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Algebraic Theory of Stacks (*)

ISTVÁN FÁRY (**)

dedicated to Jean Leray

1. – Let L be a *distributive lattice*; L could be, for example, a *ring of sets* (i.e., a family L of sets, such that $A, B \in L$ implies $A \cup B$, $A \cap B \in L$, and $\leqslant$ is $\subset$). By definition, a ***stack*** F over L associates to every $A \in L$ an abelian group $F(A)$, and to every inequality $A \geqslant B$ a homomorphism $F(A) \to F(B)$ said to be induced by $A \geqslant B$; we suppose that $A \geqslant A$ induces the identity, and that the composition of induced morphisms is induced. If $a \in F(A)$, *we denote* Ba *the image of* a under the induced morphism $F(A) \to F(B)$:

$$a \mapsto Ba \quad \textit{if} \quad a \in F(A) \qquad \textit{and} \qquad A \geqslant B\,. \tag{1}$$

With this notation the properties of the induced morphism are: $Aa = a$; if $A \geqslant B \geqslant C$, then $CBa = Ca$ (see [9], [4]).

Given any two lattice elements A, B, we introduce the morphisms

$$F(A \vee B) \xrightarrow{r} F(A) \oplus F(B) \xrightarrow{s} F(A \wedge B) \tag{2}$$

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as follows:

$$rh = (Ah,\ Bh) \qquad (h \in F(A \vee B) \tag{3}$$

$$s(a,\ b) = (A \wedge B)b - (A \wedge B)a \qquad \big(a \in F(A),\ \ b \in F(B)\big)\,. \tag{4}$$

We observe that (2) is of order two: $\operatorname{im} r \subset \ker s$. We will call a stack F ***additive***, if (2) is exact, plus $\ker r$ is isomorphic to $\operatorname{coker} s$; these will be the most important stacks for us. Of course, for proper definition, we must have group morphisms

$$d\colon F(A \wedge B) \to F(A \vee B) \qquad \big(d = d(A,\ B)\big) \tag{5}$$

for all pairs A, B in L, which commute with induced morphisms, i.e. all diagrams

$$\begin{array}{ccc} F(A \wedge B) & \to & F(A \vee B) \\ \downarrow & & \downarrow \\ F(A' \wedge B') & \to & F(A' \vee B') \end{array} \qquad (A \geqslant A',\ B \geqslant B') \tag{6}$$

are commutative, and such that all triangles

$$\begin{tikzcd} & F(A \vee B) \arrow[dl, "r"'] & \\ F(A) \oplus F(B) \arrow[rr, "s"'] & & F(A \wedge B) \arrow[ul, "d"'] \end{tikzcd} \tag{7}$$

are exact.

With this we arrive at our basic definition which will be called an axiom, in view of the remarks made later on.

ADDITION AXIOM. *The stack F over the distributive lattice L is given together with morphisms* (5) *for all ordered pairs of lattice elements, the diagrams* (6) *commute, and all triangles* (7) *are exact, where r, s are defined in* (3), (4) *and d is* (5).

REMARKS. We consider the Addition Axiom as a non-categorical axiom for the algebraic entity $\{F,\ d\}$ consisting of a stack F over a distributive lattice L, plus a family of morphisms $d(A,\ B)$, $(A,\ B) \in L \times L$. Keeping L fixed, additive stacks over L form a rather « small » subclass of the class of all stacks over L. Stacks are not to be confused with presheaves or sheaves.

If L is the lattice of open subspaces of a topological space, a stack over L is a presheaf, however, in the geometric applications we will use mainly stacks over the lattice of closed subspaces. A presheaf is a sheaf if (2) preceded by $0 \to$ is exact for all pairs of open sets. Roughly stated this gives 0 cohomology, whereas additivity is satisfied by total cohomology, as will be seen below.

2. – Let S be the space of a compact polyhedron, and L the finite family of all closed subcomplexes of the given decomposition of S. Then L is a ring of sets, and S is the largest element of L. The constructions a, b, c, d below will give important additive stacks over L.

2*a*. – Let be given a cohomology theory $\{H^q(X, A), ...\}$ satisfying the Eilenberg-Steenrod axioms with the exception of the Dimension Axiom; this could be then an exotic theory. Now L being the ring of subpolyhedra, we set, for all $A \in L$,

$$F(A) = \sum_{-\infty}^{\infty} H^q(A)\,, \tag{8}$$

and define $F(A) \to F(B)$ to be i^*, where $i: B \to A$ is the injection map. Then the Mayer-Vietoris coboundary d in (5) is defined ($\vee$ is now $\cup$), and the exactness of (7) is the Mayer-Vietoris addition theorem. This construction gives a family of additive stacks for any distributive lattice L which is isomorphic to a lattice of subpolyhedra of a compact polyhedron.

2*b*. – We consider L as in 2*a*, and suppose given a sheaf $\mathcal{A}$ on S (for notations and specific reference, we use [2], p. 65; φ is now the family of compacts of S and will be omitted from formulas). We set

$$F(A) = \sum_{p=0}^{\infty} H^p(A\,;\,\mathcal{A})\,. \tag{9}$$

Then coboundaries (5) can be introduced, so that (7) is exact; this is the Mayer-Vietoris theorem in sheaf cohomology. For the class of L's indicated in 2*a* we have now another class of additive stacks.

2*c*. – Let X be a topological space, and $f: X \to S$ a continuous map. We replace the argument on the right hand side of (8) by $f^{-1}A$, thus we set

$$F(A) = \sum_{-\infty}^{\infty} H^q(f^{-1}A)\,, \tag{10}$$

provided that the cohomology theory is defined for the spaces $f^{-1}A$, $A \in L$. This will be the case, if X is a polyhedron and f is simplicial, but even in this case we have a new family of additive stacks over L.

2*d*. – We use the conventions of 2*b*, except that $\mathcal{A}$ is now a sheaf over X, where X is as in 2*c*. We set

$$F(A) = \sum_{p=0}^{\infty} H^p(f^{-1}A\,;\,\mathcal{A})\,. \tag{11}$$

This defines a family of additive stacks over the class of L's indicated at the end of 2*a*.

2*e*. – Let S be a topological space, and L a ring of closed subsets of S; now L need not be finite, it can be the family of *all* closed subspaces of S. Using the continuity axiom, and appropriate restrictions on $\{H^q(X, A), \ldots\}$, X, f, A we can repeat all the constructions above. Let us note, however, the example on p. 177 of [8] showing that the Mayer-Vietoris theorem may fail to hold true for non-separated spaces.

3. – The author is in the process of writing a monograph in three volumes [5], [6], [7] on the foundations and elements of Algebraic Topology. The book is based on the Addition Axiom, in the same way as [3] is based on the Eilenberg-Steenrod axioms. The main difference between the two approaches is as follows. The Eilenberg-Steenrod axioms, Dimension Axiom included, are categorical in the sense of Logic for triangulable pairs; exotic theories are obtained by omitting the Dimension Axiom; sheaf cohomology is not included. The Addition Axiom, which we also call informally Mayer-Vietoris axiom, is non categorical in the first place, as already indicated in 2*a*-2*e*. To apply this axiom we have to develop an algebraic theory of this non-categorical concept. Once the algebraic consequences of the Mayer-Vietoris axiom are obtained, we can apply them to classical, exotic and sheaf cohomology.

In the present paper we indicate some algebraic results on additive stacks; the proofs and additional results will be published in [5]. We will indicate the method of [6] and [7] in forthcoming papers.

4. – For given integer $m \geqslant 1$, we consider the index set $\emptyset$, $\varepsilon = \varepsilon_1 \ldots \varepsilon_k$, $\varepsilon_i = 0, 1$, $k = 1, \ldots, m$, and a system $\{G_\varepsilon; f^\varepsilon_{\varepsilon 0}, f^{\varepsilon 0}_{\varepsilon 1}, f^{\varepsilon 1}_{\varepsilon}\}$, where the G_ε's are abelian groups, $f^\varepsilon_{\varepsilon 0}\colon G_\varepsilon \to G_{\varepsilon 0}, \ldots$ are morphisms, such that for any fixed ε

the triangle of the morphisms $f^{\varepsilon}_{\varepsilon 0}$, $f^{\varepsilon 0}_{\varepsilon 1}$, $f^{\varepsilon 1}_{\varepsilon}$ be exact. For $m = 1$, we have a single triangle

(12)
$$\begin{tikzcd} & G \arrow[dl, "f_0"'] & \\ G_0 \arrow[rr, "f_1^0"'] & & G_1 \arrow[ul, "f^1"'] \end{tikzcd}$$

as in (7). For $m = 2$, we have a diagram

(13)
$$\begin{tikzcd} & & & \cdot \arrow[dl] & & & \\ & & 0 \arrow[rr] \arrow[dl] & & 1 \arrow[ul] \arrow[dl] & & \\ & 00 \arrow[r] & 01 \arrow[u] & 10 \arrow[r] & 11 \arrow[u] & & \end{tikzcd}$$

We call these systems *interated extension diagrams.*

If F is an additive stack over the distributive lattice L, and α: $A_0, \ldots, A_m$, $A_i \in L$, is a given indexed family of lattice elements, we define an *addition diagram*

$$D(F; A_0, \ldots, A_m) \tag{14}$$

of F relative to α by an induction with respect to m. For $m = 1$, (14) is (7), by definition. We identify (7) to (12), and we call $G = F(A_0 \vee A_1)$ its *top vertex*, and the other two vertices bottom vertices. We suppose (14) defined for $m - 1$, to be an iterated extension diagram with top vertex $G = F(A_0 \vee \ldots \vee A_{m-1})$, and consider the display

(15)
$$\begin{tikzcd} & F(A_0 \vee \ldots \vee A_m) \arrow[dl] & \\ D(F; A_0, \ldots, A_{m-1}) \oplus F(A_m) \arrow[rr] & & D(F; A_0 \wedge A_m, \ldots, A_{m-1} \wedge A_m) \arrow[ul] \end{tikzcd}$$

where $\oplus F(A_m)$ means adding directly the group $F(A_m)$ to all vertices G_ε, $\varepsilon = 0 \ldots 0$. This defines then (14) by induction on m. It is easy to see that if we take the direct sum of the bottom vertices of (14), we obtain the direct sum of all groups

$$F(A_{i_0} \wedge \cdots \wedge A_{i_p}) \qquad (0 \leqslant p \leqslant m;\ 0 \leqslant i_0 < \ldots < i_p \leqslant m)\,; \tag{16}$$

we denote this direct sum by $\check{C}(\alpha; F) = \sum \check{C}^p$. With further discussion of (14) we obtain a differential δ: $\check{C} \to \check{C}$, $\delta \check{C}^p \subset \check{C}^{p+1}$.

In the author's manuscript [5] concepts of Algebraic Topology and Homological Algebra are not presupposed, but some notions are introduced in connection with (14) and are motivated by the study of this diagram. Such concepts are differential group (an abelian group X plus a morphism $\delta: X \to X$, $\delta^2 = 0$; hence $H(X)$, and $f^*: H(X) \to H(Y)$, for differential f $(\delta f = f\delta)$, or anti-differential f $(f\delta + \delta f = 0)$), Leray-Čech group $\check{C}(\alpha; F)$, spectral sequence, and related concepts. We omit presently the discussion of this motivation, and use the concepts.

For $m = 2$, the diagram (14) is (13), and gives a spectral sequence $\{E_1, E_2, E_3\}$. Here $E_1 = \check{C}(\alpha; F)$, $d_1 = \delta$ thus $E_2 = H(\check{C}(\alpha; F))$. The differential d_2 is obtained from (13): to apply it, we go from the vertex 00 up to 0, over to 1, and down to 11; this gives $E_2^0 \to E_2^2$. The group E_3 contains E_{30} such that $GdF(A_0 \vee A_1 \vee A_2)$ (graded group Gd) is an extension of E_{30} by E_3/E_{30}.

5. – In the study of stacks and additive stacks the following definition and result are useful.

DEFINITION. *Let F be a stack over the distributive lattice L. We say that $\{X; N_A, A \in L\}$ is a coordinate (group system) of F, if X is a differential group, $N_A \subset X$ differential subgroup, $N_A \subset N_B$ if $A \geqslant B$, and*

$$F(A) = H(X/N_A) \tag{17}$$

$$\text{if} \quad A \geqslant B\,, \qquad F(A) \to F(B) \quad \text{is} \quad H(X/N_A) \to H(X/N_B)\,. \tag{18}$$

If X, X/N_A are free abelian groups for every $A \in L$, we say that $\{X\}$ is a free coordinate group system.

THEOREM 1. *Over a finite, distributive lattice L, every stack has a free coordinate group system.*

In view of this result, we can use some known constructions of Algebraic Topology in the algebraic theory of stacks.

6. – Given a distributive lattice L with smallest element 0, we can form the lattice of pairs $P = \{(A, B): A, B \in L, A \geqslant B\}$; $A \to (A, 0)$ defines P as a lattice extension of L. Given a stack F over L, $F(0) = 0$, the question arises whether it can be extended to a stack over P.

THEOREM 2. *If L is finite, any stack F over L can be extended to a stack over P, in such a way that all sequences*

$$F(A, C) \to F(B, C) \xrightarrow{\delta} F(A, B) \to F(A, C) \to F(B, C) \qquad (A \geqslant B \geqslant C) \tag{19}$$

be exact. (Here $F(A, 0) = F(A)$ is the originally given stack.)

Any such extension will be called *exact extension* of F; the $F(A, B)$'s are called relative groups. Over P we have the class of *exact stacks* (F with δ as in (19)) such that all sequences (19) are exact. One exact extension is obtained by $F(A, B) = H(N_B/N_A)$, where $\{X; N_A, A \in L\}$ is a coordinate of F. With an exact extension all morphisms $F(A) \to F(B)$ of the original stack are included in an exact sequence

$$F(A, B) \to F(A) \to F(B) \to F(A, B) \to F(A)\,; \tag{20}$$

this is simply the case $C = 0$ in (19).

In case L is a ring of sets, we are particularly interested in *subtractive stacks*, i.e., exact stacks over P such that

$$\textit{if} \quad C - D = A - B\,, \qquad \textit{then} \quad F(C, D) \cong F(A, B) \tag{21}$$

SUBTRACTION AXIOM. *F is given over P, lattice of pairs of a ring of sets L, together with δ's in* (19), *is exact, and such that*

$$\textit{if} \quad C - D = A - B\,, \qquad \textit{then} \quad F(A \cup C, B \cup D) \underset{\to}{\cong} F(A, B) \tag{22}$$

holds true.

From (22) we obtain $F(C, D) \cong F(A \cup C, B \cup D) \cong F(A, B)$, thus (21) follows with a specific isomorphism this time. We agree to call a stack F over a ring of sets L subtractive, if it has an extension to pairs which satisfies the Subtraction Axiom, i.e. which is subtractive.

THEOREM 3. *A subtractive stack is additive, i.e., if F over L has an extension to P which is subtractive, then d can be introduced so that $\{F, d\}$ be an additive stack over L.*

It can be proved that an appropriate extension is also additive over P, thus we have (7), (14) for pairs.

The results above are clearly motivated by the Eilenberg-Steenrod axioms. Let us emphasize, however, that we do not consider only data for P, but we start with L and use Theorem 2 to get exact extensions, if needed. We do not have separate Exactness Axiom, but we may consider the class of exact stacks over P. The Subtraction Axiom is a combination of Exactness Axiom and Excision Axiom, but of course (22) could be considered for arbitrary stacks over P. This should indicate how the Eilenberg-Steenrod conditions can be discussed separately, and hopefully justifies our calling axioms the definitions of additive, subtractive and *simply additive stacks* (see below). Of course, many of the results obtained are known (Theorem 3 is stating that the Mayer-Vietoris addition theorem follows from the axioms, and the remark after it can be paraphrased by saying that the Mayer-Vietoris theorem holds for pairs), nevertheless, it is also correct to say that the algebraic results are more general as they also apply to sheaf cohomology.

7. – If the lattice L consists of $\{A, B, A\wedge B, A\vee B\}$, and we fix the groups $F(A)$, $F(B)$, $F(A\wedge B)$, as well as the morphisms between them, the family of all additive stacks over L is Ext (ker s, coker s) by (7). If the lattice L is generated by the indexed family α: $A_0, \ldots, A_m$, we seek to find a similar representation of the family of all additive stacks over L, and we may also ask whether this family has an algebraic structure as the group structure of Ext. We will have unified representation of stacks and additive stacks in forms of spectral sequences to be discussed below. We have no results on algebraic structure on the family of all additive stacks with prescribed $\check{C}(\alpha; F)$. This is an open problem, concerning relations between various cohomology theories, sheaf cohomology included.

7a. – Let P be the lattice of pairs of a distributive lattice L, and F an exact stack over P. Given a sequence $0 = B_{-1} \leqslant B_0 \leqslant \ldots \leqslant B_m = S$ of lattice elements, there is a spectral sequence

$$E_1 = \sum_{p=0}^{m} E_1^p, \qquad E_1^p = F(B_p, B_{p-1}) \tag{23}$$

$$E_{m+1} \cong GdF(S). \tag{24}$$

All other data of this spectral sequence can be explicitly described.

7b. – In addition to the conditions of *7a*, we suppose that F is subtractive, and that a family $\{S_i^p\}$ of lattice elements is given, such that $\cup S_i^p = B_p$, $S_i^p \neq S_j^p \subset B_{p-1}$, if $i \neq j$. Then for a term in (23) we have

$$E_1^p = \sum_i F(S_i^p, B_{p-1} \cap S_i^p). \tag{25}$$

Thus we have a more « local » E_1 term.

If the stack F is given over L only, we can extend it to pairs as in Theorem 2, and introduce the spectral sequence *7a* or *7b* for the extension. In this sense, we have a spectral sequence for any stack over a finite distributive lattice. However, this spectral sequence is not « local » in the sense (25) can be considered « local ».

8. – In some applications it is preferable to avoid the device of extension to pairs. For additive stacks another spectral sequence can be introduced. We will call this the addition spectral sequence (see [9], p. 88). It is entirely based on the diagrams (14) and does not involve relative groups. We will not describe the general case, just the case of simply additive stacks to be introduced below. This is a subclass of the class of additive stacks. The construction 2b always gives such stacks (however, this statement will not be amplified in the present paper).

Let F be an additive stack and $\{X; N_A, A \in L\}$ a coordinate group system of F. For given $A, B \in L$, we define

$$f(A \vee B)x = (Ax, Bx, 0) \qquad (x \in X,\ Ax = x + N_A,\ \text{etc.}) \tag{26}$$

which is a differential map of $(A \vee B)X$ into $AX \oplus BX \oplus CX$, $C = A \wedge B$, if we take this group with the differential

$$\delta(Ax, By, Cz) = (\delta Ax, \delta By, \delta Cz + Cy - Cx) \qquad (C = A \wedge B)\,. \tag{27}$$

We have then a diagram

$$\begin{array}{ccccc} H(CX) & \xrightarrow{d} & H((A \vee B)X) & \longrightarrow & H(AX) \oplus H(BX) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle j} \\ H(CX) & \to & H(AX \oplus BX \oplus CX) & \to & H(AX) \oplus H(BX) \end{array} \tag{28}$$

(i, j being identities). The top row here is exact by the Addition Axiom, and the bottom row is exact, being induced by an exact sequence of differential groups. The square on the right in (28) is of course commutative, but the square on the left need not be commutative. Requiring commutativity of (28) amounts to restrict $d = d(A, B)$ given with the structure of F.

AXIOM OS SIMPLE ADDITION. *The stack F over the distributive lattice L is additive, and has a coordinate group system $\{X\}$ such that* (28) *is a commutative diagram for every pair A, B in L.*

In [9] Leray generalized the Mayer-Vietoris addition theorem from two sets to $m + 1$ sets, and to sheaf cohomology (see p. 88), obtaining a spectral sequence. For simply additive stacks we have a formally identical spectral sequence below, in fact the study of this spectral sequence led us to the subject of this paper.

8*a*. – Let F be a simply additive stack over the distributive lattice L, and $\alpha\colon A_0, \ldots, A_m$ a given indexed family of lattice elements. These data determine a spectral sequence

$$E_2 = H(\check{C}(\alpha; F)) \tag{29}$$

$$E_{m+1} \cong GdF(A_0 \vee \ldots \vee A_m) \tag{30}$$

where E_2 and the differentials have the usual degree properties. Consequently, we may say that all simply additive stacks over the sublattice generated by $A_0, \ldots, A_m$ and for which the groups (16) and the induced morphisms between them are fixed, are expressed by the single group (29) and by a « variable » set of differentials $d_2, \ldots, d_m$.

8*b*. – We suppose that the $F(A)$'s are Λ-vector spaces and all morphisms are Λ-linear (Λ being a field). Now top and bottom rows in (28) are isomorphic for arbitrary additive stacks (of Λ-vector spaces), however the diagram (28) still may not commute. But in this case, the addition spectral sequence (which is defined for arbitrary additive stacks, but was not described above) has good properties and (29), (30) hold true even for additive stacks which are *not* simply additive.

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