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RIEMANNIAN MANIFOLDS WITH BOUNDED DIRICHLET FINITE POLYHARMONIC FUNCTIONS

by LUNG OCK CHUNG, LEO SARIO, and CECILIA WANG

The harmonic and quasiharmonic classification of Riemannian manifolds has yielded the following strict inclusion relations:

$$O_{HP} < O_{HB} < O_{HD} = O_{HC},$$

$$\begin{array}{ccccc} & & O_{QB} & & \\ & \angle & & \nwarrow & \\ O_{QP} & & & & O_{QC}\ . \\ & \nwarrow & & \angle & \\ & & O_{QD} & & \end{array}$$

Here for any class F of functions, O_F stands for the class of Riemannian manifolds which do not carry nonconstant functions in F; H is the class of harmonic functions; P, B, D, and C the classes of functions which are positive, bounded, Dirichlet finite, and bounded Dirichlet finite, respectively; Q the class of quasiharmonic functions q, defined by $\Delta q = 1$, where Δ is the Laplace-Beltrami operator $d\delta + \delta d$; and HX, QX designate $H \cap X$, $Q \cap X$ with $X = P, B, D$, or C.

In the present paper we turn to relations of H and Q to the class H^m of nondegenerate polyharmonic functions u, defined by $\Delta^m u = 0$, $\Delta^{m-1} u \neq 0$, m an integer ≥ 2. The first question here is: does the existence of $H^m X$ functions imply that of HX functions? We shall show that the answer is in the negative: *there exist Riemannian manifolds of any dimension which carry even $H^m C$ functions, for all $m \geq 2$, without admitting HX functions for any X.* The relation of $H^m C$ to QX is similar.

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1. For $N \geq 2$, let O^N be the subclass of N-dimensional manifolds in a given class O. Denote by $\widetilde{O}^N$ the complement of O^N.

THEOREM 1. *$\bigcap_{m=2}^{\infty} \widetilde{O}^N_{H^m G} \cap O^N_{HX} \neq \emptyset$ for $N \geq 2$, and $X = P, B, D, C$.*

PROOF. Consider the N-dimensional « beam »

$$T: \{(x, y_1, y_2, \ldots, y_{N-1}) \mid |x| < \infty, |y_i| \leq \pi\},$$

$i = 1, \ldots, N-1$, with each pair of opposite faces $y_i = \pi$, $y_i = -\pi$ identified by parallel translation perpendicular to the x-axis. Endow T with the metric

$$ds^2 = e^{-x^2} dx^2 + e^{-x^2/(N-1)} \sum_{i=1}^{N-1} dy_i^2 .$$

Since

$$\Delta f_0(x) = - e^{x^2} (e^{-x^2} e^{x^2} f_0')',$$

we have $f_0(x) \in H(T)$ if and only if $f_0(x) = ax + b$. The harmonic measure of $\{x = c > 0\}$ on $\{0 < x < c\}$ is x/c, which tends to zero as $c \to \infty$. Similarly the harmonic measure of the boundary component at $x = -\infty$ vanishes, and therefore T belongs to the class O_G of parabolic Riemannian manifolds. Since $O_G \subset O_{HP}$, we have $T \in O_{HX}$.

2. The fact that $T \in O_{HX}$ can also be proved « directly », i.e., without recourse to $O_G \subset O_{HP}$. Since the proof may offer some methodological interest, we insert it here, in the case $X = B$, $N > 2$. For integers $n_1, \ldots, n_{N-1} \geq 0$, set $n = (n_1, \ldots, n_{N-1})$, $\eta^2 = \Sigma_{i=1}^{N-1} n_i^2$, $y = (y_1, \ldots, y_{N-1})$, and $G_n(y) = \Pi_{i=1}^{N-1} {}^{\cos}_{\sin} n_i y_i$. The function $f_n(x) G_n(y)$ is harmonic if

$$0 = \Delta(f_n G_n) = \Delta f_n \cdot G_n + f_n \Delta G_n$$

$$= - e^{x^2} f_n'' G_n + e^{x^2/(N-1)} \eta^2 f_n G_n ,$$

that is,

$$f_n''(x) - \varphi(x) f_n(x) = 0$$

with $\varphi(x) = \eta^2 e^{-(N-2)x^2/(N-1)} > 0$. By a theorem of Haupt [1] and Hille [2], if $\int_0^\infty x |\varphi(x)| dx < \infty$, then the general solution of $f'' \pm \varphi f = 0$ is asymptotically $ax + b$ for some constants a, b.

A general $h(x, y) \in H(T)$ has a representation $h = \Sigma c_n f_n G_n$, where each $|f_n| \backsim |a_n x + b_n|$ as $x \to +\infty$, and the summation is under all n and all combinations of cosine and sine in G_n. If $h \in HB$, $h \neq$ const., then for each n_0 the function $G_{n_0}(y) h(x, y)$ is bounded, and the same is true of

$$f_{n_0}(x) = c \int_y G_{n_0}(y) h(x, y) \, dy.$$

For n_0 such that $f_{n_0}(x) \neq$ const., this violates $|f_{n_0}(x)| \backsim |a_{n_0} x + b_{n_0}|$. We conclude that $h \notin B$, and $T \in O_{HB}$.

3. To see that $T \in \tilde{O}_{H^m C}$, we first consider the case $m = 2$ and set

$$u_2(x) = \int_0^x e^{-x^2} dx.$$

Since $\Delta u_2 = - e^{x^2} (e^{-x^2} e^{x^2} u_2')' = 2x \in H$, we have $u_2 \in H^2$. Clearly $u_2 \in B$. Moreover,

$$D(u_2) = c \int_{-\infty}^{\infty} (u_2')^2 e^{x^2} e^{-x^2} dx = c \int_{-\infty}^{\infty} e^{-2x^2} dx < \infty.$$

Hence $u_2 \in C$.

4. We proceed to the general case $m \geq 2$ and define recursively

$$u_{m+1}(x) = \int_0^x \int_x^\infty u_m(t) e^{-t^2} dt \, dx.$$

Suppose $u_m \in H^m C(T)$. In view of

$$\Delta u_{m+1} = - e^{x^2} (u_{m+1}')' = e^{x^2} (u_m(x) e^{-x^2}) = u_m(x) \in H^m,$$

we have $u_{m+1} \in H^{m+1}$. Moreover, for all sufficiently large x,

$$\int_x^\infty u_m(t) e^{-t^2} dt < 2 \int_x^\infty t e^{-t^2} dt,$$

and $u_{m+1}(x) \leq u_2(x)$. Since u_m is odd, $u_{m+1} \in B$, and in view of $D(u_2) < \infty$,

$$D(u_{m+1}) = c \int_{-\infty}^{\infty} \left(\int_{x}^{\infty} u_m(t)\, e^{-t^2}\, dt \right)^2 dx < \infty.$$

Thus $u_{m+1} \in H^{m+1} C$. This completes the proof of Theorem 1.

5. We now exclude QX functions.

THEOREM 2. *$\tilde{O}^N_{H^m C} \cap O^N_{QX} \neq \emptyset$ for $N \geq 2$, $m \geq 1$, and $X = P, B, D, C$.*

PROOF. Consider the manifold

$$S: \{(x, y_1 y_2, \dots, y_{N-1}) \mid x > 1, \ |y_i| \leq \pi\}$$

with the opposite faces $y_i = -\pi$ and $y_i = \pi$ again identified for each $i = 1, \dots, N-1$. On S, take the metric

$$ds^2 = dx^2 + x^{2\alpha(N-1)} \sum_{i=1}^{N-1} dy_i^2,$$

α a constant. Then $u_1(x) \in H(S)$ if and only if

$$u_1(x) = ax^{-\alpha+1} + b,$$

as is seen by $u_1 = -x^{-\alpha}(x^{\alpha} u_1')'$. For $m \geq 1$ set

$$u_m(x) = x^{-\alpha+2m-1}.$$

We know that $u_1 \in H = H^1$. Suppose $u_m \in H^m$. Then

$$\Delta u_{m+1}(x) = cx^{-\alpha+2m-1} \in H^m$$

and therefore $u_{m+1} \in H^{m+1}$. Clearly $u_{m+1} \in B$ for $\alpha \geq 2m+1$. Moreover,

$$D(u_{m+1}) = c \int_{1}^{\infty} (u'_{m+1})^2 x^{\alpha}\, dx < \infty$$

for $\alpha > 4m + 1$. Thus for $\alpha > 4m - 3$, $u_m \in H^m C$, and $S \in \tilde{O}_{H^m C}$.

Note that now we do not claim the simultaneous existence of $H^m C$ functions for all m.

6. To exclude QX functions, we first note that the equation

$$\Delta q_1(x) = -x^{-\alpha}(x^\alpha q_1')' = 1,$$

is satisfied by

$$q_1(x) = -x^2/(2(\alpha+1)).$$

Every $q(x, y) \in Q$ can be written $q = q_1 + h$, $h \in H$, and therefore

$$q = q_1 + ax^{-\alpha+1} + b + \sum_n{}' c_n f_n(x) G_n(y),$$

where each $f_n G_n \in H$, and the summation Σ' excludes $n = (0, \ldots, 0)$. To see that $q \notin P$, take x_0 so large that $q_1(x_0) + ax_0^{-\alpha+1} + b < 0$. This is possible for all $\alpha > -1$, in particular for our $\alpha > 4m - 3$. Then choose y_0 such that $\Sigma' c_n f_n(x_0) G_n(y_0) = 0$. Such a y_0 exists for each x_0 since $\int_V G_n(y)\, dy = 0$, hence $\int_V \Sigma' c_n f_n(x_0) G_n(y)\, dy = 0$. For these x_0, y_0 we have $q(x_0, y_0) < 0$, that is, $q \notin P$, and we have shown that $S \in O_{QP}$, hence $S \in O_{QX}$ for all X. The proof of Theorem 2 is complete.

7. Can Theorem 2 be generalized to the class $Q^n X$ of polyquasiharmonic functions q, defined by $\Delta^n q = 1$? Let N be the class of negative functions. We shall show:

THEOREM 3. *$\tilde{O}^N_{H^m C} \cap O^N_{Q^n X} \neq \emptyset$ for $N \geq 2$; $m \geq 1$; $n \geq 1$, and $X = B, D, C$. Moreover, for $N \geq 2$, $m \geq 1$, $\tilde{O}^N_{H^m C} \cap O^N_{Q^n P} \neq \emptyset$ if $n \geq 1$ is odd and $\tilde{O}^N_{H^m C} \cap O^N_{Q^n N}$ if $n > 1$ is even.*

PROOF. On the Riemannian manifold S of No. 5, take

$$q_n(x) = \frac{(-1)^n x^{2n}}{[2 \cdot 4 \ldots (2n)]\,[(\alpha+1)(\alpha+3)\ldots(\alpha+2n-1)]}.$$

We know that $q_1 \in Q$, and $q_1 \notin P, B, D, C$ for $\alpha > -1$. Now

$$\Delta q_{n+1}(x) = -x^{-\alpha}\left\{x^{\alpha}\left[\frac{(-1)^{n+1}x^{2(n+1)}}{[2\cdot 4 \dots 2(n+1)][(\alpha+1)\dots(\alpha+2n+1)]}\right]'\right\}' = q_n \in Q^n.$$

Therefore $q_{n+1} \in Q^{n+1}$, and $q_n \notin P$ if n is odd, $q_n \notin N$ if n is even.

It remains to show that if $q(x, y) \in Q^n$, then $q \notin P, B, D, C$ for n odd, and $q \notin N, B, D, C$ for n even. The discussion is essentially the same in both cases, and we shall consider the former. Since $\Delta^n (q - q_n) = 0$, q has a representation

$$q(x, y) = q_n(x) + u(x) + \Sigma' c_j f_j(x) G_j(y),$$

with $\Delta^n u(x) = 0$, and the $G_j(y)$ trigonometric functions occurring in the presentation of a polyharmonic function. In view of $\Delta^n u(x) = 0$, u is a linear combination of functions in x belonging to $H^{\mu}(S)$ for $\mu = 1, \dots, n$. For $\alpha > -1$, all such functions grow slower than x^{2n}, as an inspection of the u_{μ} of No. 5 will show. Thus there exists an x_0 with $q_n(x_0) + u(x_0) < 0$, q_n being < 0. Choose y_0 such that $\Sigma' c_j f_j(x_0) G_n(y_0) = 0$. For these x_0, y_0, we have $q(x_0, y_0) = q_n(x_0) + u(x_0) < 0$. Thus $q \notin P, B, C$ for $\alpha > -1$.

To see that $q \notin D$, note that $|(q_n + u)'| > \varepsilon > 0$ for all sufficiently large x. This gives

$$D(q) \geq c \int_1^{\infty} [(q_n + u)']^2 x^{\alpha}\, dx = \infty .$$

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