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On Siegel's Zero.

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1. - Let \( d \) be fundamental discriminant, and let

\[
\chi(n) = \left( \frac{d}{n} \right) \quad \text{(Kronecker's symbol).}
\]

It is well known (see [1]) that \( L(s, \chi) \) has at most one zero \( \beta \) in the interval \((1 - c_1 \log |d|, 1)\) where \( c_1 \) is an absolute positive constant. The main aim of this paper is to prove:

**Theorem 1.** Let \( d, \chi \) and \( \beta \) have the meaning defined above. Then the following asymptotic relation holds

\[
1 - \beta = \frac{6}{\pi^2} \frac{L(1, \chi)}{\zeta(2)} \sum' \frac{1}{a} \left[ 1 + O \left( \frac{(\log \log |d|)^3}{\log |d|} \right) + O((1 - \beta) \log |d|) \right]
\]

where \( \sum' \) is taken over all quadratic forms \((a, b, c)\) of discriminant \( d \) such that

\[
-a < b < a < \frac{1}{4} \sqrt{|d|},
\]

and the constants in the \( O \)-symbols are effectively computable.

In order to apply the above theorem we need some information about the size of the sum \( \sum' 1/a \). This is supplied by the following.

**Theorem 2.** If \((a, b, c)\) runs through a class \( C \) of properly equivalent primitive forms of discriminant \( d \), supposed fundamental, then

\[
\sum_{\substack{|a| > |b| > -a \\ (a,b,c) \in C}} \frac{1}{a} < \begin{cases} \frac{1}{m_0} \log \epsilon_0 & \text{if } d < 0, \\ \frac{\log \epsilon_0}{\log \left( \frac{1}{2} \sqrt{d} - 1 \right)} + \frac{4}{\sqrt{d}} & \text{if } d > 676, \end{cases}
\]

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where \( m_0 \) is the least positive integer represented by \( C \) and \( e_0 \) is the least totally positive unit of the field \( Q(\sqrt{d}) \).

Theorems 1 and 2 together imply

**Corollary.** For any \( \eta > 0 \) and \( |d| > c(\eta) \) (\( d \) fundamental) we have

\[
1 - \beta > \begin{cases} \left( \frac{6}{\pi} - \eta \right) \frac{1}{\sqrt{|d|}} & \text{if } d < 0, \\ \left( \frac{6}{\pi^2} - \eta \right) \frac{\log d}{\sqrt{d}} & \text{if } d > 0, \end{cases}
\]

where \( c(\eta) \) is an effectively computable constant.

**Remark.** In the case \( d < 0 \), the constant \( 6/\pi \) could be improved by using the knowledge of all fields with class number \( < 2 \).

Similar inequalities with \( 6/\pi \) and \( 6/\pi^2 \) replaced by unspecified positive constants have been claimed by Hanecke [3], however, as pointed out by Pintz [8], Hanecke's proof is defective and when corrected gives inequalities weaker by a factor \( \log \log |d| \). Pintz himself has proved the first inequality of the corollary with the constant \( 6/\pi \) replaced by \( 12/\pi \) (see [8]).

For \( d < 0 \), the first named author [2] has obtained (1) with a better error term by an entirely different method. M. Huxley has also found a proof in the case \( d < 0 \) by a more elementary method different, however, from the method of the present paper.

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2. - The proofs of Theorems 1 and 2 are based on several lemmata.

**Lemma 1.** Let \( f(d) = (\log |d|/\log \log |d|)^2 \). Then

\[
\sum_{N_a < \frac{1}{4} \sqrt{|d|/\log |d|}} \frac{1}{N_a} = \sum_{N_a} \frac{1}{a} \left( 1 + O \left( \frac{(\log \log |d|)^2}{\log |d|} \right) \right),
\]

where the left hand sum goes over all ideals \( a \in Q(\sqrt{d}) \) with norm \( < \frac{1}{4} \sqrt{|d|} f(d) \) and the constant in the \( O \)-symbol is effectively computable.

**Proof.** Every ideal \( a \) of \( \Omega(\sqrt{d}) \) can be represented in the form

\[
a = u \left[ a, \frac{b + \sqrt{d}}{2} \right]
\]
where \( u, a \) are positive integers and \( b^2 \equiv d \pmod{4a} \) (see [5], Theorem 59). If we impose the condition that
\[
-\alpha < b < \alpha
\]
then the representation becomes unique. Since \( Na = u^2a \), it follows that
\[
\frac{1}{N^a} = \sum_{1 \leq a, b < \sqrt{d}/4a} \frac{1}{a} + O \left( \sum_{1 \leq a, b < \sqrt{d}/4a} \frac{1}{a^2} \right) = \sum_{a} \frac{1}{a} \left( \frac{\pi^2}{6} + O \left( f(d) \right) \right) + O(S).
\]

To estimate the sum \( S \), we divide it into two sums \( S_1 \) and \( S_2 \). In the sum \( S_1 \), we gather all the terms \( 1/a \) such that \( a \) has at least one prime power factor
\[
p^x > \ell(d) = d^{1/21} \log \log |d|,
\]
\[
p^x | a,
\]
and in \( S_2 \) all the other terms.

Let \( \nu(a) \) be the number of representations of \( a \) as \( Na \) where \( a \) has no rational integer divisor \( > 1 \). Then \( \nu(a) \) is a multiplicative function satisfying
\[
\nu(p^x) = \begin{cases} 1 + \left( \frac{d}{p^x} \right) & \text{if } p \nmid d \text{ or } \alpha = 1, \\ 0 & \text{otherwise.} \end{cases}
\]

Clearly
\[
S_1 < \sum_{a} \frac{1}{a} \sum \nu(p^x) p^{-x} < \sum_{a} \frac{1}{a} \sum 2p^{-x}
\]
where \( \sum \) goes over all prime powers \( p^x \) with
\[
\max (\ell(d), \sqrt{|d|}/4a) < p^x < \sqrt{|d|} f(d)/4\alpha.
\]

Now, by a well known result of Mertens,
\[
\sum_{p^x < x} p^{-x} = \log \log x + c + O((\log x)^{-1})
\]
where \( c \) is a constant.
Hence
\[ \sum_{x < p^2 < y} p^{-x} = \log \left( \frac{\log y}{\log x} \right) + O((\log x)^{-1}) \ll \frac{\log y}{\log x} - 1 + O((\log x)^{-1}) = \frac{\log y/x + O(1)}{\log x}. \]

This gives
\[ \sum_{x} p^{-x} \ll \frac{\log f(d) + O(1)}{\log l(d)} \ll \frac{(\log \log |d|)^2}{\log d} \]

and we get
\[ S_1 = O \left( \frac{(\log \log |d|)^2}{\log |d|} \right) \sum_{x} \frac{1}{\sigma} . \tag{4} \]

To estimate $S_2$, we notice that each $a$ occurring in it must have at least
\[ k_0 = \log \left( \frac{\sqrt{|d|}}{\log l(d)} \right) > 10 \log \log |d| \]
distinct prime factors. Therefore
\[ S_2 \ll \sum_{k > k_0} \left( \frac{1}{k!} \right) \left( \sum_{p^k < l(d)} \nu(p^k) p^{-x} \right) \ll (1/k_0!) \sigma^{k_0} e^\sigma \]

where
\[ \sigma = \sum_{p^k < l(d)} \nu(p^k) p^{-x} \ll 2 \log \log l(d) + O(1) = 2 \log \log |d| + O(1) . \]

Now, Stirling’s formula gives $k! > k_0^k e^{-k_0}$. Hence
\[ \log S_2 \ll -k_0 \log k_0 + k_0 (\log \sigma + 1) + \sigma \ll -k_0 [\log 10 + \log \log \log |d| - \log 2 - \log \log \log |d| - 1] + \sigma \ll -3 \log \log |d| + O(1) \]

and
\[ S_2 = O((\log |d|)^{-1}) . \]
The lemma now follows from equations (3), (4) and (5). The next lemma gives the growth conditions for the Riemann zeta-function and Dirichlet $L$-functions on the imaginary axis.

**Lemma 2.** For all real $t$

(6) \[ |\zeta(it)| \ll (|t|^\frac{3}{2} + 1) \log(|t| + 2) \]

(7) \[ |L(it, \chi)| \ll \sqrt{|d|} (|t|^\frac{3}{2} + 1) \log \left( |d|(|t| + 2) \right). \]

**Proof.** If $|t| > t_0$, the estimate

\[ |\zeta(it)| \ll |t|^\frac{3}{2} \log |t| \]

holds (see [10], p. 19). Since $\zeta(s)$ has no pole on the imaginary axis, we have

\[ |\zeta(it)| \ll 1 \quad \text{for } |t| < t_0 \]

and the inequality (6) now follows.

To prove (7), we note that

\[ |L(1 - it, \chi)| \ll \log(|d|(|t| + 2)) \]

(see [1], p. 17, lemma 2 with $q = |d|, x = 2|d|(|t| + 2)$).

Now, by the fundamental equation for $L$-functions

\[ |L(it, \chi)| = |L(1 - it, \chi)| |d|^\frac{1}{2} |\Gamma(\frac{1}{2}it + A)\Gamma(\frac{1}{2}it + A)\Gamma^{-1}(\frac{1}{2} - \frac{1}{2}it + A)| \]

where

\[ A = \frac{1}{4}(1 - \chi(-1)). \]

Using the formula

\[ |\Gamma(s)| = \sqrt{2\pi}|s|^{s-\frac{1}{2}} \exp \left( -\frac{1}{2} \pi t \right)(1 + O(|t|^{-1})) \]

valid for $s = \sigma + it, 0 < \sigma < \frac{1}{2}, |t| > 1$ (see [9], p. 395), equation (7) follows, upon noting that

\[ |\Gamma(\frac{1}{2}t + A)\Gamma^{-1}(\frac{1}{2} - \frac{1}{2}it + A)| \ll 1 \quad \text{for } |t| < 1. \]
PROOF OF THEOREM 1. By the standard argument ([4], p. 31)

\[
\frac{1}{2\pi i} \int_{2-\infty}^{2+i\infty} \frac{y^s}{s(s+2)(s+3)} ds = \begin{cases} \frac{1}{6} \frac{y^{-2}}{2} + \frac{y^{-3}}{3} & \text{if } y > 1, \\ 0 & \text{if } 0 < y < 1. \end{cases}
\]

Since for Re \(s\) > 1

\[
\zeta(s) L(s, \chi) = \sum (Na)^{-s},
\]

it follows that for any \(x > 0\)

\[
I = \frac{1}{2\pi i} \int_{2-\infty}^{2+i\infty} \zeta(s + \beta) L(s + \beta, \chi) \frac{x^s}{s(s+2)(s+3)} ds = \sum_{Na \leq x} (Na)^{-\beta} \left[ \frac{1}{6} - \frac{(Na)^2}{2x^2} + \frac{(Na)^3}{3x^3} \right].
\]

Choose \(x = \frac{1}{2} \sqrt{\log f(d)}\) with \(f(d) = (\log |d|/\log \log |d|)^2\).

If \(Na \ll x\), we have

\[(Na)^{-\beta} = (Na)^{-1} \left( 1 + O(1 - \beta) \log |d| \right).\]

Hence

\[
I = \frac{1}{6} \sum_{Na \leq x} (Na)^{-1} \left( 1 + O((1 - \beta) \log |d|) \right) + O \left( \sum_{Na \leq x} (Na)^{-1} f(d)^{-\beta} + O \left( \sum_{x/|f(d)| \leq Na \leq x} (Na)^{-1} \right) \right),
\]

and by lemma (1) (cf. formula (3))

\[
I = \frac{1}{6} \sum_{Na \leq x} \left[ \frac{1}{a} + O \left( \frac{(\log \log |d|)^2}{\log |d|} \right) \right] = O((1 - \beta) \log |d|). \tag{8}
\]

On the other hand, after shifting the line of integration to Re \(s\) = \(-\beta\)

\[
I = \frac{L(1, \chi) x^{1-\beta}}{(1-\beta)(3-\beta)(4-\beta)} + \frac{1}{2\pi i} \int_{-\beta-\infty}^{-\beta+i\infty} \zeta(s + \beta) L(s + \beta, \chi) \frac{x^s}{s(s+2)(s+3)} ds.
\]

By lemma (2), the integral on the right does not exceed

\[O(x^{-\beta} \sqrt{|d| \log |d|})\]
and since
\[ x^{1-\beta} = 1 + O((1 - \beta) \log |d|) \]
\[ (1 - \beta)(3 - \beta)(4 - \beta) = 6 + O(1 - \beta) \]
we get from (8) and (9)
\[ 1 - \beta = \frac{6}{\pi^2} \frac{L(1, \chi)}{\sum \frac{1}{1/a}} \log |d| + O(1 - \beta) \log |d| \bigg| \]

3. - Proof of Theorem 2. For \( d < 0 \) it is enough to prove that every class contains at most one form satisfying

(10) \[ -|a| < b < |a| < \frac{1}{2} \sqrt{|d|}. \]

Now, since
\[ |d| = 4ac - b^2 \]
we infer from (10) that
\[ a < \sqrt{|d|} < |d|/4a < c, \]
thus every form satisfying (10) is reduced, and it is well known that every class contains at most one such form.

For \( d > 0 \), let us choose in the class \( C \) a form (*) \((a, \beta, \gamma)\) reduced in the sense of Gauss, i.e. such that

(11) \[ \beta + \sqrt{d} > 2|\alpha| > -\beta + \sqrt{d} > 0. \]

We can assume without loss of generality that \( \alpha > 0 \). Now, for any form \( f \in C \), there exists a properly unimodular transformation
\[ T = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \]
taking \((\alpha, \beta, \gamma)\) into \( f \). The first column of this transformation can be made to consist of positive rational integers by Theorem 79 of [5]. If \( f \) satisfies (10), we infer from

(12) \[ \alpha p^2 + \beta pq + \gamma q^2 = a \]

(*) \( \beta \) is not to be confused with Siegel's zero.
that

\[ \left| p + \frac{\beta - \sqrt{d}}{2x} q \right| = a \left| xp + \frac{\beta + \sqrt{d}}{2} q \right|^{-1} \leq \frac{1}{4} \sqrt{d} \cdot 2(\sqrt{d} q)^{-1} = \frac{1}{2} q^{-1} \]

and by lemma (16), p. 175 from [5], \( p/q \) is a convergent of the continued fraction expansion for

\[ \omega = -\frac{\beta + \sqrt{d}}{2x}. \]

From this point onwards, we shall use the notation of Perron’s monograph [7]. Since by (11)

\[ \omega^{-1} > 1 \quad \text{and} \quad O > (\omega')^{-1} > -1, \]

\( \omega^{-1} \) is a reduced quadratic surd and it has a pure periodic expansion into a continued fraction. Hence

\[ \omega = [0, b_1, b_2, \ldots, b_k] \]

where the bar denotes the primitive period. The corresponding complete quotients form again a periodic sequence

\[ \omega_v = \frac{P_v + \sqrt{d}}{Q_v}, \quad \omega_0 = \omega \]

where for all \( v > 1 \), \( \omega_v \) is reduced,

\[ \omega_v = \omega_{v+1}, \]

and \( k \) is the least number with the said property.

**Lemma 3.** Let \( [0, b_1, b_2, \ldots, b_k] \) be the continued fraction for \( \omega \) defined above. Then

\[ \sum_{(a,b,c) \in \mathcal{C}} \frac{1}{|a|} \leq \frac{2}{\sqrt{d}} \sum_{n=2}^{[k/2]} \min \left( \frac{\sqrt{d}}{2}, b_n + 1 \right) \]

where the sum on the left is taken over all \( (a, b, c) \) in the class \( \mathcal{C} \) satisfying (10).
PROOF. If $A_j/B_j$ is the $j$-th convergent of $\omega$, we have by formula (18), § 20 of [7]

\[
(A_{v-1} Q_0 - B_{v-1} P_v)^2 - d(B_{v-1})^2 = (-1)^v Q_v Q_v'
\]

which gives on simplification

\[
(14) \quad \alpha A_{v-1}^2 + \beta A_{v-1} B_{v-1} + \gamma B_{v-1}^2 = (-1)^v Q_v/2.
\]

Similarly, eliminating $Q_v$ from formulae (16) and (17) in § 20 of [7], we get

\[
(15) \quad 2\alpha A_{v-1} A_{v-2} + \beta (A_{v-1} B_{v-1} + B_{v-2} A_{v-2}) + 2\gamma B_{v-1} B_{v-2} = (-1)^{v-1} P_v.
\]

Let $p = A_{v-1}$, $q = B_{v-1}$ ($v \geq 1$). By (12)

\[
a = (-1)^v Q_v/2.
\]

Hence, by formula (1) of § 6 of [7]

\[
\begin{vmatrix} A_{v-1} & A_{v-2} \\ B_{v-1} & B_{v-2} \end{vmatrix} = (-1)^v.
\]

and since

\[
\begin{vmatrix} A_{v-1} & r \\ B_{v-1} & s \end{vmatrix} = 1
\]

it follows that

\[
T = \begin{pmatrix} A_{v-1} & A_{v-2} \\ B_{v-1} & B_{v-2} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & (-1)^v \end{pmatrix}, \quad t \in \mathbb{Z}.
\]

Thus we find using (14) and (15)

\[
f = (\alpha, \beta, \gamma) \begin{pmatrix} A_{v-1} & A_{v-2} \\ B_{v-1} & B_{v-2} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & (-1)^v \end{pmatrix} =
\]

\[
= \begin{pmatrix} (-1)^v Q_v/2, (-1)^{v-1} P_v, (-1)^v Q_{v-1}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & t \end{pmatrix} (-1)^v.
\]

In order to make $f$ satisfy (10) we must choose

\[
t = (-1)^v \left[ \frac{P_v}{Q_v} + \frac{1}{2} \right].
\]
Thus $f$ is uniquely determined by $\omega_r$ and in view of (13), we have
\[
\sum_{(a, b) \in \mathbb{C}} \frac{1}{|a|} < \sum_{r=1}^{[k, 2]} 2(Q_r)^{-1}.
\]
Since $\omega_r$ is reduced, we have further for $v$ in question
\[
\sqrt{d} > \frac{2\sqrt{d}}{Q_r} > \frac{P_r + \sqrt{d}}{Q_r} > \sqrt{d} > 2.
\]
Hence for
\[
b_r = [\omega_r],
\]
we get the inequalities
\[
\sqrt{d} > b_r > 2, \quad b_r + 1 > \sqrt{d}/Q_r,
\]
and by (16), lemma (3) follows.

Now, let $\varepsilon_0$ be the least totally positive unit $\varepsilon_0 > 1$ of the ring $Z(\sigma)$ where
\[
\sigma = \begin{cases} 
\frac{1}{2} \sqrt{d} & \text{if } d \equiv 0 \pmod{4}, \\
\frac{1 + \sqrt{d}}{2} & \text{if } d \equiv 1 \pmod{4}.
\end{cases}
\]
By Theorem (7) of Chapter IV of [6]
\[
\varepsilon_0 = \frac{u + v\sqrt{d}}{2},
\]
where for $l = [k, 2],
\[
v = (q_{l-1}, p_{l-1} - q_{l-2}, p_{l-3}), \quad u = p_{l-1} + q_{l-2}
\]
and $p_j, q_j$ are the numerator and denominator, respectively, of the $j$-th convergent for $\omega^{-1}$. Moreover, since $\omega^{-1}$ satisfies the equation
\[-\gamma\omega^2 - \beta\omega - \alpha = 0, \quad (\gamma > 0)
\]
we find from formula (1) of § 2 of Chapter IV of [6] that
\[
q_{l-1} - p_{l-1} = -\beta v, \quad -p_{l-2} = -\alpha v.
\]
Hence
\[
\varepsilon_0 = \frac{p_{l-1} + q_{l-2}}{2} + \frac{p_{l-2} \sqrt{d}}{2\alpha} = q_{l-1} + \frac{\beta + \sqrt{d}}{2\alpha} - p_{l-2}.
\]
Since \( p_j = B_{t+1} \), \( q_j = A_{t+1} \), we get

\[
(17) \quad \varepsilon_\theta = B_{t-1} \left( \frac{A_{t-1}}{B_{t-1}} + \frac{\beta + \sqrt{d}}{2\alpha} \right) > B_{t-1} \left( \omega + \frac{\beta + \sqrt{d}}{2\alpha} \right) = \frac{\sqrt{d}}{\alpha} B_{t-1}.
\]

Now,

\[
\omega_t = b_t + \omega_{t+1} = b_t + \omega_t^{-1} = b_t + \omega, \quad \omega_t' = b_t + \omega'
\]

and since \( \omega_t \) is reduced \( 0 > b_t + \omega' > -1 \)

\[
b_t = \left[-\omega'\right] = \left[\frac{\beta + \sqrt{d}}{2\alpha}\right] = \frac{\sqrt{d}}{\alpha}.
\]

Thus (17) gives

\[
\varepsilon_\theta > b_t B_{t-1} \geq \prod_{r=1}^{l} b_r,
\]

and by (16)

\[
(18) \quad \sum_{(a,b) \in \mathcal{C}} \frac{1}{|a|} \leq \frac{2}{\sqrt{d}} \max \sum (x_i + 1) = \frac{2}{\sqrt{d}} M
\]

where maximum is taken over all non-decreasing sequence of at most \( l \) numbers satisfying

\[
2 < x_i \leq \frac{1}{2} \sqrt{d} - 1 = D, \quad \prod x_i < \varepsilon_\theta.
\]

Let \((x_1, x_2, \ldots, x_m)\) be a point in which the maximum is taken with the least number \( m \). We assert that the sequence contains at most one term \( x \) with \( 2 < x < D \). Indeed, if we had \( 2 < x_i < x_{i+1} < D \), we could replace the numbers \( x_i, x_{i+1} \) by

\[
\frac{x_i}{\min(x_i/2, D/x_{i+1})}, \quad x_{i+1} \min\left(\frac{x_i}{2}, \frac{D}{x_{i+1}}\right)
\]

and the sum \( \sum (x_i + 1) \) would increase. Also, if we had \( x_1 = x_2 = x_3 = 2 \), we could replace them by \( x_1 = 8 \), and the sum \( \sum (x_i + 1) \) would remain the same while \( m \) would decrease.

Let

\[
\frac{\varepsilon_\theta}{4} = D^\theta, \quad \text{where } \theta = \log \left(\frac{\varepsilon_\theta}{4}\right).
\]

Using \( d > 676 \), we get

\[
M = \begin{cases} 
\frac{1}{2} \varepsilon \sqrt{d} + \max(4\theta + 1, 2\theta + 4) & \text{if } 4\theta < D, \\
\frac{1}{2} \varepsilon \sqrt{d} + 2\theta + 4 & \text{if } 2\theta < D < 4\theta, \\
\frac{1}{2} \varepsilon \sqrt{d} + \theta + 7 & \text{if } D < 2\theta.
\end{cases}
\]
Now,

\[ e = \frac{\log \varepsilon_0}{\log D} - \frac{\log 4\theta}{\log D}. \]

Since for \( 1 < x < y, \ y(\log x/\log y) > x - 1 \), and for \( d > 676, \ D/\log D > 12/\log 12 > 4.8 \), we obtain if \( 4\theta < D \).

\[
M - \frac{1}{2} \sqrt{d} \frac{\log \varepsilon_0}{\log D} = \max (4\theta + 1, 2\theta + 4) - D \frac{\log 4\theta}{\log D} = \frac{\log 4\theta}{\log D} < \\
< \max (4\theta + 1, 2\theta + 4) - \max (4\theta - 1, 6) < 2,
\]

if \( 2\theta < D \leq 4\theta \)

\[
M - \frac{1}{2} \sqrt{d} \frac{\log \varepsilon_0}{\log D} = 2\theta + 4 - D \frac{\log 2\theta}{\log D} - D \frac{\log 2}{\log D} - \frac{\log 4\theta}{\log D} < \\
< 2\theta + 4 - 2\theta + 1 - 3 - 1 = 1,
\]

if \( D \leq 2\theta \)

\[
M - \frac{1}{2} \sqrt{d} \frac{\log \varepsilon_0}{\log D} = \theta + 7 - D \frac{\log \theta}{\log D} - D \frac{\log 4}{\log D} - \frac{\log 4\theta}{\log D} < \\
< \theta + 7 - \theta + 1 - 6 - 1 = 1.
\]

This together with (18) gives the theorem.

4. - PROOF OF COROLLARY. We can assume \( 1 - \beta < (\log |d|)^{-\frac{1}{2}} \). It then by Theorem (1) that for every \( \eta > 0 \), there exists \( c(\eta) \) such that if \( d > c(\eta) \)

\[
1 - \beta > \frac{6}{\pi^2} \frac{L(1, \chi)}{\sum' \frac{1}{a}} \left( 1 - \frac{\eta}{2} \right).
\]

Let \( h_0 \) be the number of classes of forms in question. For \( d < -4 \), we have

\[
L(1, \chi) = \frac{\pi h_0}{\sqrt{|d|}},
\]

and by Theorem (2)

\[
\sum' \frac{1}{a} < h_0.
\]

Hence by (19)

\[
1 - \beta > \frac{6}{\pi^2} \frac{h_0 \pi}{h_0 \sqrt{|d|}} \left( 1 - \frac{\eta}{2} \right) > \left( \frac{6}{\pi} - \eta \right) \frac{1}{\sqrt{|d|}}.
\]
For \( d > 0 \), we have
\[
L(1, \chi) = \frac{h_d \log \varepsilon_0}{\sqrt{d}}.
\]

Now, for any class \( C \) of forms
\[
\sum_{(a,b,c) \in C} \frac{1}{|a|} = \sum_{\{d \geq a \geq b > -a \}} \frac{1}{|a|} + \sum_{\{d > -a \geq b > -a \}} \frac{1}{|a|}.
\]

If \((a, b, c)\) runs through \( C \), \((-a, b, -c)\) runs through another class which we denote by \(-C\) (it may happen that \(-C = C\)). If \( C_1 \neq C_2 \), then \(-C_1 \neq -C_2\). Hence
\[
\sum_{C} \sum_{\{d \geq a \geq b > -a \}} \frac{1}{|a|} = 2 \sum_{C} \frac{1}{a}
\]
and by Theorem (2)
\[
\sum \frac{1}{a} \leq \frac{h_d}{2} \left( \frac{\log \varepsilon_0}{\log \left( \frac{1}{2} \sqrt{d} - 1 \right)} + \frac{4}{\sqrt{d}} \right) \leq \frac{h_d \log \varepsilon_0}{\log d} \left( 1 + O \left( \frac{1}{\sqrt{d}} \right) \right),
\]
where the constant in the \( O \)-symbol is effective. (Note that \( \varepsilon_0 > \frac{1}{2} \sqrt{d} \)). This together with (19) gives the corollary.

REFERENCES