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IN Variant PSEUDO-DIFFERENTIAL OPERATORS
ON A LIE GROUP

by ROBERT S. STRICHARTZ

ABSTRACT. - The invariant pseudo-differential operators on a Lie group $G$ with Lie algebra $\mathfrak{g}$ are characterized in terms of a function on $\mathfrak{g}^*$ called the Lie symbol. A calculus of Lie symbols is developed in terms of the algebraic structure of $\mathfrak{g}$. We also define global Sobolev spaces for non-compact groups.

§ 1. Introduction.

Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. The universal enveloping algebra $\mathcal{U}$ of $\mathfrak{g}$, which is the free algebra over the complexification of $\mathfrak{g}$ modulo the ideal generated by elements of the form $XY - YX - [XY]$, is naturally isomorphic to the algebra of left invariant partial differential operators on $G$. In this paper we define an analogous algebraic structure corresponding to classes of left invariant pseudo differential operators. Thus to every left invariant operator on $G$ we associate a function on $\mathfrak{g}^*$ (the dual space of $\mathfrak{g}$ considered as a vector space) called its Lie symbol. The correspondence is bi-unique if we consider operators modulo smooth operators and Lie symbols modulo functions in $\mathcal{S}(\mathfrak{g}^*)$. Composition of operators corresponds to a rather complicated product operation on the Lie symbols. This operation is described in § 3, and it is effectively computable from the Lie algebra structure of $\mathfrak{g}$. In particular we may identity $\mathcal{U}$ with the algebra of polynomials on $\mathfrak{g}^*$ under this composition law.

The usual symbolic calculus for pseudo differential operators is of course still valid in this context. It has the disadvantage, however, that left invariance of the operator is not conveniently expressible in properties of the symbol, except for the top order symbol, and even then it is not clear that to every top order symbol, which is invariant under the induced action of $G$ on its cotangent bundle, there corresponds an invariant operator.
We refer the reader to Stetkaer-Hansen [20] and Rothschild [14] who prove the non-existence of bi-invariant pseudo-differential operators which are not differential operators for most non-compact non-abelian groups.

In Section 2 we review the facts about pseudo-differential operators we will need. In Section 3 we construct the algebra of Lie symbols and prove the main theorems of the paper. In Section 4 we extend the calculus of Section 3 to operators with mixed homogeneity. In Section 5 we prove that a compactly supported pseudo-differential operator of order zero is bounded in $L^p(G)$ and use this fact to define global Sobolev spaces. We show that the usual properties of Euclidean Sobolev spaces are true also in this context. The results of Section 5 actually depend only on the top order calculus. An appendix is devoted to sketching some aspects of the local $L^p$ theory of pseudo-differential operators that do not seem to be adequately dealt with in the literature.

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A $C^\infty$ function $p(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support in the $x$ variable belongs to $S^m_\theta$ for $m$ real and $\frac{1}{2} < \theta \leq 1$ if

$$ |D^\alpha_x D^\beta_\xi p(x, \xi)| \leq c_{\alpha, \beta} (1 + |\xi|)^{m-\theta} |\beta|^{(1-\theta)} |\alpha| $$

for all multi-indices $\alpha, \beta$. We denote $S^\infty_\theta = \bigcup_m S^m_\theta$ and $S^{-\infty}_\theta = \bigcap_m S^m_\theta$. Two functions in $S^m_\theta$ are equivalent if they differ by a function in $S^{-\infty}_\theta$. Associated with $p(x, \xi)$ is the operator $p(x, D)$ given by

$$ p(x, D) u(x) = (2\pi)^{-n} \int p(x, \xi) \hat{u}(\xi) e^{ix \cdot \xi} \, d\xi 
= (2\pi)^{-n} \int \int u(y) e^{i(x-y) \cdot \xi} p(x, \xi) \, dy \, d\xi $$

where the iterated integral is taken first with respect to $y$ and then $\xi$.

The classes $S^m_\theta$ and associated operators were introduced by Hörmander in [6] and the reader is referred to that paper for the proofs of the properties we state below:
1. \( p(x, D) \) is a continuous operator on \( C_\text{com}^\infty (\mathbb{R}^n) \)

2. If \( p \in S_\psi^{-\infty} \) then \( p(x, D) u(x) = \int k(x, y) u(y) \, dy \) for some \( C^\infty \) function \( k(x, y) \) with compact support in \( x \), and conversely.

3. The operator \( p(x, D) \) is quasi-local, i.e. if \( u \) is \( C^\infty \) on any open set \( \Omega \) then \( p(x, D) u \) is \( C^\infty \) on \( \Omega \).

4. If \( p \in S_\psi^0 \) then \( \| p(x, D) u \|_2 \leq C_k \| u \|_2 \) for \( u \in C_\text{com}^\infty (|x| < k) \) for each \( k \). The same result holds for \( L^q \) norms for \( 1 < q < \infty \) provided \( q = 1 \), as will be shown in the appendix.

5. Given any sequence \( m_j \downarrow -\infty \) and \( p_j \in S_\psi^{m_j} \) there exists \( p \in S_\psi^m \), unique up to equivalence, such that \( p = \sum_{j=1}^{k-1} p_j \in S_\psi^m \). We write \( p \equiv \Sigma p_j \).

6. If \( p \in S_\psi^m \) and \( q \in S_\chi^m \) there exists \( r \in S_\psi^{m_1 + m_2} \) such that \( p(x, D) \cdot q(x, D) = r(x, D) \). In fact \( r \) is given up to equivalence by

\[ r \equiv \sum_a \frac{1}{a!} \left( \frac{\partial}{i \partial \xi} \right)^a p \left( \frac{\partial}{\partial x} \right)^a q. \]

7. The class of operators \( p(x, D) \) for \( p \in S_\psi^m \) is invariant under diffeomorphism.

We also consider a subspace of \( S_\psi^m \), denoted \( S_h^m \) of symbols which are homogeneous in the following sense: \( p \equiv \Sigma p_j \) where \( p_j(x, t \xi) = t^{m_j} p_j(x, \xi) \) for \( t \geq 1 \) and \( |\xi| \geq 1 \) for \( m_j \downarrow -\infty \). These symbols and the corresponding operators were the first class (locally) of pseudo-differential operators considered in Kohn-Nirenberg [10]. They are preserved under the operations in 5, 6, 7 above. If \( p \in S_h^m \) is a polynomial in \( \xi \) for each \( x \), then \( p(x, D) \) is a differential operator.

Now let \( M \) be a paracompact \( C^\infty \) manifold with smooth measure \( dx \) equivalent to Lebesgue measure in every coordinate patch. We say an operator \( T: C_\text{com}^\infty (M) \rightarrow C^\infty (M) \) is smooth if it has the form \( Tu(x) = \int k(x, y) u(y) \, dy \) for some \( k \in C^\infty (M \times M) \). An operator \( T: C_\text{com}^\infty (M) \rightarrow C^\infty (M) \) is called a general pseudo-differential operator of class \( S_\psi^m \) or \( S_h^m \) if for any coordinate patch \( U \) and \( \varphi \in C_\text{com}^\infty (U) \) the operator \( T(\varphi u) \) is
the sum of a smooth operator and an operator \( p(x, D) \) for \( p \in S^m \) or \( S^m_h \) in the local coordinates of \( U \). Because of property 7 this definition is independent of the choice of local coordinates.

A pseudo-differential operator is said to be compactly supported if given any compact neighborhood \( \Omega \) there exists a compact neighborhood \( \Omega \) such that \( T \) vanishes in \( \Omega \) whenever \( u \) vanishes in \( \Omega \), and \( T \) has support in \( \Omega \) whenever \( u \) has support in \( \Omega \). It is easily seen that a compactly supported operator preserves \( C^\infty(M) \) and by extension preserves \( C^\infty(M) \). Thus two such operators may be composed, while two general pseudo-differential operators need not have a well-defined composition. Note, however, that a general operator may be composed with a compactly supported operator in either order.

While many pseudo-differential operators naturally arising on non-compact manifolds are not compactly supported, there is no real loss in generality in restricting consideration to compactly supported operators in view of the following observation: every pseudo-differential operator differs from a compactly supported one by a smooth operator. This may be seen by considering a partition of unity \( \varphi_1, \ldots \) and observing that
\[
T = \sum \sum M_{\varphi_j} T M_{\varphi_k}
\]
where \( M_{\varphi_j} \) denotes the operator \( M_{\varphi_j} u(x) = \varphi_j(x) u(x) \). If we take the sum over those \( j \) and \( k \) such that \( \varphi_j \varphi_k = 0 \) we obtain a smooth operator, and the remaining summands give a compactly supported operator. Thus restricting operators to be compactly supported is merely a technical device to simplify the theory on non-compact manifolds.

§ 3. The calculus of Lie symbols.

Let \( G \) be a Lie group and \( \mathfrak{g} \) its Lie algebra. We use lower case roman letters to denote elements of \( G \), upper case for elements of \( \mathfrak{g} \), and greek letters for elements of \( \mathfrak{G}^* \). Let \( X_1, \ldots, X_n \) be a basis for \( \mathfrak{g} \), and \( \xi_1, \ldots, \xi_n \) the dual basis for \( \mathfrak{G}^* \). The pairing between \( \mathfrak{g} \) and \( \mathfrak{G}^* \) is denoted by the dot product \( X \cdot \xi \).

We shall deal only with Lie algebra products which are associated \( [..[XY]Z]..] \) \( W \) and which we denote bracket \( (XYZ \ldots W) \). For \( X \in \mathfrak{g} \) we let \( ad_X Y = (XY) \) \( X \) \( (XY) \); \( ad_X X \) should be regarded as a polynomial map of \( \mathfrak{g} \) to itself.

There exists \( \varepsilon > 0 \) such that the exponential map \( \exp: \mathfrak{g} \rightarrow G \) is a diffeomorphism of \( |X| < \varepsilon \) onto a neighborhood of the identity of \( G \). For \( X \in \mathfrak{g} \) we associate the left invariant first order partial differential operator
\[
X u(x) = \frac{d}{dt} u(x \exp tX) |_{t=0}.
\]
We shall use the Campbell-Hausdorff formula (see [7]) which states that for sufficiently small $X$ and $Y$ we have $\exp X \exp Y = \exp Z$ where

$$Z = \sum_m \sum_{p_i, q_i} \frac{(-1)^{m-1}}{m \Sigma (p_i + q_i)} \frac{\text{bracket}(X^{p_1} Y^{q_1} X^{p_2} Y^{q_2} \cdots Y^{q_m})}{p_1! q_1! p_2! q_2! \cdots p_m! q_m!}$$

where $m$ varies over the positive integers and, for each $m$, $p_i$ and $q_i$ for $i = 1, \ldots, m$ vary over the positive integers subject to $p_i + q_i > 0$. As a consequence we have $\exp tX \exp Y = \exp Z_t$ where

$$Z_t = Y + tX + t \sum_{k=1}^{\infty} c_k \text{bracket}(XY^k) + O(t^2)$$

as $t \to 0$. The actual determination of the constants $c_k$ from (3.1) is a combinatorial problem.

Let us now consider a left invariant compactly supported pseudo-differential operator $T$ on $G$ of class $S^m$. Now $G$ has a distinguished set of coordinate patches, namely for each $x \in G$ we may coordinatize a neighborhood of $x$ by $x \exp X$ for $|X| < \varepsilon$. Thus for each $x \in G$ there exists $p_x(X, \xi) \in S^m$ such that

$$T u(x \exp X) = (2\pi)^{-n} \int u(x \exp Y) e^{i(X-Y) \cdot \xi} p_x(X, \xi) \psi(Y) \, dY \, d\xi$$

where $\psi \in C^\infty(|X| < \varepsilon)$ and $\psi = 1$ in a neighborhood of the origin.

Now the left invariance of $T$ means we can choose $p_x$ to be independent of $x$ by modifying the smooth operator in (3.3). The requirement that (3.3) define the same operator in every coordinate neighborhood places additional restrictions on $p_x(X, \xi)$ which we shall not study directly. Instead we set $X = 0$ in (3.3) and obtain

$$T u(x) = (2\pi)^{-n} \int u(x \exp Y) e^{-iX \cdot \xi} \psi(Y) \, dY \, d\xi + u \ast \varphi$$

where $\psi \in C^\infty(G)$ and $u \ast \varphi = \int_0^1 u(xy^{-1}) \varphi(y) \, dy$. Here we have deduced the form of the smooth operator from the fact that $T$ is invariant and compactly supported. We note that $p(\xi) \in S^m$ or $S^m$ if $p(X, \xi)$ is. By $p(\xi) \in S^m$ we mean (2.1) holds without $X$ derivatives.
Thus we see that every left invariant pseudo-differential operator must have the form (3.4). The converse is also true:

**Theorem 1:** $T$ is a left invariant compactly supported pseudo-differential operator of class $S^m_0$ or $S^m_0$ if and only if it has the form (3.4). The correspondence $T \leftrightarrow p(\xi)$ is bi-unique between classes of operators modulo convolutions with $C^\infty_0$ functions and classes of functions in $S^m_0$ modulo functions in $\mathscr{S}(\mathbb{S}^n)$. $T$ is a differential operator if and only if $p(\xi)$ may be chosen to be a polynomial.

**Proof:** We must show how to go from (3.4) back to (3.3), deducing the form of $p(X, \xi)$ from $p(\xi)$. To do this we replace $x$ by $x \exp X$ in (3.4) and set $\exp X \exp Y = \exp Z$. To do this we must take $X$ sufficiently small, and $Y$ may be forced to be small by shrinking the support of $\psi$. We note that changing $\psi$ away from the origin only results in a change in the smooth operator $u \ast \varphi$. Thus we have

\begin{equation}
(3.5) \quad Tu(x \exp X) = (2\pi)^{-n} \int u(x \exp Z) e^{-iY \cdot \xi} p(\xi) \psi(Y) J(X, Z) dZ d\xi
\end{equation}

where $Y$ may be determined from $\exp Y = \exp(-X) \exp Z$ by the Campbell-Hausdorff formula (3.1). Now we claim that for small $X$ and $Z$ that $-Y = W(X, Z)(X - Z)$ where $W(X, Z)$ is an invertible linear transformation on $\mathcal{G}$ which depends on $X$ and $Z$ in a $C^\infty$ manner. Indeed (3.1) gives $-Y = X - Z + \frac{1}{2} [X, Z] + \ldots$ where each subsequent term is obtained by multiplying by a constant and applying $adX$ and $adY$ in various orders to $[XZ] = [(X - Z)Z] + [ZZ] = -adZ(X - Z)$. Thus $-Y = X - Z + Q(adX, adZ)(X - Z)$ where $Q$ is a non-commutative power series with values in the space of linear transformations on $\mathcal{G}$. If we take $X$ and $Z$ small enough then $\|adX\|$ and $\|adZ\|$ will be small and the power series will converge because of the decrease of the coefficients in (3.1). Since $Q$ has no constant terms we may make $\|Q(adX, adZ)\| < 1$ and obtain the invertibility of $W(X, Z) = I + Q(adX, adZ)$. Thus

$$
\psi(X) Tu(x \exp X) = (2\pi)^{-n} \int u(x \exp Z) e^{i(X - Z) \cdot \xi} w(X, Z)^{\xi} p(\xi).
$$

$$
J(X, Z) \psi(X) \psi(-W(X, Z)(X - Z)) dZ d\xi
$$

$+a$ smooth operator.
Changing variables in the $\xi$ integration we obtain

$$\psi (X) T u (x \exp X) = (2\pi)^{-n} \int \int u (x \exp Z) e^{i(x - Z) \cdot \xi} p \left( W (X, Z)^{\ast -1} \xi \right)$$

$$R (X, Z) dZ d\xi$$

$$+ a \text{ smooth operator}$$

where $R (X, Z) \in C^\infty (G \times G)$. We note that

Thus we may apply a lemma of Kuranishi (see [13], p. 155) which allows us to conclude from (3.6) and (3.7) that $T$ is a pseudo-differential operator of class $S^m$, and

$$p (X, \xi) \in S^m \text{ if } p (\xi) \text{ does, and } p (X, \xi) \text{ may be taken to be a polynomial in } \xi \text{ if } p (\xi) \text{ may. Also } p (\xi) \in S^m \text{ if and only if } p (\xi) \in \mathfrak{g}.$$ The theorem follows immediately from the above and the properties of pseudo-differential operators outlined in § 2. Q. E. D.

We call $\mathcal{C}_p = \text{the space of Lie symbols of class } \mathfrak{g}$, and loosely speaking we call $p (\xi)$ the Lie symbol of $T$. We call $p (X, \xi)$ the full symbol of $T$.

We turn now to the question of what operation on the Lie symbols corresponds to composition of operators. From property 6 of § 2 it is clear that the key step in answering this question is to compute $\left( \frac{\partial}{\partial X} \right)^a p (X, \xi) |_{X=0}$ in terms of $p (\xi)$. While it is in principle possible to deduce this from (3.8), we have found it easier to do this by computing $X_j T u$ using (3.3) and (3.4) and comparing the two results.

**Lemma 1:** If $T$ has full symbol $p (X, \xi)$ then $X_j T$ has full symbol $q_j (X, \xi)$ where

$$q_j (X, \xi) = i \left( X_j + \sum_{k=1}^\infty (-1)^k e_k a d_k X_j (X) \right) \cdot \xi p (X, \xi) +$$

$$\left( X_j + \sum_{k=1}^\infty (-1)^k e_k a d_k X_j (X) \right) \cdot F_X p (X, \xi)$$
and Lie symbol \( q_j(\xi) \) given by

\[
q_j(\xi) = iX_j \cdot \xi p(\xi) + \frac{\partial}{\partial \xi_j} p(X, \xi) \big|_{x=0}.
\]

Here \( c_k \) are the constants appearing in (3.2).

**Proof:** We have, using (3.3),

\[
X_j Tu(x \exp X) = \frac{d}{dt} Tu(x \exp X \exp tX_j) \big|_{t=0} =
\]

\[
\frac{d}{dt} (2\pi)^{-n} \int u(\exp X) e^{i(x-Y)^j \xi} p(Z, \xi) \psi(Y) dY d\xi \big|_{t=0}
\]

\[
+ a \text{ smooth operator}
\]

where \( Z_t = X + tX_j + t \sum_{i=1}^{\infty} (-1)^i c_k a d_k X_j(X) + o(t^2) \) from (3.2). We may now differentiate, ignoring the \( o(t^2) \) terms, and set \( t = 0 \) to obtain (3.9). Setting \( X = 0 \) we obtain (3.10).

**Lemma 2:** If \( T \) has Lie symbol \( p(\xi) \) then \( X_j T \) has Lie symbol \( q(\xi) \) where

\[
q(\xi) \sim iX_j \cdot \frac{\partial}{\partial \xi_j} p(\xi) + i\xi \cdot \sum_{k=1}^{\infty} c_k a d_k X_j \bigg( \frac{\partial}{\partial \xi_j} \bigg) p(\xi).
\]

**Proof:** We have, using (3.4),

\[
X_j Tu(x) = \frac{d}{dt} (2\pi)^{-n} \int u(x \exp tX_j \exp Y) e^{-X \cdot \xi} p(\xi) \psi(\xi) dY d\xi \big|_{t=0}
\]

\[
+ a \text{ smooth operator.}
\]

For fixed \( t \) we set \( \exp tX_j \exp Y = \exp Z \) or \( \exp Y = \exp (-tX_j) \exp Z \) and change the \( Y \) integration to a \( Z \) integration, absorbing the change in \( \psi \) into the smooth operator:

\[
X_j Tu = \frac{d}{dt} (2\pi)^{-n} \int u(\exp Z) e^{(-Z+tX_j+t \sum_{k=1}^{\infty} c_k a d_k X_j(Z)+o(t^2)) \cdot \xi}
\]

\[
\cdot p(\xi) \det \left( \frac{\partial}{\partial \xi} \left( Z - tX_j - t \sum_{k=1}^{\infty} c_k a d_k X_j(Z) + o(t^2) \right) \right),
\]

\[
\cdot \psi(Z) dZ d\xi \big|_{t=0} + a \text{ smooth operator}
\]
where we have used (3.2). Using the fact that 
\[ \frac{d}{dt} \det (I + tA + O(t^2)) \bigg|_{t=0} = \text{tr } A \]
we have

\[ (3.12) \quad X_j Tu(x) = \frac{(2\pi)^n}{n} \int u(x \exp Z) e^{-iZ \cdot \xi} p(\xi) \left( iX_j \cdot \xi + \sum_{k=1}^{\infty} c_k a d_k X_j(Z) \cdot \xi - \text{tr } \frac{\partial}{\partial Z} \sum_{k=1}^{\infty} c_k a d_k X_j(Z) \right) \psi(Z) dZ d\xi. \]

Since this is a Fourier integral we may integrate by parts and neglect boundary terms to replace the polynomials in \( Z \) by differentiations of \( p(\xi) \).

Indeed \( \left( \frac{\partial}{\partial \xi} \right)^1 e^{-iZ \cdot \xi} = (-1)^{a_1} Z^{a_1} e^{-iZ \cdot \xi} \) so \( a d_k X_j \left( \frac{\partial}{\partial \xi} \right) e^{-iZ \cdot \xi} = (-1)^{a_1} a d_k \cdot X_j(Z) e^{-iZ \cdot \xi} \). By Leibnitz' rule we have

\[ i a d_k X_j \left( \frac{\partial}{\partial \xi} \right)^1 e^{-iZ \cdot \xi} = (-1)^{a_1} \left( i a d_k X_j(Z) - \text{tr } \frac{\partial}{\partial Z} a d_k X_j(Z) \right) e^{-iZ \cdot \xi} \]

so when we integrate by parts we have

\[ \int u(x \exp Z) e^{-iZ \cdot \xi} p(\xi) \left( i a d_k X_j(Z) - \text{tr } \frac{\partial}{\partial Z} a d_k X_j(Z) \right) \psi(Z) dZ d\xi = \int \int u(x \exp Z) e^{-iZ \cdot \xi} i \xi \cdot a d_k X_j \left( \frac{\partial}{\partial \xi} \right) p(\xi) \psi(Z) dZ d\xi. \]

Substituting this in (3.12) for \( k = 1, \ldots, N \) and handling the remainder in the usual way we deduce (3.11).

Q. E. D.

**Lemma 3:** There exist infinite order partial differential operators \( A(\alpha) \) on \( G^* \) with polynomial coefficients such that

\[ (3.13) \quad \left( \frac{\partial}{\partial X} \right)^\alpha p(X, \xi) \bigg|_{X=0} = A(\alpha) p(\xi). \]

If we write \( A(\alpha) = \sum_{\beta, \gamma} c(\alpha, \beta, \gamma) \xi^\beta \left( \frac{\partial}{\partial \xi} \right)^\gamma \) then non-zero terms occur only when \( |\beta| \leq |\gamma| \) and \( |\hat{\beta}| \leq |\alpha| \). Thus \( A(\alpha) \) preserves the classes \( S^m_\alpha \) and \( S^m_\alpha \). \( A(\alpha) \) may be computed recursively from the following relation:

\[ (3.14) \quad \prod_{r=1}^{m} \left( i \xi_{(r)} + \frac{\partial}{\partial X_{(r)}} \right) p(X, \xi) \bigg|_{X=0} = \text{sym} \prod_{r=1}^{m} \left( i \xi_{(r)} + B(X_{(r)}) \right) p(\xi) \]
where the symmetric product $\text{sym} \Pi$ is the average of the product for all $m!$ orderings of the factors and

\begin{equation}
B(X_j) = i \tilde{\xi} \cdot \sum_{k=1}^{\infty} c_k \, \text{ad}_{\tilde{\xi}} \, X_j \left( \frac{\partial}{\partial \tilde{\xi}} \right).
\end{equation}

An explicit algorithm for computing $A(\alpha)$ is: 1) expand out the right side of (3.14); 2) commute all terms where $B(X_j)$ stands to the left of $\tilde{\xi}_k$ by replacing $B(X_j) \tilde{\xi}_k$ by $\tilde{\xi}_k B(X_j) + [B(X_j), \tilde{\xi}_k]$; 3) similarly commute any multiple commutator of $B(X_j)$ which stands to the left of $\tilde{\xi}_k$; 4) delete all terms in which any $\tilde{\xi}_k$ occurs outside a commutator.

**Proof:** We deduce relation (3.14) by computing the Lie symbol of $\text{sym} \Pi \sum_{r=1}^{m} X_{j(r)} T$ by iterating Lemma 1 and 2. Indeed by iterating Lemma 2 we obtain easily the right side of (3.14). In iterating Lemma 1 we must be careful to apply all the differential operators before setting $X = 0$; thus we obtain that the Lie symbol is

$$
\text{sym}\sum_{r=1}^{m} \left[ i \left( X_{j(r)} + \sum_{k=1}^{\infty} (-1)^{k} c_k \, \text{ad}_{\tilde{\xi}} X_{j(r)}(X) \right) : \tilde{\xi} + \left( X_{j(r)} + \sum_{k=1}^{\infty} (-1)^{k} c_k \, \text{ad}_{\tilde{\xi}} X_{j(r)}(X) \right) \cdot \mathcal{P}(X, \tilde{\xi}) \right]_{X = 0}.
$$

Now it appears that in addition to the commuting terms on the left side of (3.14) there are many more terms arising from applying the $X$ derivatives to the coefficients. We claim, however, that all those terms that survive setting $X = 0$ will cancel when the product is symmetrized. To see this we observe that $\text{ad}_{\tilde{\xi}} X_j(X)$ is a homogeneous polynomial of degree $k$ in $X$ and

$$
\frac{\partial}{\partial X_{j(1)}} \ldots \frac{\partial}{\partial X_{j(k)}} \text{ad}_{\tilde{\xi}} X_j(X) = \text{bracket (} X_j X_{j(1)} \ldots X_{j(k)} \text{)} + \text{similar brackets.}
$$

All terms that arise contain such expressions, and by interchanging the role of $X_j$ and $X_{j(1)}$ we obtain exactly the negative since each index $j(r)$ occurs only once in each term. Thus they all cancel in the symmetric product.

Once (3.14) has been established the rest of the Lemma follows easily. We observe that $B(X_j)$ contains terms of the form $\xi^\beta \left( \frac{\partial}{\partial \xi} \right)^\gamma$ for $|\beta| = 1$ and $|\gamma| \geq 1$ so the corresponding observations about the terms in $A(\alpha)$ follow.

The algorithm for computing $A(\alpha)$ is easily established by induction.

Q. E. D.
THEOREM 2: Let $S$ and $T$ be left invariant compactly supported pseudo-differential operators of class $S^m_e$ and $S^{m'}_e$ respectively. Let $S$ have Lie symbol $p(\xi)$ and $T$ have Lie symbol $q(\xi)$. Then $ST$ is a left invariant compactly supported pseudo-differential operators of class $S^{m+m'}_e$ with Lie symbol

$$r(\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \left( \frac{\partial}{i \partial \xi} \right)^\alpha p(\xi) A(\alpha) q(\xi).$$

PROOF: It is obvious that $ST$ has all the properties except perhaps (3.16). Now the terms $\left( \frac{\partial}{i \partial \xi} \right)^\alpha p(\xi) \xi^\beta \left( \frac{\partial}{i \partial \xi} \right)^\gamma q(\xi)$ appearing in (3.16) are in $S^{m+m'+|\beta|-e(|\alpha|+|\gamma|)}_e$ so there are only a finite number of order $> M$ for any real $M$ (here we use $|\beta| \leq |\alpha|, |\beta| \leq |\gamma|$ and $e > \frac{1}{2}$). If we denote the sum of these terms by $r_M(\xi)$ and the Lie symbol of $ST$ by $r(\xi)$ it suffices to show $r_M(\xi) - r_M(\xi) \in S^M_e$ to establish (3.16). But we know from Lemma 3 and property 6 of § 2 that there exists a pseudo differential operator $R_M$ (which need not be left invariant) with symbol $r_M(X, \xi)$ in the coordinate neighborhood about the identity with $r(X, \xi) - r_M(X, \xi) \in S^M_e$ and $r_M(\xi) - r_M(0, \xi) \in S^M_e$. Indeed we take $r_M(X, \xi) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \left( \frac{\partial}{i \partial \xi} \right)^\alpha p(X, \xi).$

Thus (3.16) is established by setting $X = 0$.

Q. E. D.

REMARK: It is clear that the product (3.16) preserves the class $S^m_e$ and also the subclass of homogeneous polynomials. Thus we have obtained a faithful representation of $\mathcal{U}$ as the polynomials on $\mathfrak{G}^*$ with the product (3.16), in this case $\infty$ may be replaced by $= .$

We next consider adjoints. We do not use the standard adjoint formula for pseudo-differential operators because it gives the adjoint with respect to the wrong measure.

THEOREM 3: Let $T$ be a left invariant compactly supported pseudo-differential operators of class $S^m_e$ with Lie symbol $p(\xi)$. Then the adjoint operator $T^*$ with respect to left Haar measure is also a left invariant compactly supported pseudo-differential operators of class $S^m_e$ and $T^*$ has symbol $q(\xi) = \overline{p(\xi)}$ if $G$ is unimodular or

$$q(\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} (\xi^0)^{\alpha} \left( \frac{\partial}{i \partial \xi} \right)^\alpha p(\xi).$$
in general where

\[ \Lambda(\exp X) = e^{X \cdot \xi^0}, \]

\( \Lambda \) the modular function.

**PROOF:** By definition, \( \int_\mathbb{G} Tu(x) v(x) \, dx = \int_\mathbb{G} u(x) T^*v(x) \, dx \) so we compute, neglecting smooth operators,

\[
\int_\mathbb{G} Tu(x) v(x) \, dx = (2\pi)^{-n} \int_\mathbb{G} \int \int u(x \exp Y) e^{-iY \cdot \xi} \psi(Y) dY d\xi \, dx
\]

\[ = (2\pi)^{-n} \int_\mathbb{G} \int \int u(x) e^{-iY \cdot \xi} \psi(Y) \Lambda(\exp Y) dY d\xi \, dx \]

\[ = (2\pi)^{-n} \int_\mathbb{G} \int \int v(x \exp Y) e^{-iY \cdot \xi} \Lambda(\exp Y) \psi(Y) dY d\xi \, dx. \]

Now \( \Lambda(\exp X) = e^{X \cdot \xi^0} \) for some \( \xi^0 \in \mathfrak{g}^* \); in fact \( X \cdot \xi^0 = \text{tr} \, adX \) (see Helgason [5] pp. 366-367). Thus we conclude that

\[
T^*v(x) = (2\pi)^{-n} \int_\mathbb{G} \int \int v(x \exp Y) e^{-iY \cdot \xi} e^{Y \cdot \xi^0} \psi(Y) dY d\xi \, dx.
\]

Now this appears to require an analytic continuation of \( \overline{p(\xi)} \) to \( \overline{p(\xi + i\xi^0)} \) to obtain the form (3.4). Since the Lie symbol is only defined modulo functions in \( \mathcal{D} \), however, we have available a substitute for analytic continuation even when \( \overline{p} \) isn't analytic. Indeed we let \( \overline{p_k(\xi)} = \sum \frac{1}{\alpha!} (\xi^0)^\alpha \cdot \overline{p_{k-\alpha}(\xi)} \); note this is just the partial power series expansion of \( \overline{p(\xi + i\xi^0)} \) if it existed. In order to establish (3.17) we must show

\[ (2\pi)^{-n} \int_\mathbb{G} \int \int v(x \exp Y) e^{-iY \cdot \xi} (e^{Y \cdot \xi^0} \overline{p(\xi)} - \overline{p_k(\xi)}) \psi(Y) dY d\xi \]

is a pseudo-differential operator of class \( \mathcal{S}_M^\infty \) for any real \( M \) provided \( k \) is large enough. But by integration by parts we easily transform (3.19) into

\[ (2\pi)^{-n} \int_\mathbb{G} \int \int v(x \exp Y) e^{-iY \cdot \xi} \overline{p(\xi)} \left( e^{Y \cdot \xi^0} - \sum_{j=0}^k \frac{1}{j!} (Y \cdot \xi^0)^j \right) \psi(Y) dY d\xi. \]
We now repeat the argument given in the proof of Theorem 1, replacing (3.4) by (3.20). We may thus conclude that (3.20) defines a pseudo-differential operator whose symbol is given by a variant of (3.8). Note that since

\[ e^{X_0} = \sum_{j=0}^{k} \frac{1}{j!} (Y \cdot \xi)^j \]

vanishes to order \( k \), the function \( R(X, Z) \) in (3.8) will contain a factor that vanishes to order \( k \) in \( X - Z \). Thus we must take \( \alpha \geq k \) in (3.8) in order to obtain a non-zero contribution, so in fact the symbol is of class \( S_{\phi}^{m-k(2k-1)} \). Since \( \phi > \frac{1}{2} \) we may make \( m - k(2k-1) \) arbitrarily small by taking \( k \) large. Q. E. D.

All the above results have obvious extensions to systems of operators. It is not necessary to formulate results for vector bundles with group action over \( G \) because all such bundles are trivial. Indeed if \( \pi : E \rightarrow G \) is a vector bundle on which \( G \) acts by \( L_\pi \) covering left multiplication \( \pi(L_\pi e) = x \pi(e) \) then we map \( E \) to \( G \times \pi^{-1}(1) \) by \( e \rightarrow (\pi(e), L_{\pi(e)}^{-1} e) \). It is clear this a bundle isomorphism and \( L_\pi e \rightarrow (x \pi(e), L_{(x \pi(e))^{-1}} L_\pi e) = (x \pi(e), L_{\pi(e)}^{-1} e) \) so the induced action on \( G \times \pi^{-1}(1) \) is left multiplication cross the identity. Note, however, if we are also given an action \( R_\pi \) of \( G \) on \( E \) covering right multiplication it need not be trivialized by this map.

For our next result we do not assume \( T \) compactly supported, which means the convolution kernel \( \phi \) in (3.4) need not have compact support.

**Theorem 4:** Let \( T \) be a left invariant pseudo-differential operator of class \( S_{\phi}^{m} \) with Lie symbol \( p(\xi) \). If we have \( \phi = 1 \) and

\[
|p(\xi)| \geq c|\xi|^m \quad \text{for} \quad |\xi| \geq c
\]

then \( T \) is elliptic.

If we assume merely that

\[
|p(\xi)| \geq c|\xi|^{m'} \quad \text{for} \quad |\xi| \geq c
\]

and

\[
\left| (i \xi \partial_{\xi})^{\alpha} p(\xi) \right| \leq c_{\alpha} |p(\xi)| (1 + |\xi|)^{-\alpha} \quad \text{for} \quad \alpha \geq 1
\]

then \( T \) is hypoelliptic.

**Proof:** This is a simple variant of Theorem 4.2 of Hörmander [6]. The point is that we are assuming only estimates on the Lie symbol and not on the full symbol.

We construct a parametrix \( S \) of class \( S_{\phi}^{m} \) for the elliptic case and class \( S_{\phi}^{m} \) for the hypoelliptic case. The theorem then follows by standard arguments. We construct only a right parametrix because the hypotheses of the theorem are easily seen to hold for \( T^{*} \).
Thus we need to construct a Lie symbol \( q(\xi) \) with \( \sum_{a} \frac{1}{\alpha!} \left( \frac{\partial}{\partial \xi^a} \right)^{\alpha} p(\xi) \cdot A(\alpha) q(\xi) \sim 1 \). We do this by setting \( q_0(\xi) = p(\xi)^{-1} \) for large \( \xi \) and \( C^\infty \) elsewhere and then solving \( p(\xi) q_j(\xi) \sim -\sum_{a \neq 0} \frac{1}{\alpha!} \left( \frac{\partial}{\partial \xi^a} \right)^{\alpha} p(\xi) A(\alpha) q_j(\xi) \).

We have \( q_0 \in S_{\theta}^\infty \) and by induction \( q_j \in S_{\theta}^{\infty} \) for \( j \leq 2 \). Thus we may set
\[
q(\xi) \sim \sum_{j=0}^{\infty} q_j(\xi) \in S_{\theta}^{\infty}
\]
and obtain
\[
\sum_{a} \frac{1}{\alpha!} \left( \frac{\partial}{\partial \xi^a} \right)^{\alpha} p(\xi) A(\alpha) q(\xi) \sim \sum_{a} \frac{1}{\alpha!} \left( \frac{\partial}{\partial \xi^a} \right)^{\alpha} p(\xi) A(\alpha) q_j(\xi) \sim p(\xi) q_0(\xi)
\]
\[
+ \sum_{j=1}^{\infty} \left( p(\xi) q_j(\xi) + \sum_{a \neq 0} \frac{1}{\alpha!} \left( \frac{\partial}{\partial \xi^a} \right)^{\alpha} p(\xi) A(\alpha) q_{j-1}(\xi) \right).
\]
The last terms cancel and \( p(\xi) q_0(\xi) \sim 1 \). Q. E. D.

**Remark:** While the theorem applies to differential operators given by non commutative polynomials in \( X_1, \ldots, X_n \), except in the elliptic case it may be difficult to verify the hypotheses since computing the Lie symbol of such an operator requires using (3.4).


In this section we indicate how to extend the results of Section 3 to operators with mixed homogeneity. The Euclidean theory of singular integral operators with mixed homogeneity may be found in Fabes and Riviere [4]. The compatibility requirements we place on the structure constant of the Lie algebra may be verified for many examples of nilpotent groups. However they do not apply to the singular integral operators studied by Knapp and Stein [9].

Let \( a = (a_1, \ldots, a_n) \) be a fixed \( n \)-tuple of positive reals. Let \( \delta(r) x = (r^{a_1} x_1, \ldots, r^{a_n} x_n) \) and let \( w(x) \) be a \( C^\infty \) positive valued function on \( \mathbb{R}^n \) which satisfies \( w(\delta(r) x) = r \cdot w(x) \). It is easy to see that such functions exist and any two give equivalent theories. We then define the class \( S^m_\theta \), in analogy with \( S_1^m \), to be the set of \( C^\infty \) functions on \( \mathbb{R}^n \times \mathbb{R}^n \) with compact support in the \( x \) variable, and satisfying
\[
|D_x^\alpha D_\xi^\beta p(x, \xi)| \leq C_{\alpha, \beta}(1 + w(\xi))^{m-\alpha, \beta}
\]
for all \( \alpha, \beta \). We may then define an operator \( p(x, D) \) by (2.2) and verify properties 1-6 of Section 2. Property 7 does not hold in general since \( w(\xi) \) is not invariant under rotations. Nevertheless if we are given a fixed set of coordinate neighborhoods and local coordinates for a manifold \( M \) we may define pseudodifferential operators of class \( \mathcal{S}_2^m \) as in Section 2; the definition now will depend on the choice of local coordinates.

We will also need a version of Kuranishi's lemma: if \( p(x, \xi, y) \) is a \( C^\infty \) function on \( R^n \times R^n \times R^n \) with compact support in the \( x \) and \( y \) variables and satisfying

\[
|D_\xi^a D_\xi^b D_y^c p(x, \xi, y)| \leq C_{\alpha, \beta, \gamma} (1 + w(\xi))^{m-\alpha, \beta}
\]

then

\[
(2\pi)^{-n} \int u(y) e^{i(x-y) \cdot \xi} p(x, \xi, y) \, dy \, d\xi
\]
defines a pseudo-differential operator of class \( \mathcal{S}_2^m \) with symbol \( q(x, \xi) \) satisfying

\[
q(x, \xi) \sim \sum_a \frac{1}{a!} \left( \frac{\partial}{i\partial \xi} \right)^a \left( \frac{\partial}{\partial y} \right)^a \left. p(x, \xi, y) \right|_{x=y=x}.
\]

The proof is a trivial modification of the proof given in Nirenberg [13].

Now let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \). We say that a basis \( X_1, \ldots, X_n \) of \( \mathfrak{g} \) with dual basis \( \xi_1, \ldots, \xi_n \) of \( \mathfrak{g}^* \) is compatible with \( a \) if the structure constants \( c_{ijk} = [X_i, X_j] \cdot \xi_k \) vanish whenever \( a_k > a_j \).

**Theorem 5**: Assume there exists a basis \( X_1, \ldots, X_n \) for \( \mathfrak{g} \) compatible with \( a \). Then the class of left-invariant compactly supported pseudo-differential operators of class \( \mathcal{S}_2^m \) with respect to the canonical coordinate systems may be described in analogy with the class \( \mathcal{S}_1^m \) as follows: they are of the form

\[
Tu(x) = (2\pi)^{-n} \int u(x \exp Y) e^{-iY \cdot \xi} p(\xi) \psi(Y) \, dY \, d\xi + u \ast \varphi
\]

where \( \varphi, \psi \) are as in theorem 1 and \( p(\xi) \in \mathcal{S}_2^m(\mathfrak{g}^*) \), that is \( p \in C^\infty(\mathfrak{g}^*) \) and

\[
|D_\xi^\beta p(\xi)| \leq c_\beta (1 + w(\xi))^{m-\alpha, \beta}
\]

for all \( \beta \).

The correspondence \( T \longmapsto p \) is bi-unique between operators mod smooth operators and Lie symbols mod symbols in \( \mathcal{S}(\mathfrak{g}^*) \).
PROOF: We may use the same argument as in the proof of theorem 1 with the appropriate modifications except for one point: to establish the analogue of (3.7) we must use the fact that \( X_1, \ldots, X_n \) is compatible with \( a \).

Indeed the compatibility implies \( W(X, Z) X_j \cdot \xi_k = 0 \) if \( a_k > a_j \), hence \( b_{jk} = X_j \cdot W(X, Z)^{s-1} \xi_k = 0 \) if \( a_k > a_j \) because matrices of block triangular form are preserved under inversion. Thus we have

\[
|p(W(X, Z)^{s-1} \xi)| \leq c(1 + \nu (\Sigma b_{jk} \xi_k))^m
\]

\[
\leq c'(1 + \Sigma |\xi_k|^{1/a_j})^m
\]

\[
\leq c''(1 + \Sigma |\xi_k|^{1/a_j})^m \leq c'''(1 + \nu(\xi))^m
\]

since \( b_{jk} \neq 0 \) only if \( \frac{1}{a_j} \leq \frac{1}{a_k} \). Reasoning similarly we obtain

\[
\left| \left( \frac{\partial}{\partial X} \right)^a \left( \frac{\partial}{\partial Z} \right)^b \right| (P W(X, Z)^{s-1} \xi) R(X, Z) \leq c_{a, \beta, y} (1 + \nu(\xi))^{m-a-\beta}
\]

which is the desired analogue of (3.7).

Q. E. D.

REMARK: A similar argument may be used to show that the class of \( S^m_a \) operators is invariant under those diffeomorphisms whose Jacobian matrices at each point are block triangular in the sense that \( J(x)_{jk} = 0 \) if \( a_j > a_k \).

Theorems 2 and 3 are easily seen to hold in this context also.

§ 5. Global Sobolev Spaces.

In this section we construct Banach space \( L^p_a(G) \) for real \( a \) and \( 1 < p < \infty \) with the property that \( L^p_a(G) = L^p(G, dx) \) for left Haar measure \( dx \), and if \( T \) is a left invariant compactly supported pseudo-differential operator of class \( S^m_a \) then \( T \) is continuous from \( L^p_{a+m} \) to \( L^p_a \). Here and throughout we assume that \( p = 2 \) unless \( q = 1 \). The requirement that \( T \) be compactly supported is important, since the theory of \( L^p \) estimates for convolutions with kernels which are not compactly supported is quite involved (see Stein [17]).
If $G$ is compact we do not need the group structure and the theory is well-known [16]. If $G$ is not compact the best we can do with these local methods is to define Frechet spaces $L^p_{a,\text{com}}$ and $L^p_{a,\text{loc}}$ of compactly supported and locally $L^p$ functions. We will of course want $L^p_{a,\text{com}} \subseteq L^p_a \subseteq L^p_{a,\text{loc}}$.

Using the results of Section 3 it will be easy to construct the spaces $L^p_a$ using the local theory (property 4 of Section 2) together with the following simple principle of uniform localization:

**Lemma 5:** Let $\varphi$ be any non-zero continuous function with compact support on $G$. Then $f \in L^p$ if and only if \( \left( \int_G |f(x)\varphi(y)|^p \, dy \right)^{1/p} \in L^p_a \) with equivalence of norms.

**Proof:**

\[
\left\| \left( \int_G |f(x)\varphi(y)|^p \, dy \right)^{1/p} \right\|^p_p = \int \int |f(x)\varphi(y)|^p\, dy\, dx
\]

\[
= \int \int |f(x)\varphi(y)|^p\, dx\, dy = C \|f\|_p^p.
\]

**Theorem 6:** Let $T$ be a left invariant compact supported pseudo-differential operator of class $S^0_a$. Then $T$ is bounded in $L^p$ for $p = 2$ and $1 < p < \infty$ if $G = G$. The $L^p$ boundedness for $1 < p < \infty$ remains true for operators of class $S^0_a$.

**Proof:** Let $\varphi \in C_0^{\infty}(G)$. Since $T$ is compactly supported there exists $\psi \in C_0^{\infty}(G)$ with $\psi = 1$ on a neighborhood of the support of $\varphi$ such that $\varphi Tu = \psi T u$. From the local theory we have $\|\varphi Tu\|_p \leq C \|\psi u\|_p$. Applying this to $L^p_a u$ and using the invariance of $T$ we obtain $\int |T u(x)\varphi(y)|^p dy \leq \psi \int |u(x)\varphi(y)|^p dy$. The theorem follows by integrating with respect to $x$ and applying the lemma.

**Definition:**

Let $P_a u(x) = (2\pi)^{-n} \int \int u(x\exp Y) e^{-i\xi \cdot Y} (1 + |\xi|^2)^{a/2} \psi(Y) \, dY \, d\xi$.  

---

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This will play the role of the Riesz potential of order \( \alpha \) in the Euclidean case [1]. We note that \( P_a \) is of class \( S^1_{\alpha} \) and is elliptic. These are the only properties of \( P_a \) we will use. If we could show that \( P_a \) (or some substitute) were invertible then it could play the role of the Euclidean Bessel potential and much of what follows could be simplified.

For \( \alpha \geq 0 \) we define \( L^p \subseteq L^p(\mathbb{R}^n) \) to be the completion of \( C^\infty_0(G) \) in the norm \( \| u : L^p \| = (\| u \|_p^p + \| P_a u \|_p^p)^{1/p} \). For \( \alpha < 0 \) we define \( L^p \subseteq D' \) to be the completion of \( C^\infty_0(G) \) in the norm \( \| u : L^p \| = \inf (\| f \|_p + \| g \|_p^{1/p}; u = f + g) \). Note that the \( L^p \) norm is invariant under left translations.

**Theorem 7:** Let \( T \) be a left invariant compactly supported pseudodifferential operator of class \( S^m_\alpha \). Then

\[
\| Tu : L^p_a \| \leq c \| u : L^p_{a+m} \|
\]

for any \( \alpha \) and \( p = 2 \) or \( 1 < p < \infty \) if \( \alpha = 1 \). Thus \( T \) extends to a bounded operator from \( L^p_{a+m} \) to \( L^p_a \).

**Proof:** Since \( P_a \) is elliptic for all \( \alpha \) we may write \( T = \sum_{n=0}^\infty Q_0 P_{a+n+m} + R \) where \( Q_0 \) is of class \( S^0 \) and \( R \) is convolution with a \( C^\infty_0 \) function. It is easy to show that \( R \) is bounded from \( L^p_a \) to \( L^p_a \) for any \( \alpha \) and \( \beta \), so it suffices to show

\[
\| P_{-a} u : L^p_a \| \leq \| u : L^p_0 \| \quad \text{and}
\]

\[
\| P_a u : L^p_a \| \leq \| u : L^p_0 \| \quad \text{for all } \alpha.
\]

First let \( \alpha \geq 0 \). Then (5.3) is obvious and if \( u \in L^p \) then \( P_{-a} u \in L^p \) and \( P_a P_{-a} u \in L^p \) since both \( P_{-a} \) and \( P_a P_{-a} \) are of class \( S^0_\alpha \). Thus (5.2) holds.

Next let \( \alpha < 0 \). Then (5.2) is obvious and if \( u = f + P_{-a} g \) with \( f, g \in L^p \) then \( P_a u = P_a f + P_a P_{-a} g \in L^p \) since both \( P_a \) and \( P_a P_{-a} \) are of class \( S^0_\alpha \). Thus (5.3) holds.

Q. E. D.

**Corollary:**

1. \( L^p_a \subseteq L^p_b \) if \( \beta < \alpha \).
2. \( \| u : L^p_a \| \) and \( \| u : L^p_{a-1} \| + \sum_{j=1}^n \| X_j u : L^p_{a-1} \| \) are equivalent norms.
3. There exists a smooth convolution operator \( R_a u = q(a \cdot u) \), \( q \in C^\infty_0 \), such that \( \| u : L^p_a \| \) and \( \| P_a u \|_p + \| R_a u \|_p \) are equivalent norms.
In this way $L^p_\alpha$ and $L^{p'}_\alpha$ are dual spaces.

(e) If $k$ is a positive integer then $u \in L^p_k$ if and only if $u \in L^p$ and $X_j(1)\ldots X_j(k)u \in L^p$ in the distribution sense for all sequences $j(1), \ldots, j(k)$.

**Proof:** (a) The identity operator is of class $S^0_1$ hence of class $S^{k-1}_1$.

(b) We have $\|X_j u\|_{L^p_{k-1}} \leq c \|u\|_{L^p_k}$ since $X_j$ is of class $S^1_1$, and $\|u\|_{L^p_{k-1}} \leq c \|u\|_{L^p_k}$ by (a). Conversely, since $\sum_{j=1}^n X_j^2$ is elliptic by theorem 4, we may write $I = \sum_{j=1}^n T_j X_j + R$ where $T_j$ is of class $S^{-1}_1$ and $R$ is smooth, and deduce

$$\|u\|_{L^p_k} \leq c \left( \sum_{j=1}^n \|X_j u\|_{L^p_{k-1}} + \|u\|_{L^p_{k-1}} \right).$$

(c) We have $\|P_a u\|_p + \|R_a u\|_p \leq c \|u\|_{L^p_k}$ immediately from the theorem. For the converse we write $I = P_{-a} Q_0 P_a + R_a$. If $\alpha < 0$ we have $u = R_a u + P_{-a} (Q_0 P_a u)$ so $\|u\|_{L^p_k} \leq \left( \|R_a u\|_p + \|Q_0 P_a u\|_p \right)^{1/p}$. If $\alpha > 0$ $\|u\|_p \leq \|P_{-a} Q_0 (P_a u)\|_p + \|R_a u\|_p \leq c \|P_a u\|_p + \|R_a u\|_p$ since $P_{-a} Q_0$ is of order $\leq 0$ so $\|u\|_{L^p_k} \leq c' \|P_a u\|_p + \|R_a u\|_p$.

(d) We may as well assume $\alpha > 0$. Write $v = f + P_{-a} g$ with $f, g \in L^{p'}$. Then

$$\int u (x) \overline{v(x)} \, dx = \int u (x) \overline{f(x)} \, dx + \int P_{-a} u (x) \overline{g(x)} \, dx \leq \|u\|_p \|f\|_{L^{p'}} + \|P_{-a} u\|_p \|g\|_{L^{p'}} \leq c \|u\|_{L^p_k} \|v\|_{L^{p'}}.$$
in the \( L^{p'} \) metric by \( f_k, g_k \in C_{\text{com}} \) and so \( \lambda (u) = \lim_{k \to \infty} \int u(x)(f_k(x) + P_ug_k(x))dx. \)

But \( f_k + P_ug_k \in C_{\text{com}} \) since \( P_ug \) is compactly supported hence \( \lambda \in L^{p'}_{\text{com}}. \)

Next let \( \lambda \in (L^{p'}_{\text{com}})^{\ast}. \) Then \( |\lambda (f)| \leq c \|f: L^{p'}_{\text{com}}\| \leq c \|f\|_{p'} \) so we may identify \( \lambda \) with \( u \in L^p \), \( \lambda (v) = \int v(x)u(x)dx \) for every \( v \in C_{\text{com}} \). It suffices to show \( P_ug \in L^p \). Note \( P_ug \in L^{p}_{\text{com}} \) is defined by taking \( u_k \to u \) in \( L^p, u_k \in C_{\text{com}} \), and \( P_ug = \lim_{k \to \infty} P_ug_k \) in \( L^p \). Thus for \( \varphi \in C_{\text{com}} \) we have

\[
\lim_{k \to \infty} \int P_ug_k(x)\varphi(x)dx = \lim_{k \to \infty} \int u_k(x)P_\varphi^*u(x)dx = \int u(x)P_\varphi^*\varphi(x)dx
\]

and since \( |\lambda (P_\varphi^*\varphi)| \leq c \|P_\varphi^*\varphi: L^{p'}_{\text{com}}\| \leq c \|\varphi\|_{p'} \) we conclude that \( \lim_{k \to \infty} P_ug_k \)
may be identified with a function in \( L^p \).

(e) This follows by induction from (b) and (d). Q.E.D.

REMARK: Part (d) above shows that by defining \( L^p_\alpha \) to be the completion of \( C_{\text{com}} \) we have not made the space too small; indeed \( (L^{p}_{\text{com}})^{\ast} \) is the largest space that might be considered to define \( L^p_\alpha \).

We next consider the extension of Lemma 5 to \( L^{p}_{\alpha}. \)

THEOREM 8: Let \( \varphi \in C_{\text{com}} \) be not identically zero. Then \( \|u: L^p_\alpha\| \) and

\[
\left( \int \| \varphi L_\alpha u: L^p_\alpha \|^p dx \right)^{1/p}
\]

are equivalent norms for \( \alpha \geq 0. \)

PROOF: Let \( \varphi_0 \in C_{\text{com}} \) be identically one on a large enough neighborhood of the support of \( \varphi \) so that \( \varphi_0 P_\alpha (\varphi f) = P_\alpha (\varphi f) \). We may then write \( P_\alpha (\varphi f) = QP_\alpha (f) + Rf \) where \( Q \) and \( R \) are compactly supported pseudodifferential operators of class \( S^0_{\text{loc}} \) and \( S^{-\infty} \) respectively (not left invariant of course) such that \( \varphi_0 Q = Q \) and \( \varphi_0 R = R \). From the local theory we have

\[
\| P_\alpha (\varphi f) \|_p + \| \varphi f \|_p \leq c (\| \varphi_0 P_\alpha (f) \|_p + \| \varphi_0 f \|_p).
\]

If we substitute \( L_\alpha u \) for \( f \) in (5.4) and then take the \( L^p \) norm in \( x \) we obtain \( \left( \int \| \varphi L_\alpha u: L^p_\alpha \|^p dx \right)^{1/p} \leq c \| u: L^p_\alpha \| \) by Lemma 5.

For the converse we choose \( \varphi_1, \varphi_2 \in C_{\text{com}} \) so that \( \varphi_1 \) is supported on a compact set on which \( \varphi \) is bounded away from zero, and \( \varphi \varphi_2 = 1 \) on a
neighborhood of the support of \(\varphi_1\). If necessary we modify the definition of \(P_\alpha\) by changing the function \(\psi(Y)\) to have so small a support that 
\[ \varphi_1 P_\alpha f = \varphi_1 \psi(Yf) ; \]
this change does not affect the \(L^p_a\) spaces. Again from the local theory we obtain

\[
\| \varphi_1 P_\alpha f \|_p + \| \varphi_1 f \|_p \leq c (\| P_\alpha (\psi f) \|_p + \| \psi f \|_p)
\]

and the proof is completed as before. Q.E.D.

From this theorem we may pass from local properties of functions in 
\(L^p_a\) which do not depend on the group structure to global properties of the class \(L^p_a\). In this way virtually all properties of the Euclidean \(L^p_a\) spaces proved in [3] and [19] generalize to \(L^p_a(G)\). We state a few of these, proving only the first since the other proof are analogous:

**COROLLARY:**

(a) \(L^p_a \subseteq L^\beta_a \) continuously if \(\alpha > \beta \geq 0\) and \(\frac{1}{q} - \frac{1}{p} = \frac{\alpha - \beta}{n}, \) \(1 < p < q < \infty\).

(b) \(L^p_a\) forms an algebra under pointwise multiplication if \(\alpha > n/p\).

(c) \(f \in L^2_a\) for \(0 < \alpha < 1\) if and only if \(f \in L^2\) and
\[
\int \int |f(x \exp Y) - f(x)|^2 Y^{n+2a} dx < \infty \text{ with equivalence of norms.}
\]

(d) \(f \in L^p_a\) for \(0 < \alpha < 1\) if and only if \(f \in L^p\) and \(S_\alpha f \in L^p\) with equivalence of norms, where
\[
S_\alpha f = \left( \int_0^\infty \left( \int_{|Y| \leq 1} |f(x \exp tY) - f(x)|^2 Y^{n-2a} dt \right)^{1/2} \right).
\]

**PROOF:**

(a) Let \(\varphi, \varphi_0 \in C_0^\infty\) with \(\varphi_0 = 1\) on a neighborhood of the support of \(\varphi\). It follows from the local theory (see Calderon [3] and Seely [16]) that
\[
\| \varphi f : L^p_a \| \leq c \| \varphi_0 f : L^p_a \| . \]
Thus
\[
\| \varphi L_\alpha u : L^p_a \| \leq c \| \varphi_0 L_\alpha u : L^p_a \| \leq c' \| u : L^p_a \| . \]
If \(u \in L^p_a\) we have \(\| \varphi L_\alpha u : L^p_a \| \) in \(L^p\) and \(L^\infty\) hence \(L^q\) since \(p < q < \infty\).

(b) (c) and (d): Use the local results in [19] and reason similarly.

Q. E. D.

A more general study of global Sobolev spaces on a Riemannian manifold is given in [2] with less precise results.
Most works on pseudo-differential operators deal only with the $L^2$ theory. The $L^p$ theory for the class $S^m_\nu$ is developed by Seeley in [15]; his method depends on spherical harmonic expansions and appears unlikely to generalize to the class $S^m_\nu$. Results for more general classes than $S^m_\nu$ are given in Kagan [8] and extended in Kumao-go and Nagase [12]. However P. Szeptycki, in reviewing Kagan's paper (MR 37 # 4392), has cast doubt on its validity. In any case Kagan's paper is not available in translation so we present here a brief proof which covers the classes $S^m_\nu$ and the more general $S^m_\nu$ classes defined in Section 4. The reader will note that our method is really a straightforward generalization of the usual $L^2$ theory. In contrast with Seeley [15] we require more rapid decrease in the $x$ variable for our symbols.

Let $p(x, \xi)$ be a symbol of class $S^\nu_1$ with rapid decrease in $x$. Thus we assume $p \in C^\infty(R^n \times R^n)$ and

$$\left| x \right|^N \left| \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial \xi} \right)^\alpha \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial \xi} \right)^\beta p(x, \xi) \right| \leq c_{\alpha, \beta, N} (1 + \left| \xi \right|)^{\gamma - \left| \beta \right|}$$

for all $\alpha, \beta$ and all non-negative $N$. For $f \in \mathcal{S}(R^n)$, $1 < p < \infty$, $\alpha$ real, define $\|f\|: L^p_a = \| \mathcal{F}^{-1} (1 + |\xi|)^{\alpha} \hat{f}(\xi) \|_p$ and let $L^p_a$ denote the completion of $\mathcal{S}$ in the $L^p_a$ norm. We wish to prove:

**Theorem 9:** Under the above hypotheses the operator $p(x, D)$ is bounded from $L^p_a$ to $L^p_a$, for $1 < p < \infty$ and all real $\alpha$.

Our main tool, which replaces the Plancherel theorem in the $L^2$ theory, is the Marcinkiewicz multiplier theorem. There are many versions of this theorem; the following will suffice for our purposes (see [18]):

**Marcinkiewicz Multiplier Theorem:** Let $M^p_\mu$ denote the space of bounded functions $m(\xi)$ on $R^n$ such that $\mathcal{F}^{-1} (m(\xi) \hat{f}(\xi))$ is a bounded operator on $L^p$, with the operator norm. A sufficient condition for $m(\xi)$ to belong to $M^p_\mu$ is that $m$ be bounded and $C^\infty$ on the complement of the set
\[ \{\xi : \xi_1, \xi_2, \ldots, \xi_n = 0\} \text{ and } \xi^a \left( \frac{\partial}{\partial \xi} \right)^a \mathcal{m}(\xi) \text{ be bounded for all } a \text{ such that each } a \text{ is zero or one. We then have } \| \mathcal{m} : M_p^p \| \leq c \left( \sum \| \xi^a \left( \frac{\partial}{\partial \xi} \right)^a \mathcal{m}(\xi) \|_\infty \right). \]

**Lemma 6.** (Peetre's inequality): For every real \( s \) there exist positive constants \( c \) and \( t \) such that

\[ \| (1 + |\xi|^2)^{-\frac{nt}{2}} (1 + |\xi + \eta|^2)^{\frac{nt}{2}} : M_p^p(\xi) \| \leq c (1 + |\eta|)^t. \]

**Proof:** In fact we may take \( t = |s| + n \). We use the usual form of Peetre's inequality, \( (1 + |\xi|^2)^{-\frac{nt}{2}} (1 + |\xi + \eta|^2)^{\frac{nt}{2}} \leq c (1 + |\eta|)^{|s|} \), and apply the Marcinkiewicz multiplier theorem; it is easy to see that the differentiation at most produces an additional factor of \( (1 + |\eta|)^n \)

**Lemma 7:** Let \( R(\xi, \eta) \in C^0(R^n \times R^n) \) satisfy

\[ \| R(\xi, \eta + \xi) (1 + |\xi + \eta|^2)^{-\frac{nt}{2}} : M_p^p(\xi) \| \leq c_N (1 + |\eta|)^{-N} \]

for every positive \( N \). Then the operator \( T \) given by \( (Tf)(\xi) = \int R(\xi, \eta) f(\eta) dy \) is bounded from \( L^p_{+r} \) to \( L^p \) for every real \( s \).

**Proof:** Let \( f \in L^p_{+r} \). Then \( (1 + |\xi|^2)^{(s+r)/2} \hat{f}(\xi) = \hat{g}(\xi) \) for \( g \in L^p \) with \( \| f : L^p_{+r} \| = \| g : L^p \|. \) To show \( T \in L^p \) we must show \( \mathcal{F}^{-1}((1 + |\xi|^2)^{\frac{nt}{2}} (Tf)^{\xi} \cdot (\xi)) \in L^p. \) Thus we must show that the operator

\[ Sg = \mathcal{F}^{-1} \left( \int R(\xi, \eta) \hat{g}(\eta) (1 + |\eta|^2)^{-\frac{nt}{2}} (1 + |\xi|^2)^{\frac{nt}{2}} d\eta \right) \]

is bounded in \( L^p \). Replacing \( \eta \) by \( \eta + \xi \) and applying the hypotheses we obtain

\[ \| Sg \|_p \leq c_N \int (1 + |\eta|)^{-N} \| \mathcal{F}^{-1}(\hat{g}(\eta + \xi) (1 + |\eta + \xi|^2)^{-\frac{nt}{2}} (1 + |\xi|^2)^{\frac{nt}{2}} \|_p d\eta. \]

From Peetre's inequality we then obtain

\[ \| Sg \|_p \leq c_N c \int (1 + |\eta|)^{-N} (1 + |\eta|^t) \| g \|_p d\eta \leq c' \| g \|_p \]

if we take \( N \) large enough.

Q. E. D.
PROOF OF THEOREM: Since

\[
(P(x, D)f)(\xi) = \int f(\eta) \widehat{\rho}(\xi - \eta, \eta) \, d\eta
\]

where the Fourier transform \( \widehat{\rho} \) is taken in the \( x \) variables alone, it suffices to verify the conditions of lemma 7 for \( R(x, \eta) = \widehat{\rho}(-\eta, \eta + \xi) \). After a change of variables this amounts to estimating \( \| \widehat{\rho}(\eta, \xi)(1 + |\xi|^2)^{-\gamma/2} : M_p^\beta(\xi) \| \). But from (A1) we obtain

\[
\left| \left( \frac{\partial}{\partial \xi} \right)^\beta \widehat{\rho}(\eta, \xi) \right| \leq C_N (1 + |\eta|)^{-N} (1 + |\xi|)\gamma/2 - |\beta|.
\]

Thus an application of the Marcinkiewicz multiplier theorem gives

\[
\| \widehat{\rho}(\eta, \xi)(1 + |\xi|^2)^{-\gamma/2} : M_p^\beta(\xi) \| \leq C_N (1 + |\eta|)^{-N}
\]
as desired. Q. E. D.

The above arguments may be modified to deal with the classes \( S^m_\alpha \) defined in Section 4. We let \( \lambda(\xi) \in C^\infty(\mathbb{R}^n) \) be chosen equal to \( \varphi(\xi) \) in \( |\xi| \geq 1 \) and non-vanishing in \( |\xi| \leq 1 \). We define \( L_p^\alpha \) to be the closure of \( \mathcal{S} \) in the norm

\[
\| f : L_p^\alpha \| = \| \mathcal{F}^{-1}(\lambda(\xi)^\alpha \widehat{f}(\xi)) \|_p.
\]

We may then show that an operator with symbol of class \( S^m_\alpha \) satisfying

\[
|x|^N \left| \left( \frac{\partial}{\partial x} \right)^\alpha \left( \frac{\partial}{\partial \xi} \right)^\beta p(x, \xi) \right| \leq C_{\alpha, \beta, N} \lambda(\xi)^{\mu - \alpha - \beta}
\]
is bounded from \( L_p^{\alpha + m} \) to \( L_p^\alpha \). The key point is that \( |\xi|^N \left( \frac{\partial}{\partial \xi} \right)^\alpha \lambda(\xi) \| \leq \sigma \lambda(\xi) \) so the Marcinkiewicz multiplier theorem may be applied as before.
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