $\Phi \in K^*(a, b)$.

b) Similarly we define $u(t) \in L^2_{\text{loc}}(R; H)$ as a weak solution of (1) on R_1 if:

$$(3) \qquad \int_{-\infty}^{\infty} \Big(u(t), (-1)^n \Phi^{(n)}(t) + \sum_{j=1}^{n} (-1)^{n-j} (A_j^* \Phi)^{(n-j)}\Big)\, dt = \int_{-\infty}^{\infty} (f(t), \Phi(t))\, dt$$

hold for all $\Phi \in K^*$ where $f(t)$ is given in $L^2_{\text{loc}}(R; H)$.

In [1] and [2], S. Zaidman considered the following equations:

$$(4) \qquad u'(t) + A\, u(t) = f(t)$$

$$(5) \qquad u'' + A\, u(t) = f(t)$$

with A a closed operator with dense domain $\mathcal{D}_A$ in H. Upon certain condition on A, S. Zaidman has shown that a weak solution $u(t)$ in $L^2_{\text{loc}}(R; H)$ exists in the sense of (3) for every given function $f(t)$ in $L^2_{\text{loc}}(R; H)$.

The purpose of this paper is to generalize the method of S. Zaidman to get a weak solution of (1) in the sense of (3) for every given function $f(t)$ in $L^2_{\text{loc}}(R; H)$.

DEFINITION II: [3] Let j be a positive integer and s a positive real and F a family of vertical lines of the complex plane given by $\operatorname{Re} \lambda = \sigma_n$ and $\operatorname{Re} \lambda = \sigma'_n$, $\sigma_n \to +\infty$, $\sigma'_n \to -\infty$.

We shall say that the operators: $A_1, A_2, \ldots, A_{k_0}, \ldots, A_n$ satisfy the condition S on F if:

$$(6) \qquad \left\| \left[(-1)^n \lambda^n I + \sum_{j=0}^{n-1} (-1)^j \lambda^j A^*_{n-j}\right]^{-1} \right\| \leq M$$

hold on every line of F except possibly for j intervals of length s.

We will say that $\{A_1, A_2, \ldots A_n\}$ are (j, s) bounded on F.

We will prove the following theorem:

THEOREM: Let the equation (1), with $A_1, A_2, \dots, A_n$ continuous except for A_{k_0} which is a (generally) unbounded closed operator with dense domain and suppose moreover $\{A_1, A_2, \dots, A_n\}$ satisfying the condition S above, then for any given $f(t) \in L^2_{\text{loc}}(R; H)$ there exist a $u(t) \in L^2_{\text{loc}}(R; H)$ solution of (3).

LEMMA I. Let the operators $A_1, A_2, \dots, A_{k_0}, \dots . A_j, \dots, A_n$ be continuous except for A_{k_0} which is closed with dense domain $\mathcal{D}_{A_{k_0}}$ and $\{A_1, A_2, \dots, A_n\}$ (j, s)-bounded on the line $\operatorname{Re} \lambda = \sigma$.

Then for every bounded interval $(a, b) \subset R$ and for every $u(t) \in K^*(a, b)$ we have:

$$\int_a^b \|u(t)\|^2 dt \leq C e^{2\sigma(b-a)} \int_a^b \left\| (-1)^n u^{(n)}(t) + \sum_{j=1}^n (-1)^{n-j} (A_j^* u(t))^{(n-j)} \right\|^2 dt. \tag{7}$$

PROOF: Let $V(t) = e^{\sigma t} u(t)$.

From:

$$(-1)^n u^n(t) + \sum_{j=1}^n (-1)^{n-j} (A_j^* u)^{(n-j)} = f(t)$$

we deduce:

$$(-1)^n \left[\sum_{l=0}^n C_l^n (-1)^l \sigma^l V^{(n-l)}(t) \right]$$

$$+ \sum_{j=1}^n (-1)^{n-j} \left[\sum_{l=0}^{n-j} C_l^{n-j} (-1)^l \sigma^l (A_j^* V)^{(n-j-l)} \right] = f(t) e^{\sigma t} = g(t)$$

which can be written as:

$$(-1)^n \left[\sum_{l=0}^n C_l^n (-1)^l \sigma^l V^{(n-l)}(t) \right] + \sum_{j=0}^{n-1} (-1)^j \left[\sum_{l=0}^j C_l^j (-1)^l \sigma^l (A_{n-j}^* V)^{(j-l)} \right] = g(t).$$

Let us take the Fourier transform on both sides: we obtain:

$$(-1)^n \left[\sum_{l=0}^n C_l^n (-1)^l \sigma^l (i\tau)^{n-l} \widehat{V}(\tau) \right] + \sum_{j=0}^{n-1} (-1)^j [C_l^j (-1)^l \sigma^l (i\tau)^{j-l} A_{n-j}^* \widehat{V}(\tau)] = \widehat{g}(\tau)$$

i. e., if we set: $\lambda = -\sigma + i\tau$

$$\left[(-1)^n \lambda^n I + \sum_{j=0}^{n-1} (-1)^j \lambda^j A_{n-j}^* \right] \widehat{V}(\tau) = \widehat{g}(\tau)$$

where $\widehat{V}(\tau)$ and $\widehat{g}(\tau)$ are the Fourier transform of $V(t)$ and $g(t)$.

Let Γ be the real axis $-\infty < \tau < +\infty$, from which we delete j intervals of length s.

Then for $\tau \in \Gamma$; by hypothesis:

$$\left\| \left[(-1)^n \lambda^n I + \sum_{j=0}^{n-1} (-1)^j \lambda^j A_{n-j}^* \right]^{-1} \right\| \leq M \qquad \tau \in \Gamma \qquad \operatorname{Re} \lambda = -\sigma$$

$$\Longrightarrow \| \widehat{V}(\tau) \| \leq M \| \widehat{g}(\tau) \|, \qquad \tau \in \Gamma$$

$$\Longrightarrow \int_{\Gamma} \| \widehat{V}(\tau) \|^2 d\tau \leq M^2 \int_{-\infty}^{\infty} \| \widehat{g}(\tau) \|^2 d\tau.$$

Since $V(t)$ has compact support in (a, b) by a result of S. Agmon-L. Nirenberg [3], there exist $k = k(j, s)$ such that:

$$\int_{-\infty}^{\infty} \| \widehat{V}(\tau) \|^2 d\tau \leq k \int_{\Gamma} \| \widehat{V}(\tau) \|^2 d\tau \Longrightarrow \int_{-\infty}^{\infty} \| \widehat{V}(\tau) \|^2 d\tau \leq k M^2 \int_{-\infty}^{\infty} \| \widehat{g}(\tau) \|^2 d\tau.$$

Using the vector form of Parseval's Theorem we get:

$$\int_{-\infty}^{\infty} \| V(t) \|^2 dt = \int_a^b e^{2\sigma t} \| u(t) \|^2 dt \leq k M^2 \int_a^b \| f(t) \|^2 e^{2\sigma t} dt$$

If we suppose $\sigma < 0$, we have,

$$e^{2\sigma b} \int_a^b \| u(t) \|^2 dt \leq k M^2 e^{2\sigma a} \int_a^b \| (-1)^n u^{(n)}(t) + \sum_{j=1}^{n} (-1)^{n-j} (A_j^* u(t))^{(n-j)} \|^2 dt.$$

Hence (7) with $C = kM^2$.

LEMME 2 (Local existence):

Under the same hypothesis as in Lemma 1, for every $f(t) \in L^2((a, b); H)$ there exist a function $u(t) \in L^2((a, b); H)$ satisfying (2).

PROOF: Consider the linear subspace

$$\left[(-1)^n \frac{d^n}{dt^n} + \sum_{j=0}^{n-1} (-1)^j \frac{d^j}{dt^j} \, 0 \, A_{n-j}^* \right] (K^*(a, b))$$

in $L^2((a, b); H)$. We can define a linear form F by

$$F\left[(-1)^n \Phi^{(n)}(t) + \sum_{j=0}^{n-1}(-1)^j (A_{n-j}^* \Phi)^{(j)}\right] = \int_a^b (f, \Phi)_H \, dt, \qquad \Phi \in K^*(a, b)$$

which is well defined by (7),

F is continuous since:

$$\left|\int_a^b (f, \Phi)\, dt\right| \le C\left\{\int_a^b \|\Phi\|^2 \, dt\right\}^{\frac{1}{2}} \le$$

$$\le C_1\left\{\int_a^b \left\|(-1)^n \Phi^{(n)} + \sum_{j=0}^{n-1}(-1)^j (A_{n-j}^* \Phi^{(j)}\right\|^2 dt\right\}^{\frac{1}{2}}$$

$$= C_1 \left\|(-1)^n \Phi^{(n)}(t) + \sum_{j=0}^{n-1}(-1)^j (A_{n-j}^* \Phi)^{(j)}\right\|_{L^2(a, b)\,;\, H)}.$$

Hence by the Hahn-Banach theorem F has an extension to $L^2((a, b); H)$ and there exist $u(t) \in L^2(a, b); H)$ such that:

$$F((-1)^n \Phi^{(n)}(t) + \sum_{j=0}^{n-1}(-1)^j (A_{n-j}^* \dot{\Phi})^{(j)}) =$$

$$= \int_a^b (u(t), (-1)^n \Phi^{(n)}(t) + (-1)^j (A_{n-j}^* \Phi)^{(j)})\, dt = \int_a^b (f, \Phi)\, dt.$$

for every $\Phi \in K^*(a, b)$.

LEMME 3 (Unicity): Let $\{A_1, A_2, \ldots, A_n\}$ satisfy the condition S and $u(t)$ defined on (a, b) with values in $D_{A_{k_0}^*}$ such that $u^{(n-k_0)} \in \mathcal{D}_{A_{k_0}^*}$ and:

$$(8) \qquad (-1)^n u^{(n)}(t) + \sum_{j=0}^{n-1}(-1)^j (A_{n-j}^* u)^{(j)} = 0 \qquad t \in (a, b),$$

and: $0 \le j \le n-1 : u^{(j)}(c) = 0$ for each $j = 0, \ldots, n-1$, then $u \equiv 0$ in (a, b).

PROOF: We prove that $u \equiv 0$ for $c \le t < b$.

For this, let:

$$\zeta(t) \in C^{\infty}(c, b)$$

such that:

$$\zeta(t) = \begin{cases} 1 & c \leq t \leq \alpha < b \\ 0 & \alpha + \delta \leq t \leq b \end{cases}$$

and set:

$$V(t) = \begin{cases} 0 & t < c, t > b \\ e^{\sigma t} \zeta(t)\, u(t) & t \in (c, b) \end{cases}$$

$$\Longrightarrow V(t)\, e^{-\sigma t} = \zeta(t)\, u(t).$$

Then equation (8) on this function gives:

$$(9)\quad (-1)^n \sum_{k=0}^{n} C_k^n (-1)^k \sigma^k V^{(n-k)} e^{-\sigma t} + \sum_{j=0}^{n-1} \left[\sum_{k=0}^{j} C_k^j (-1)^{k+1} \sigma^k (A_{n-j}^* V(t))^{(j-k)} e^{-\sigma t} \right]$$

$$= (-1)^n \sum_{k=0}^{n} C_k^n \zeta^{(k)} u^{(n-k)} + \sum_{j=0}^{n-1} \left[\sum_{k=0}^{j} C_k^j (-1)^j \zeta^{(k)} (A_{n-j}^* u)^{(j-k)} \right].$$

The right hand side of (9) can be rewritten as:

$$\left[(-1)^n u^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* u)^{(j)} \right] \zeta + \dots +$$

$$+ \zeta^{(k)} \left[(-1)^n u^{(n-k)} C_k^n + \sum_{j \geq k}^{n-1} C_k^j (-1)^j (A_{n-j}^* u)^{(j-k)} \right] + \dots$$

And in behalf of (8), the right hand side of (9) can be written in the form:

$$(10)\qquad \sum_{k=1}^{n-1} \zeta^{(k)} \left[(-1)^n C_k^n u^{(n-k)} + \sum_{j \geq k}^{n-1} C_k^j (-1)^j (A_{n-j}^* u)^{(j-k)} \right] + \zeta^{(n)} u.$$

So (9) becomes:

$$(11)\quad (-1)^n \left[\sum_{k=0}^{n} C_k^n (-1)^k \sigma^k V^{(n-k)} \right] + \sum_{j=0}^{n-1} \left[(-1)^j \sum_{k=0}^{j} C_k^j (-1)^k \sigma^k (A_{n-j}^* V^{(j-k)} \right]$$

$$= e^{\sigma t} \left\{ \sum_{k=1}^{n} \zeta^{(k)} \left[(-1)^n C_k^n u^{(n-k)} + \sum_{j \geq k}^{n-1} C_k^j (-1)^j (A_{n-j}^* u)^{(j-k)} \right] + \zeta^{(n)} u \right\}$$

$$= f(t) \text{ for } c \leq t \leq b.$$

If we set $f(t) = 0$ outside (c, b). The hypothesis on u and $u^{(j)}$ imply that the equation (11) is valid for all $-\infty < t < +\infty$. We can then consider the Fourier transform of $f(t)$.

So we get:

$$(12) \qquad (-1)^n \left[\sum_{k=0}^{n} C_k^n (-1)^k \sigma^k (+i\tau)^{n-k} \right] \widehat{V}(\tau) +$$

$$\sum_{j=0}^{n-1} \left[(-1)^j \sum_{k=0}^{j} (-1)^k \sigma^k (i\tau)^{j-k} A_{n-j}^* \right] \widehat{V}(\tau) = \widehat{f}(\tau)$$

$$\Longrightarrow \left[(-1)^n \lambda^n I + \sum_{j=0}^{n-1} (-1)^n \lambda^j A_{n-j}^* \right] \widehat{V}(\tau) = \widehat{f}(\tau), \qquad \text{with } \lambda = -\sigma + i\tau.$$

Γ being as in lemma 1, the condition S on $\{A_1, \ldots, A_n\}$ gives for $\tau \in \Gamma$

$$\| \widehat{V}(\tau) \| \le M \| \widehat{f}(\tau) \| \Longrightarrow \int_{\Gamma} \| \widehat{V}(\tau) \|^2 d\tau \le M^2 \int_{-\infty}^{\infty} \| \widehat{f}(\tau) \|^2 d\tau.$$

And with the same argument as in Lemma 1:

$$\int_{-\infty}^{\infty} \| V(t) \|^2 dt \le k M^2 \int_{-\infty}^{\infty} \| f(t) \|^2 dt$$

$$\Longrightarrow \int_{c}^{a} e^{2\sigma t} \| u(t) \|^2 dt \le k M^2 \int_{a}^{a+\delta} \| \sum_{k=1}^{n} \zeta^{(k)} \left[(-1)^n C_k^n u^{(n-k)} \right.$$

$$\left. + \sum_{j \ge k}^{n-1} C_k^j (-1)^j (A_{n-j}^* u)^{(j-k)} \right] + \zeta^{(n)} u \|^2.$$

If we suppose: $\sigma < 0$ and take $\beta < \alpha$, we get:

$$e^{2\sigma\beta} \int_{c}^{\beta} \| u(t) \|^2 dt \le c_u k M^2 e^{2\sigma\alpha} \quad \forall \sigma < 0.$$

In particular for: $\sigma = \sigma_n' \to -\infty$.

$$\Longrightarrow \int_{c}^{\beta} \| u(t) \|^2 dt \le c^u k M^2 e^{2\sigma_n' (\alpha - \beta)} \to 0 \quad \text{as} \quad n \to \infty.$$

$\Longrightarrow u(t) \equiv 0$ for $c \leq t \leq \beta$. As $\beta < \alpha$ and β and α where arbitrary it follows that

$$u(t) = 0 \qquad c \leq t < b.$$

A similar method using the sequence $\sigma_n \to \infty$, gives

$$u(t) = 0 \qquad a < t \leq c \Longrightarrow u(t) = 0 \qquad t \in (a, b).$$

COROLLARY. If $\{A_1, A_2, \dots, A_n\}$ satisfy condition S, $u(t) \in K^*$ and

$$(-1)^n u^{(n)}(t) + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* u)^{(j)} = 0 \quad \text{outside } [a, b].$$

Then $u(t) = 0$ outside $[a, b]$.

LEMMA 4 (regularisation): Let $A_1, A_2, \dots, A_n$ be all continuous on H except for some A_{k_0} which is a closed operator of dense domain in H.

Then for given $f(t) \in L^2_{\text{loc}}(R; H)$ and $u(t) \in L^2_{\text{loc}}(R; H)$ satisfying (3), i. e.:

$$\int_{-\infty}^{\infty} (u(t), (-1)^n \Phi^{(n)}(t) + \sum_{j=1}^{n} (-1)^{n-j} (A_j^* \Phi)^{(n-j)})\, dt = \int_{-\infty}^{\infty} (f(t), \Phi(t))^{(n-j)}\, dt.$$

for all $\Phi \in K^*$.

We have for every $\alpha(t) \in C_0^\infty(R)$, if: $u * \alpha(t) = \int_{-\infty}^{\infty} u(\tau)\, \alpha(t - \tau)\, d\tau$, then

$$(13) \quad (-1)^n \frac{d^n}{dt^n}(u_* \alpha) + A_1 (u_* \alpha)^{(n-1)} + A_2 (u_* \alpha)^{(n-2)} + \dots + A_n (u_* \alpha) = f_* \alpha.$$

with

$$(u^* \alpha)^{n-k_0} \in \mathcal{D}_{A_{k_0}}.$$

PROOF: We have by hypothesis:

$$\int_{-\infty}^{\infty} (u(t), (-1)^n \Phi^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* \Phi)^{(j)}\, dt = \int_{-\infty}^{\infty} (f(t), \Phi(t))\, dt \quad \Phi \in K^*.$$

Denote by $\check{*}$ the operation:

$$\Phi(t) \to \int_{-\infty}^{\infty} \alpha(\zeta)\, \Phi(t + \zeta)\, d\zeta \qquad \text{with} \quad \alpha \in C_0^\infty(R).$$

So we have:

$$\int_{-\infty}^{\infty}\left((u_*\alpha)(t),\ (-1)^n\,\Phi^{(n)}+\sum_{j=0}^{n-1}(-1)^j\,(A^*_{n-j}\,\Phi)^{(j)}\right)dt$$

$$=\int_{-\infty}^{\infty}\left(u(t),\ \left[(-1)^n\,\Phi^{(n)}+\sum_{j=0}^{n-1}(-1)^j\,(A^*_{n-j}\,\Phi)^{j}\right]\check{*}\ \alpha\right)dt$$

$$=\int_{-\infty}^{\infty}\left(u,\ (-1)^n\,\Phi^{(n)}\check{*}\ \alpha+\sum_{j=0}^{n-1}(-1)^j\,(A^*_{n-j}\,\Phi)^{(j)}\check{*}\ \alpha\right)dt$$

$$=\int_{-\infty}^{\infty}\left(u,\ (-1)^n\,(\Phi\check{*}\ \alpha)^{(n)}+\sum_{j=0}^{n-1}(-1)^j\,(A^*_{n-j}\,\Phi\check{*}\ \alpha)^{(j)}\right)dt$$

$$=\int_{-\infty}^{\infty}(f(t),\ \Phi\check{*}\ \alpha)\,dt$$

since $\Phi\longrightarrow\Phi\check{*}\ \alpha\in K^*$

$$\Longrightarrow\int_{-\infty}^{\infty}\left(u*\alpha,\ (-1)^n\,\Phi^{(n)}+\sum_{j=0}^{n-1}(-1)^j\,(A^*_{n-j})^{(j)}\right)dt=\int_{-\infty}^{\infty}(f*\alpha,\ \Phi)\,dt.$$

And since $u*\alpha$ is infinitely differentiable in H, we get:

$$\int_{-\infty}^{\infty}(u*\alpha,\ (-1)^n\,\Phi^{(n)})\,dt=\int_{-\infty}^{\infty}((u*\alpha)^{(n)},\ \Phi(t))\,dt.$$

So we have:

$$\sum_{j=0}^{n-1}\int_{-\infty}^{\infty}(u*\alpha,\ (-1)^j\,(A^*_{n-j}\,\Phi)^{(j)})\,dt=\int_{-\infty}^{\infty}(f*\alpha,\ \Phi)\,dt-\int_{-\infty}^{\infty}((u*\alpha)^{(n)},\ \Phi)\,dt$$

$$\sum_{j=0}^{n-1}\int_{-\infty}^{\infty}((u*\alpha)^{(j)},\ A^*_{n-j}\,\Phi)\,dt=\int_{-\infty}^{\infty}(f*\alpha,\ \Phi)\,dt-\int_{-\infty}^{\infty}((u*\alpha)^{(n)},\ \Phi)\,dt.$$

As the operator $A_1, A_2, \dots, A_n$ are continuous except for one of them A_{k_0} which is supposed to be a closed operator, we have:

$$\int_{-\infty}^{\infty}((u*\alpha)^{(n-k_0)}, A^*_{k_0}\Phi) = \int_{-\infty}^{\infty}(f*\alpha, \Phi)\,dt - \int_{-\infty}^{\infty}((u*\alpha)^{(n)}, \Phi)\,dt$$

$$- \sum_{n-j\neq k_0} \int_{-\infty}^{\infty}((u*\alpha)^{j}, A^*_{n-j}\Phi).$$

Let now $\Phi(t) = \nu(t)\,V$ where $\nu(t) \in C_0^{\infty}(R)$ and $V \in \mathcal{D}_{A^*_{k_0}}$, so we obtain:

$$\int_{-\infty}^{\infty}((u*\alpha)^{(n-k_0)}, A^*_{k_0}\nu(t)\,V)\,dt = \int_{-\infty}^{\infty}(f*\alpha, \nu(t)\,V)\,dt - \int_{-\infty}^{\infty}((u*\alpha)^{(n)}, \nu(t)\,V)\,dt$$

$$- \sum_{n-j\neq k_0} \int_{-\infty}^{\infty}((u*\alpha)^{(j)}, A^*_{n-j}\nu(t)\,V)\,dt$$

$$(14)\qquad \Longrightarrow \left(\int_{-\infty}^{\infty}\nu(t)\,(u*\alpha)^{(n-k_0)}(t)\,dt,\ A^*_{k_0}V\right)$$

$$= \left(\int_{-\infty}^{\infty}\nu(t)\left[(f*\alpha) - (u*\alpha)^{(n)} - \sum_{n-j\neq k_0} A_{n-j}(u*\alpha)^{(j)}\right]dt,\ V\right).$$

As the A_j, $j \neq k_0$ are continuous.
Since (14) is valid for all $V \in \mathcal{D}_{A^*_{k_0}}$ it follows that

$$\int_{-\infty}^{\infty}\nu(t)\,(u*\alpha)^{(n-k_0)}\,dt\ \mathcal{D}_{A^{**}_{k_0}} = \mathcal{D}_{A_{k_0}} \quad \forall\ \nu(t) \in C_0^{\infty}(R).$$

And since $(u*\alpha)^{(n-k_0)}$ is continuous there exists a sequence $v_p(t)$ such that:

$$\int_{-\infty}^{\infty}\nu_p(t)\,(u*\alpha)^{(n-k_0)}(t)\,dt \rightarrow (u*\alpha)^{(n-k_0)}(t) \quad \forall\ t \in R.$$

From (14) we have:

$$A_k\int_{-\infty}^{\infty}\nu(t)\,(u*\alpha)^{(n-k)}(t)\,dt = \int_{-\infty}^{\infty}\nu(t)\,[(f*\alpha) - (u*\alpha)^{(n)} - \sum_{n-j\neq k_0} A_{n-j}(u*\alpha)^{(j)}]\,dt.$$

And since A_{k_0} is closed:

$$A_{k_0}(u*\alpha)^{(n-k_0)} = (f*\alpha) - (u*\alpha)^{(n)}(t) - \sum_{n-j\neq k_0} A_{n-j}(u*\alpha)^{(j)}(t) \qquad \forall\, t \in R$$

so the relation (13)

$$(u*\alpha)^{(n)} + \sum_{j=0}^{n-1} A_{n-j}(u*\alpha)^{(j)} = f*\alpha \qquad \forall\, t \in R.$$

LEMMA 5 (Unicity): Let $A_1^*, \ldots, A_n^*$ satisfy condition S and let $u(t) \in L^2_{\mathrm{loc}}(R;H)$ with compact support in R be such that:

$$\int_R \left(u(t), (-1)^n + \sum_{j=0}^{n-1}(-1)^j (A_{n-j}\,\Phi^{(j)}\right) dt = \int_R (f(t), \Phi(t))\, dt, \quad \Phi \in K$$

with $f \in L^2_{\mathrm{loc}}(R;H)$ and $\operatorname{supp} f \subset [a,b]$.

Then $\operatorname{supp} u \subset [a,b]$.

PROOF: Let $\alpha_n(t) \in C_0^\infty$ such that $\alpha_n(t) \to \delta$ (the Dirac function) with $\alpha_n(t) = 0,\ |t| > \dfrac{1}{n}$.

Then by the preceding Lemma 4 we have for

$$u_k = u*\alpha_k, \qquad k = 1, 2, \ldots.$$

$$(u*\alpha_k)^{(n)} + \sum_{j=0}^{n-1} A_{n-j}^*(u*\alpha_k)^j = 0 \quad \text{for} \quad t \notin \left(a - \frac{1}{k},\ b + \frac{1}{k}\right)$$

by corollary of lemma: $u_k(t) = 0$ for $t \notin \left(a - \dfrac{1}{k},\ b + \dfrac{1}{k}\right)$.

Since $u(t)$ has compact support, $u_k(t)$ has compact support:
And $u_k(t) \to u(t)$ in $L^2_{\mathrm{loc}}(R;H)$ implies

$$u(t) = 0 \text{ outside } [a,b] \text{ a. e.}$$

DEFINITION. For $T > 0$, let V_T be the set of functions $u(t) \in L^2(-T,T;H)$ such that:

$$\int_{-T}^{T} (u(t), (-1)^n\,\Phi^{(n)} + \sum_{j=0}^{n-1}(A_{n-j}^*\,\Phi)^{(j)})\, dt = 0$$

or $\forall\, \Phi \in K^*(-T,T)$.

LEMMA 6 (Density): Let $\{A_1, \ldots, A_n\}$ satisfy conditon S and $0 < T_1 < < T_2 < T_3$ three arbitrary positive numbers.

Then V_{T_3} is dense in V_{T_2} for the $L^2(-T_1, T_1; H)$ topology.

PROOF: Let $\psi(t) \in L^2(-T_1, T_1; H)$ such that: $\int_{-T_1}^{T_1} (\psi, V)\, dt = 0$ for all $v \in V_{T_3}$. We shall show that:

$$\int_{-T}^{T_1} {}_1(\psi, h)\, dt = 0 \text{ for all } h(t) \in V_{T_2}.$$

For this, let $\psi(t) = 0$ outside $[-T_1, T_1]$.

Let

$$M = \left\{ (-1)^{n k^{(n)}} + \sum_{j=0}^{n-1} (-1)^{(j)} (A_{n-j}^* k)^{(j)} \right\}; \quad k(t) \in K^*(-T_3, T_3).$$

Then $\psi(t) \in \overline{M}$[the closure in $L^2(-T_3, T_3; H)$]

For if $U(t) \in L^2(-T_3, T_3; H)$ satisfies

$$\int_{-T_3}^{T_3} (U, (-1)^{n k^{(n)}} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* k)^{(j)})\, dt = 0:$$

$k(t) \in K^*(-T_3, T_3)$

then $U(t) \in V_{T_3}$ and

$$\int_{-T_3}^{T_3} (\psi, U)\, dt = \int_{-T_1}^{T_1} (\psi, U)\, dt = 0.$$

It follows that there exist a sequence $\{k_m\} \in K^*(-T_3, T_3)$ such that: $\lim_{m \to \infty} \left((-1)^n k_m^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* k_m)^{(j)} \right) = \psi$ in $L^2(-T_3, T_3; H)$ and since $\{k_m\}$ and ψ have their support in $[-T_3, T_3]$, the limit is valid in $L^2(-\infty, \infty; H)$.

But by the Lemma 1, the sequence $\{k_m\}$ is also convergent in $L^2(-T_3, T_3; H)$ (and hence in $L^2(R; H)$).

Let

$$\lim_{m \to \infty} k_m = \chi(t)$$

Furthermore for any $\Phi(t) \in K$, we have

$$\int_{-\infty}^{\infty}\left(x(t), \Phi^{(n)} + \sum_{j=0}^{n-1} A_{n-j} \Phi^{(j)}\right) dt = \lim_{m \to \infty} \int_{-\infty}^{\infty}\left(k_m, \Phi^{(n)} + \sum_{j=0}^{n-1} A_{n-j} \Phi^{(j)}\right) dt$$

$$= \lim_{m \to \infty} \int_{-\infty}^{\infty} ((-1)^n k_m^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* k_m)^{(j)}, \Phi) \, dt.$$

Hence there exists $\chi(t) \in L^2(R; H)$ such that

$$\int_{-\infty}^{\infty}\left(\chi(t), \Phi^{(n)} + \sum_{j=0}^{n-1} A_{n-j} \Phi^{j}\right) dt = \int_{-\infty}^{\infty} (\psi, \Phi) \, dt \qquad \forall \Phi \in K. \tag{15}$$

And since χ has compact support, it follows by Lemma 5 that χ has support in $[-T_1, T_1]$. Hence there exists $\chi(t) \in L^2_{\text{loc}}(R; H)$ and $\chi(t)$ satisfies (15):

To complete the proof, it remains to show that for $h(t) \in V_{T_2}$ we have:

$$\int_{-T_1}^{T_1} (\psi, h) \, dt = 0.$$

For this let $\{\alpha_n\} \to \delta$, $\alpha_n \in C_0^\infty(R)$. Then the function $\psi * \alpha_n$ has its support contained in $(-T_2, T_2)$ for sufficiently large n. We have for large n:

$$\int_{-T_2}^{T_2} (\psi * \alpha_n, h) \, dt = 0.$$

$$\int_{-\infty}^{\infty}\left(\chi, \Phi^{(n)} + \sum_{j=0}^{n-1} A_{n-j} \Phi^{(j)}\right) dt = \int_{-\infty}^{\infty} (\psi, \Phi) \, dt \qquad \forall \Phi \in K.$$

By Lemma 4:

$$(-1)^n (\chi * \alpha_m)^{(n)} + \Sigma (-1)^j \cdot (A_{n-j}^* \chi * \alpha_m)^{(j)} = \psi * \alpha_m \qquad \forall t \in R.$$

Then:

$$\int_{-\infty}^{\infty} ((-1)^n (\chi * \alpha_m)^{(n)} + \Sigma (-1)^j (A_{n-j}^* \chi * \alpha \;\;)^{(j)}, h) \, dt = \int_{-\infty}^{\infty} (\chi * \alpha_m, h) \, dt$$

$$= 0 \text{ for } h \in V_{T_2}.$$

And since $\chi * \alpha_m \in K^*(-T_2, T_2)$, $(\operatorname{supp} \chi \subset [-T_1, T_1])$ for large m. It follows that for large m:

$$\int_{-T_2}^{T_2} (\psi * \alpha_m, h)\, dt = 0.$$

Hence:

$$\int_{-T_2}^{T_2} (\psi, h)\, dt = \int_{-T_1}^{T_1} (\Phi, h)\, dt = 0.$$

And the Lemma is proved.

LEMMA 7. (approximation): $\{A_1, \ldots, A_n\}$ satisfy condition S and $0 < T_1 < T_2$. Then, the set of functions:

$u(t) \in L^2_{\text{loc}}(R; H)$ such that:

$$\int_{-\infty}^{\infty} \left(u, (-1)^n \Phi^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A^*_{n-j} \Phi)^{(j)} \right) dt = 0, \ \Phi \in K^*$$

is dense in V_{T_2} for the $L^2(-T_1, T_1; H)$ topology.

PROOF: Let $u_0(t) \in V_{T_2}$ and $\varepsilon > 0$. By Lemma 6, there exists $u_1(t) \in V_{T_2+1}$ such that:

$$\int_{-T_1}^{T_1} \| u_0(t) - u_1(t) \|^2\, dt < \frac{\varepsilon^2}{4}.$$

And there exist $u_2(t) \in V_{T_2+2}$ such that

$$\left\{ \int_{-T_2}^{T_2} \| u_2 - u_1 \|^2\, dt \right\}^{\frac{1}{2}} \leq \frac{\varepsilon}{2^2}$$

$$\Longrightarrow \left\{ \int_{-T_1}^{T_1} \| u_2 - u_0 \|^2\, dt \right\}^{\frac{1}{2}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} < \varepsilon.$$

So we can find a sequence $\{u_n\}$ $u_n(t) \in V_{T_2+n}$ such that

$$\left\{ \int_{-(T_2+n)}^{+T_2+n} \| u_{n+2} - u_{n+1} \|^2\, dt \right\}^{\frac{1}{2}} \leq \frac{\varepsilon}{2^{n+2}},$$

and

$$\left\{ \int_{-T_1}^{T_1} \| u_{n+2} - u_0 \|^2 \, dt \right\}^{\frac{1}{2}} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \dots + \frac{\varepsilon}{2^{n+2}} < \varepsilon.$$

Consider the series

$$u_1 + (u_2 - u_1) + \dots$$

this series converges in $L^2_{\text{loc}}(R\,;H)$

$$\text{And so} \lim_{n \to \infty} u_n = u_\varepsilon \text{ in } L^2_{\text{loc}}(R; H).$$

and we have:

$$\int_{-T_1}^{T_1} \| u_\varepsilon - u_0 \|^2 \, dt \leq \varepsilon^2.$$

Furthermore since: $u_k \in V_{T_2+k}$ satisfy:

$$\int_{-\infty}^{\infty} (u_k\,, (-1)^n \Phi^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A^*_{n-j} \Phi)^{(j)}) \, dt = 0$$

$$\forall\ \Phi \in K^* (-T_2 - k, T_2 + k).$$

u_ε satisfy the same equation for all $\Phi \in K^*$.

Proof of the theorem. $\{A_1\,, \dots, A_n\}$ satisfy condition S, and $f(t) \in L^2_{\text{loc}}(R\,;H)$. Let $f_n(t)$ be the restriction of $f(t)$ to $(-n, +n)$.

Then by Lemma 2, there exist a function $u_n(t) \in L^2(-n, n\,; H)$ such that:

$$\int_{-n}^{n} (u(t), (-1)^n \Phi^n + \sum_{j=0}^{n-1} (-1)^j (A^*_{n-j} \Phi)^{(j)}) \, dt = \int_{-n}^{n} (f_n(t), \Phi) \, dt\,; \ \forall \Phi \in K^*(-n, n)$$

Let us consider the series:

$$u_1 + (u_2 - u_1) + (u_3 - u_2) + \dots$$

The function: $u_n - u_{n-1} \in V_{n-1}$. So by lemma 6, there exist $h_n \in L^2_{\text{loc}}(R; H)$ such that:

$$\int_{-\infty}^{\infty} (h_n\,, (-1)^n \Phi^{(n)} + \Sigma (-1)^j (A_{n-j}\, \Phi)^{(j)}) \, dt = 0, \qquad \forall\ \Phi \in K^*$$

and

$$\left\{ \int_{-n+2}^{n-2} \| u_n - u_{n-1} - h_{n-1} \|^2 dt \right\}^{\frac{1}{2}} < \frac{\varepsilon}{2^n} .$$

Then the series:

$$u_1 + (u_2 - u_1 - h_1) + \dots + (u_n - u_{n-1} - h_{n-1}) + \dots$$

is convergent in $L^2_{\mathrm{loc}}(R; H)$ to a function $u(t)$ in $L^2_{\mathrm{l\,c}}(R; H)$ which satisfy:

$$\int_{-\infty}^{\infty} (u(t), (-1)^n \Phi^{(n)} + \Sigma (-1)^j (A^*_{n-j} \Phi)^{(j)}) \, dt = \int_{-\infty}^{\infty} (f(t), \Phi(t)) \, dt$$

for all $\Phi \in K^*$, i. e., $u(t)$ is a solution of (1) in the sense of (3).

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REFERENCES

[1] ZAIDMAN, S., *Equations différentielles abstraites.* Séminaire de Mathématiques Supérieures (1965), Les Presses de l'Université de Montréal.

[2] ZAIDMAN, S., *Un teorema di esistenza globale per alcune equazioni differenziali astratte.* Ricerche di Matematica, vol. XIII (1964).

[3] AGMON, S., & NIRENBERG, L., *Properties of solutions of ordinary differential equations in Banach space.* Comm. Pure Appl. Math., vol. XVI, No 2, 1963, p. 121.