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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 2, n° 3 (1975), p. 469-478

<http://www.numdam.org/item?id=ASNSP_1975_4_2_3_469_0>
Quasiharmonic \( L^p \)-Functions on Riemannian Manifolds (*).

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Let \( Q \) be the class of quasiharmonic functions \( q \), defined by \( \Delta q = 1 \), with \( \Delta = d\delta + \delta d \) the Laplace-Beltrami operator. Denote by \( P, B, D, C \) the classes of functions which are positive, bounded, Dirichlet finite, or bounded Dirichlet finite, respectively. For \( X = P, B, D, C \), let \( Q^N_X \) be the classes of Riemannian \( N \)-manifolds, \( N > 2 \), for which \( QX = Q \cap X = \emptyset \). These classes are known to be related by the strict inclusion relations

\[
\begin{aligned}
O^N_{QB} & \subset O^N_P \subset O^N_Q \subset O^N_C \subset O^N_Q \setminus O^N_{QB}
\end{aligned}
\]

for each \( N \), whereas there is no inclusion between \( O^N_{QP} \) and \( O^N_Q \) [3, 5].

In the present paper we introduce the class \( QL^p \) of quasiharmonic functions in \( L^p \), with \( 1 < p < \infty \); the value \( p = \infty \) will be excluded since \( O^N_{QL} \) is nothing but \( O^N_{QB} \). If \( \tilde{O}^N \) signifies the complement of \( O^N \) with respect to the totality of Riemannian \( N \)-manifolds, then we shall show that \( O^N_{QLX} \cap \tilde{O}^N_{QX} \neq \emptyset \), \( \tilde{O}^N_{QLX} \cap \tilde{O}^N_{QX} \neq \emptyset \), \( \tilde{O}^N_{QLD} \cap O^N_{QX} \neq \emptyset \) for \( p > 1 \), \( X = P, B, D, C \); \( O^N_{QLX} \cap \tilde{O}^N_{QX} \neq \emptyset \) for \( p > 1 \), \( X = P, B \); and \( O^N_{QLD} \cap \tilde{O}^N_{QD} \neq \emptyset \) for \( p > 1 \). In striking contrast with these noninclusions, we shall establish the strict inclusions \( O^N_{QL} < O^N_Q \), and \( O^N_{QL} < O^N_Q \) for \( p > 1 \).

1. We first prove the existence of \( N \)-manifolds which carry neither \( QL^p \) nor \( QX \) functions:

(*) The work was sponsored by the U.S. Army Research Office, Grant DA-ARO-31-124-73-G39, University of California, Los Angeles.
MOS Classification: 31B30.
Pervenuto alla Redazione il 10 maggio 1974 e in forma definitiva il 9 luglio 1974.
THEOREM 1. \( O_{Q^p}^N \cap O_{Q^X}^N \neq \emptyset \) for \( N \geq 2, 1 < p < \infty, X = P, B, D, C. \)

PROOF. An example is simply the Euclidean \( N \)-space \( E^N \). We first show that \( QL^1 = \emptyset \). Every \( q \in Q \) can be written as \( q = q_0 + h \), where \( q_0 = -r^2/2N \in Q \) and \( h \) belongs to the class \( H \) of harmonic functions. Set \( h = h(0) + k \), with \( h(0) \) the value of \( h \) at the origin. Since \( \int_{E^n} k dV = ek(0) = 0 \), we have

\[
\|q\|_1 = \left| \int_{E^n} (q_0 + h(0) + k) dV \right| = \left| \int_{E^n} (q_0 + h(0)) dV \right| = \infty;
\]

in fact, \( q_0 + h(0) \sim e^{r^2} \) and \( dV = r^{N-1} \lambda(\theta) d\theta_1 \cdots d\theta_{N-1} \), with \( \lambda \) a trigonometric function of \( \theta = (\theta_1, \ldots, \theta_{N-1}) \).

To see that \( QL^p = \emptyset \) for \( p > 1 \), take a function \( \varphi \in C^{\infty}(E^N) \) with \( \varphi = r^x \) on \( \{ r > 1 \} \), \( x \) a real constant to be specified later. If \( p' \) is determined by \( 1/p + 1/p' = 1 \), then

\[
\|\varphi\|_{p'} = a + b \int_1^{\infty} r^{a + x - N} dr < \infty,
\]

that is, \( \varphi \in L^p \), for \( x < 1/N \). Suppose there exists a function \( q \in QL^p \). Then \( |(q, \varphi)| < \infty \). For some \( h \in H \),

\[
|(q, \varphi)| = |(q_0 + h(0) + k, \varphi)| = |(q_0 + h(0), \varphi)| = \infty
\]

if \( x > -N - 2 \). Thus any \( x \in [-N - 2, -1/N'] \) gives a contradiction, and we have \( QL^p = \emptyset \) for every \( p > 1 \).

We proceed to show that \( E^N \in O_{Q^X}^N \) for all \( X \). It suffices to establish \( QP = \emptyset \). Write again an arbitrary \( q \in Q \) as \( q = q_0 + h \). Take an increasing sequence \( \{ r_n \}_{n=1}^{\infty} \) with \( r_n \to \infty \). For every \( n \) there exists an \( \theta^* = (\theta^*_1, \ldots, \theta^*_{N-1}) \) such that \( h(r_n, \theta^*) = h(0) \). Therefore \( q(r_n, \theta^*) = q_0(r_n) + h(0) \to -\infty \), and \( q \notin QP \).

2. The relation \( \bar{O}_{Q^p}^N \cap \bar{O}_{Q^X}^N \neq \emptyset \) is trivial in view of the Euclidean \( N \)-ball. We proceed to give a Riemannian \( N \)-manifold which carries \( QL^p \) functions but no \( QX \) functions.

THEOREM 2. \( \bar{O}_{Q^p}^N \cap \bar{O}_{Q^X}^N \neq \emptyset \) for \( N \geq 2, 1 < p < \infty, X = P, B, D, C. \)

PROOF. Consider the manifold

\[ T: \{(x, y_1, \ldots, y_{N-1})| |x| < \infty, |y_i| < 1, i = 1, \ldots, N-1\}, \]
with the opposite faces $y_i = 1$ and $y_i = -1$ identified for each $i$ by a parallel translation perpendicular to the $x$-axis. Endow $T$ with the metric

$$ds^2 = e^{-x^2} dx^2 + e^{-x^2} \sum_{i=1}^{N-1} dy_i^2.$$  

For a quasiharmonic function $q_0(x)$ we have

$$\Delta q_0(x) = -e^{x^2} (e^{-x^2} e^{x^2} q_0'(x))' = 1,$$

which is satisfied by

$$q_0(x) = -\int_0^x \int_0^t e^{-\tau} \, d\tau \, dx.$$

To estimate

$$\|q_0\|_p^p = \int_{-\infty}^{\infty} \left( \int_0^x \int_0^t e^{-\tau} \, d\tau \right)^p e^{-x^2} \, dx$$

we set $a = \int_0^{\infty} e^{-\tau} \, d\tau$ and obtain

$$\|q_0\|_p^p < a^p \int_{-\infty}^{\infty} |x|^p e^{-x^2} \, dx < \infty.$$  

Therefore $T \in \bar{O}^N_{QX}$ for all $p$.

A harmonic function $h_0(x)$ satisfies $\Delta h_0(x) = -e^{x^2} (e^{-x^2} e^{x^2} h_0'(x))' = 0$, which gives $h_0(x) = ax + b$. The harmonic measure $\omega$ of the boundary component at $x = \infty$ on $\{x > 0\}$ is $\omega = \lim_n \omega_n$, with $\omega_n = x/n$ harmonic on $\{0 < x < n\}$. Thus $\omega = 0$, the same is true of the harmonic measure of the boundary component at $x = -\infty$, and therefore $T$ belongs to the class $O^N_G$ of parabolic $N$-manifolds. In view of $O^N_G \subset O^N_{QX}$ (loc. cit.), we obtain $T \in O^N_{QX}$ for all $X$.

3. – Our next problem is to find an $N$-manifold which admits $QX$ functions for $X = P, B$, but no $QL^p$ functions.

**Theorem 3.** $O^N_{QX} \cap O^N_{QX} \neq \emptyset$ for $N \geq 2$, $1 < p < \infty$, $X = P, B$.

**Proof.** Let $M$ be the $N$-space equipped with the metric

$$ds^2 = dr^2 + \psi(r)^2 e^{x(N-1)} \sum_{i=1}^{N-1} \lambda_i(\theta) \, d\theta_i^2,$$

where $\psi(r)$
where \( \psi \in C^\infty(0, \infty) \),

\[
\psi(r) = \begin{cases} 
  r^{N-1} & \text{for } r < \frac{1}{2}, \\
  e^r e^r & \text{for } r \geq 1,
\end{cases}
\]

and the \( \lambda_i \) are trigonometric functions of \( \theta \) such that the metric is Euclidean on \( \{ r < \frac{1}{2} \} \). The function

\[
q_0(r) = -\int_0^r \psi^{-1}(t) \psi(s) \, ds \, dt
\]

satisfies the quasiharmonic equation \( \Lambda q_0 = -\psi^{-1}(\psi q_0')' = 1 \). For \( r > 1 \),

\[
q_0(r) = q_0(1) - \int_1^r e^{-t} e^{-t} \left( a + \int_1^t e^s e^s \, ds \right) \, dt
\]

\[
= q_0(1) - \int_1^r (e^{-t} + a_1 e^{-t} e^{-t}) \, dt = O(1)
\]
as \( r \to \infty \). Therefore \( q_0 \in QB \), and \( M \in \bar{O}_{QB}^N < \bar{O}_{QP}^N \).

We next prove that \( M \in O_{QB}^N \). Suppose there exists a \( q \in QL^1 \). Then \( |(q, e^{-r})| < \infty \). We may again write \( q = q_0 + c + k \), \( k \in H \), \( k(0) = 0 \), and

\[
(q, e^{-r}) = (q_0 + c, e^{-r}) = a + b \int_1^\infty (q_0 + c) e^{-r} e^r \, dr.
\]

Set

\[
c_0 = -\psi^{-1}(\psi) = \int_0^\infty \psi^{-1} \psi \, ds \, dt.
\]

If \( c \neq c_0 \), then \( \lim_{r \to \infty} (q_0 + c) = a \neq 0 \), and \( |(q, e^{-r})| = \infty \). If \( c = c_0 \), then for \( r > 1 \)

\[
q_0 + c_0 = \int_1^r (e^{-t} + a_1 e^{-t} e^{-t}) \, dt
\]

and

\[
(q, e^{-r}) = a + b \int_1^\infty (q_0 + c_0) e^r \, dr
\]

\[
= a + b \int_1^\infty \left( e^{-r} + a_1 e^{-r} e^{-r} \right) e^r \, dr
\]

\[
= a + b \int_1^\infty e^{-r} e^r \, dr + a_1 b \int_1^\infty \left( e^{-r} e^{-r} \right) e^r \, dr.
\]
Here

\[ \lim_{r \to \infty} \left( \int_r^\infty e^{-t} e^{-t} \, dt \right) e^r e^r = \lim_{r \to \infty} \frac{-e^{-r} e^{-r}}{e^{-r} (1 + e^{-r})} = 0 , \]

so that for some \( R > 0 \),

\[ (q, e^{-r}) = c + b \int_1^\infty e^{-r} e^r \, dr + a_1 \int_1^\infty o(1) e^{-r} \, dr . \]

The last integral converges, the first diverges, and therefore \( |(q, e^{-r})| = \infty \). This contradiction shows that \( q \not\in L^1 \), that is, \( M \in O_{DL}^e \).

To see that \( M \in O_{DL}^e \) for \( p > 1 \), let \( p' \) be determined by \( 1/p + 1/p' = 1 \). For a function \( \varphi \in C^\infty(M) \) with

\[ \varphi |(r \geq 1) = (e^{-r} e^{-r} e^{-r})^{1/p'} \]

we have

\[ \| \varphi \|_p^{p'} = a + b \int_1^\infty e^{-r} e^{-r} e^r \, dr < \infty , \]

hence \( \varphi \in L^p \). If there exists a \( q \in QL^p \), then \( |(q, \varphi)| < \infty \). But \( (q, \varphi) = (q_0 + c, \varphi) \), and if \( c \neq c_0 \), the integrand in \( (q_0 + c, \varphi) \) is asymptotically

\[ c_1 (e^{-r} e^{-r} e^{-r})^{1/p} e^r e^r = c_1 e^{r/p} e^{-r/p} e^{-r/p} \]

so that \( |(q_0 + c, \varphi)| = \infty \), a contradiction.

In the case \( c = c_0 \) we observe that for \( r > 1 \)

\[ q_0 + c_0 = e^{-r} + a \int_r^\infty e^{-t} e^{-t} \, dt , \]

with

\[ \int_r^\infty e^{-t} e^{-t} \, dt < e^{-r} \int_r^\infty e^{-t} \, dt = e^{-r} e^{-r} , \]

so that \( q_0 + c_0 \sim e^{-r} \) as \( r \to \infty \). It follows that the integrand in \( (q_0 + c_0, \varphi) \) is asymptotically

\[ e^{-r} (e^{-r} e^{-r} e^{-r})^{1/p} e^r e^r = e^{r/p} e^{-r/p} e^{-r/p} , \]
and therefore \(|(q_0 + a_0, \varphi)\) = \(\infty\). This contradiction shows that \(M \in O_{QL}^N\), and the proof of Theorem 3 is complete.

In the proofs of Theorems 1-3 we have actually shown somewhat more:

\[
\cap \bigcup_{p} O_{QL}^N \cap O_{QX}^N \neq \emptyset,
\]

\[
\cap \bigcup_{p} O_{QL}^N \cap O_{QX}^N \neq \emptyset
\]

for \(N \geq 2, 1 < p < \infty, X = P, B, D, C\); and

\[
\cap \bigcup_{p} O_{QL}^N \cap O_{QX}^N \neq \emptyset
\]

for \(N \geq 2, 1 < p < \infty, X = P, B\).

4. - Next we are to find an \(N\)-manifold which carries \(QD\) functions but no \(QL^p\) functions. First we consider the case \(p > 1\).

**Theorem 4.** \(O_{QL}^N \cap O_{QL}^N \neq \emptyset\) for \(N \geq 2, 1 < p < \infty\).

**Proof.** Take the manifold

\[ T: \{(x, y_1, \ldots, y_{N-1}) | 0 < x < 1, |y_i| < \pi, i = 1, \ldots, N-1\}, \]

with the opposite faces \(y_i = \pi, y_i = -\pi\) identified as in No. 2, and choose the metric

\[
ds^2 = x^{-2\alpha} dx^2 + x^{2\beta(N-1)} \sum_{i=1}^{N-1} dy_i^2,
\]

where \(\alpha, \beta\) are real constants to be specified later; they will depend on \(p\), so that we shall not have a generalization of the theorem in the same manner as at the end of No. 3.

The function

\[
q_0(x) = -(\beta - \alpha + 1)^{-1}(-2\alpha + 2)^{-1}x^{-2\alpha + 2}
\]

satisfies the quasiharmonic equation

\[
\Delta q_0 = x^{\alpha-\beta}(x^{\alpha-\alpha}x^2q_0)' = 1
\]

provided

\[
\beta - \alpha + 1 \neq 0, \quad \alpha \neq 1.
\]
The Dirichlet integral is
\[ D(q_0) = \int_0^1 q_0^2 x^{\alpha} dx = e \int_0^1 x^{\beta-3\alpha+2} dx < \infty \]
if
\[ \beta - 3\alpha + 3 > 0. \]

For the \( L^p \) norm we have
\[ \|q_0\|_p^p = e \int_0^1 x^{(\beta-3\alpha+2)p} dx = \infty \]
if
\[ \beta - (2p + 1)\alpha + 2p + 1 < 0. \]

An inspection of the last two inequalities shows that \( p = 1 \) is ruled out. For
\[ p > 1, \quad \alpha > \frac{2}{3}, \quad \beta \in (3(\alpha - 1), (2p + 1)(\alpha - 1)], \]
all four inequalities are satisfied. In particular, \( T \in \mathcal{H}_D^\varphi \).

The exponent \( \beta - \alpha \) in the volume element is positive, and the constant function 1 belongs to \( L^{p'} \) for our \( p' > 1 \). Suppose there exists a \( q \in QL^p \). Then \( |\langle q, 1 \rangle| < \infty \).

Every \( h \in H \) can be written
\[ h(x, y) = h_0(x) + \sum_n f_n(x) G_n(y), \]
where \( h_0 \in H, f_n G_n \in H, n = (n_1, ..., n_{N-1}), \) the \( n_i \) integers \( > 0, \) the \( G_n \) products of the form
\[ G_n(y) = \frac{\cos n_1 y_1 \cos n_2 y_2 \cdots \cos n_{N-1} y_{N-1}}{\sin n_1 y_1 \sin n_2 y_2 \cdots \sin n_{N-1} y_{N-1}}, \]
and the prime in \( \sum' \) indicates that in each term at least one \( n_i \) does not vanish. The harmonic equation \( \Delta h_0(x) = -x^{\beta-3\alpha} e^{i2\pi \beta} B h_0(x) = 0 \) is satisfied by
\[ h_0(x) = a x^{\beta-3\alpha + 1} + b. \]

Suppose first \( a \neq 0. \) Since \( -2\alpha + 2 > -\alpha - \beta + 1, \)
\[ q_0(x) + h_0(x) \sim h_0(x) \quad \text{as} \quad x \to 0. \]
It follows that the integrand in \((q, 1) = (q_0 + h_0, 1)\) is asymptotically
\(x^{-a-\beta+1+\beta-a} = x^{-2a+1}\). A fortiori, \(|(q, 1)| = \infty\), a contradiction.

Now let \(a = 0\), \(h_0 = b\). Since
\[ q(x) = x^{(-2a+2)/p'} \in L^p, \]
\(|(q, \psi)| < \infty\). On the other hand,
\[ |(q, \psi)| = |(q_0 + b, \psi)| = a_1 + b_1 \int_0^1 x^{-2a+2} \psi x^{-\beta} dx = \infty \]
if
\[ -2(a-1) \left(1 + \frac{1}{p} \right) + \beta - (\alpha-1) < 0, \]
i.e.,
\[ \beta < \left(3 + \frac{2}{p'} \right)(a-1). \]
Since \(2p + 1 > 3 + 2/p'\) for \(p > 1\), the choice
\[ \beta \in \left(3(a-1), \left(3 + \frac{2}{p'} \right)(a-1) \right) \]
gives the contradiction \(|(q, \psi)| = \infty\) while preserving the earlier inequalities.
We conclude that \(\text{O}_{QL}^N \cap \overline{\text{O}_{QB}^N} \neq \emptyset\) for all \(p > 1\).

5. – For \(p = 1\), \(\text{O}_{QL}^N \cap \overline{\text{O}_{QB}^N} \neq \emptyset\) is no longer true. In fact, we even have a strict inclusion:

**Theorem 5.** \(\text{O}_{QL}^N < \text{O}_{QB}^N\) for \(N > 2\).

**Proof.** To prove the inclusion relation \(\text{O}_{QL}^N \subset \text{O}_{QB}^N\), suppose \(u \in QD\). For any regular subregion \(\Omega\), the Riesz decomposition yields (cf. e.g. Nakai-Sario [3])
\[ u(x) = h_\Omega(x) + \int_\Omega g_\Omega(x, y) dy \]
on \(\Omega\), where \(h_\Omega(x)\) is the harmonic function on \(\Omega\) with \(h_\Omega = u\) on \(\partial \Omega\),
and \(g_\Omega(x, y)\) is the Green’s function on \(\Omega\) with pole \(y\). By Stokes’ formula,
\[ \int_{\partial \Omega} \int_\Omega g_\Omega(x, y) dy dx \lesssim D_\Omega(u). \]
On letting $\Omega \to R$ we obtain

$$\int_R \int_R g(x, y) \, dy \, dx < D(u) < \infty,$$

where $g$ is the Green function on $R$. Since $A_x \int_R g(x, y) \, dy = 1$, we have

$$\int_R g(x, y) \, dy \in QL^1,$$

and therefore $O_{QL}^N \subset O_{QB}^N$. By Theorem 2, $O_{QL}^N \cap O_{QB}^N \neq \emptyset$, hence $O_{QL}^N \subset O_{QB}^N$.

6. It remains to consider the class $QC$. Here we have the most elegant case, as there is strict inclusion for all $p$:

**Theorem 6.** $O_{QL}^N \subsetneq O_{QC}^N$ for $N > 2$, $p > 1$.

**Proof.** In view of Theorem 2, it suffices to show that $O_{QL}^N \subsetneq O_{QC}^N$. Suppose $R \notin O_{QC}^N$, and take a $u \in QC$. The Riesz decomposition of $u$ on $\Omega$ implies

$$\int_R g(x, y) \, dy < |u(x)| + |h_{\Omega}(x)| < 2 \sup_\Omega |u(x)|.$$

On letting $\Omega \to R$ we obtain $\int_R g(x, y) \, dy \in B$. From the proof of Theorem 5, we conclude that $\int_R g(x, y) \, dy \in C$. Let

$$R_1 = \{x \in R \mid \int_R g(x, y) \, dy > 1\}.$$

Then

$$V(R_1) = \int_{R_1} \int_R g(x, y) \, dy \, dx < \infty.$$

For $p > 1$,

$$\int_R \left( \int_R g(x, y) \, dy \right)^p \, dx = \int_R \left( \int_R g(x, y) \, dy \right)^p \, dx + \int_R \left( \int_R g(x, y) \, dy \right)^p \, dx$$

$$< MV(R_1) + \int_R \int_R g(x, y) \, dy \, dx < \infty,$$

and therefore $\int_R g(x, y) \, dy \in QL^p$. 


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