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Analytic Regularity for the Dirichlet Problem in Domains with Conic Singularities.

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dedicated to Jean Leray

0. – Introduction and main result.

In this paper we shall study the analytic regularity of the Dirichlet problem for certain (degenerate) elliptic equations of second order in a domain in \mathbf{R}^n whose boundary may present certain singularities of conic type. Our main result (Theorem 0.1) will be local and it is therefore convenient to work in a neighborhood of the origin in \mathbf{R}^n .

Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$ be an open set such that $0 \in \bar{\Omega}$. We shall always assume that Ω has a « conic » singularity at 0, or more precisely that:

(0.1) *There exists a real analytic diffeomorphism $\kappa: V_1 \rightarrow V_2$ between two, neighborhoods of the origin, such that $\kappa(0) = 0$ and $\kappa(\Omega \cap V_1) = \Omega_0 \cap V_2$, where Ω_0 is an open cone in \mathbf{R}^n .*

After composition with a linear transformation, we can get a transformation as in (0.1) which satisfies $d\kappa(0) = I$. The corresponding Ω_0 will then be independent of the choice of κ (satisfying $d\kappa(0) = I$) and we denote it by $\mathfrak{C}_0(\Omega)$; the « tangent cone » of Ω at 0.

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Let $P(x, D)$ be a second order differential operator with analytic coefficients defined near 0. We assume that P has the form

$$(0.2) \quad P(x, D) = P_0(x, D) + P_1(x, D) = \sum_{|\alpha| \leq 2} \alpha_\alpha(x) D^\alpha + \sum_{|\alpha| \leq 2} a'_\alpha(x) D^\alpha,$$

where $a_\alpha(x)$ and $a'_\alpha(x)$ satisfy the following condition:

(0.3) *There exists an integer $K \geq -2$ such that $a_\alpha(x)$ are homogeneous polynomials of degree $|\alpha| + K$ and $a'_\alpha(x)$ vanish at least to the order $|\alpha| + K + 1$ at the origin (if $|\alpha| + K + 1 > 0$).*

Here the notations are the usual ones; $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \sum \alpha_j$, $D^\alpha = (i^{-1} \partial / \partial x_1)^{\alpha_1} \dots (i^{-1} \partial / \partial x_n)^{\alpha_n}$. Let $p_0(x, \xi)$ be the principal symbol of P_0 . We assume that

(0.4) *P_0 is elliptic on $\overline{\mathfrak{C}_0(\Omega)} \setminus \{0\}$. When $n = 2$ we also assume that P_0 is properly elliptic, or more precisely that $\text{var arg } p_0(x, \cdot) = 0$ for every closed curve γ in $T_x^*(\mathbf{R}^2) \setminus \{0\}$, $x \in \overline{\mathfrak{C}_0(\Omega)} \setminus \{0\}$.*

Notice that the conditions (0.2), (0.3) and (0.4) are invariant under analytic diffeomorphisms preserving 0. When $d\kappa(0) = I$ then P_0 will not change under such a diffeomorphism. We also notice that (0.2), (0.3) and (0.4) are satisfied when P is (properly) elliptic at the point 0; in fact, we choose $K = -2$ and then $P_0(x, D) = p(0, D)$, where p is the principal symbol of P .

Introducing polar coordinates, we can write $\mathfrak{C}_0(\Omega) = \{(r, \theta); r > 0, \theta \in \omega\}$, where $\omega \subset S^{n-1}$ is an open subset of the unit sphere. The operator P_0 takes the form

$$(0.5) \quad P_0 = r^K Q_0(\theta, D_\theta, rD_r) = r^K \sum_{j=0}^2 A_j(\theta, D_\theta) \left(r \frac{\partial}{\partial r} \right)^{2-j}$$

where A_j is of order $\leq j$. Let $a_j(\theta, \eta)$ be the principal symbol of order j of A_j and put

$$(0.6) \quad a(\theta, \eta, z) = \sum_{j=0}^2 a_j(\theta, \eta) z^{2-j}.$$

Notice that $q_0(\theta, \eta, \tau) = a(\theta, \eta, i\tau)$ is the principal symbol of the operator Q_0 at the point $\theta \in S^{n-1}$.

Put

$$(0.7) \quad \Gamma = \{z \in \mathbf{C}; a(\theta, \eta, z) = 0 \text{ for some } (\theta, \eta) \in T^*S^{n-1} \setminus \{0\}, \theta \in \bar{\omega}\}.$$

The ellipticity of P_0 in $\overline{\mathfrak{C}_0(\Omega)} \setminus \{0\}$ implies that $A_2(\theta, D_\theta)$ is elliptic and $A_0(\theta) = a_0(\theta)$ is non-vanishing on $\bar{\omega}$. Then clearly Γ is a closed cone in \mathbf{C} , such that

$$(0.8) \quad \Gamma \cap i\mathbf{R} = \emptyset.$$

Put $\Gamma_+ = \{z \in \Gamma; \operatorname{Re} z > 0\}$ and let $\hat{\Gamma}_+$ be the convex hull of $\Gamma_+ \cup \mathbf{R}_+$. Notice that Γ is not invariant under arbitrary diffeomorphisms, preserving the origin (except those with $d\kappa = I$ at 0). We introduce the following condition:

(H) *After a suitable local analytic diffeomorphism, preserving the origin, the angle of $\hat{\Gamma}_+$ is strictly smaller than π/n .*

If P is elliptic at 0 with real principal symbol at that point, then (H) is satisfied. In fact, we can make a linear change of variables so that P_0 becomes $\pm\Delta$, and then $\Gamma_+ = \hat{\Gamma}_+ = \mathbf{R}_+$. More generally, if $\varphi(x)$ is analytic near the origin and satisfies

$$(0.9) \quad C^{-1} < |\varphi(x)|/|x|^K < C, \quad x \in \Omega \setminus \{0\}, \quad |x| < C^{-1},$$

for some constant C and a non-negative integer K , and $p(x, \xi) = \varphi(x)r(\xi)$, where r is real, elliptic and homogeneous of degree 2, then (H) is satisfied.

Let $C^\infty(\bar{\Omega}) = \{u|_{\bar{\Omega}}; u \in C^\infty(\mathbf{R}^n)\}$. We can now state the main result of this paper:

THEOREM 0.1. *Let Ω satisfy (0.1) and let P satisfy (0.2), (0.3), (0.4) and (H). If $u \in C^\infty(\bar{\Omega})$ and Pu and $u|_{\partial\Omega}$ have analytic extensions to a full neighborhood of the origin, then the same is true for u .*

We say that an open set $\Omega \subset \mathbf{R}^n$ is an analytic polyhedron, if in a neighborhood of each point $x_0 \in \partial\Omega$ we can define Ω by k inequalities: $\varphi_1(x) > 0, \dots, \varphi_k(x) > 0$, where $\varphi_1(x_0) = \dots = \varphi_k(x_0) = 0$ and $\varphi_1, \dots, \varphi_k$ are real, analytic and have independent differentials. (The number k will depend on x_0 , as well as the functions φ_j). In a neighborhood V of x_0 we can then choose an analytic diffeomorphism $\kappa: V \ni x \mapsto (y_1, \dots, y_n) \in \kappa(V)$ such that $y_j = \varphi_j(x)$ for $1 \leq j \leq k$. Then $\kappa(V_1 \cap \Omega) = \kappa(V_1) \cap \Omega_0$, where Ω_0 is the cone $y_1 > 0, \dots, y_k > 0$. Disregarding the fact that x_0 is not necessarily 0 we conclude that the condition (0.1) is satisfied. From our preceding remarks we then deduce the following consequence of Theorem 0.1.

COROLLARY 0.2. *Let $\Omega \subset \mathbf{R}^n$ be an analytic polyhedron and let $P(x, D)$ be a second order elliptic operator with real principal symbol and analytic coefficients, defined in a neighborhood of $\bar{\Omega}$. If $u \in C^\infty(\bar{\Omega})$ and Pu and $u|_{\partial\Omega}$ have analytic extensions to neighborhoods of $\bar{\Omega}$ and $\partial\Omega$ respectively, then u has an analytic extension to a neighborhood of $\bar{\Omega}$.*

It is obvious from Theorem 0.1 that $\partial\Omega$ can also have some isolated singularities. As an example, let

$$\Omega = \left\{ x \in \mathbf{R}^n \mid x_n^2 > \sum_{i=1}^{n-1} x_i^2, 0 < x_n < 1 \right\}.$$

It is clear that Ω satisfies a condition similar to (0.1) at each point of $\bar{\Omega}$, and hence the conclusion of Corollary (0.2) holds in this case.

The main idea of the proof of Theorem 0.1 is similar to the one in [4]. We work with the Mellin transform in the radial direction and apply certain estimates for a holomorphic 1-parameter family of elliptic operators on a subset of the unit sphere, together with the Phragmén-Lindelöf principle. Notice that in the case when $\Omega \cup \{0\}$ is a full neighborhood of 0, then Theorem 0.1 is essentially contained in Theorem 3 in [4].

Several authors have treated boundary problems in domains with conic singularities and some of them have also used the Mellin transform, which is very natural in this context. We refer to Kondratiev [7] and Grisvard [6], where further references are given. To our knowledge our analyticity results obtained in Theorem 0.1 and Corollary 0.2 are new even for the Laplacian, (of course when the boundary is analytic such results are well known [9]).

Finally we would like to mention that it is also possible to obtain non-regularity results in C^∞ if the boundary has only isolated conic singularities. We will treat these questions in a separate paper.

1. - Function spaces.

We recall here some more or less well known facts. Let $\omega \subset S^{n-1}$, be an open subset. (The following discussion is also valid when $\omega \subset \mathbf{R}^{n-1}$ is bounded and open.) We denote by $\mathcal{H}_0^1(\omega)$ the closure of $C_0^\infty(\omega)$ for the induced topology of the Sobolev space $H^1(S^{n-1})$. We denote by $\mathcal{H}^{-1} = \mathcal{H}^{-1}(\omega)$ the dual space and we think of $\mathcal{H}^{-1}(\omega)$ as a subspace of $\mathcal{D}'(\omega)$, where the duality is given by the usual (extension of the) L^2 -product:

$$(1.1) \quad \langle u, v \rangle = \int uv \, d\sigma, \quad u \in \mathcal{H}_0^1(\omega), \quad v \in \mathcal{H}^{-1}(\omega).$$

Here $d\sigma$ is the Euclidean volume density on S^{n-1} . We write $\mathcal{H}^0(\omega)$ for $L^2(\omega)$

and use the notation

$$(u, v)_0 = (u, v)_{\mathcal{H}^0(\omega)} = \langle u, \bar{v} \rangle.$$

We have the compact and dense inclusions:

$$C^\infty(\omega) \subset \mathcal{H}_0^1(\omega) \subset \mathcal{H}^0(\omega) \subset \mathcal{H}^{-1}(\omega) \subset \mathcal{D}'(\omega)$$

and if A is a differential operator of order $j = 1$ or 2 with coefficients in $C^\infty(\bar{\omega})$, then A is continuous $\mathcal{H}_0^1 \rightarrow \mathcal{H}^{1-j}$. (Here Au is computed as a distribution in ω).

Now $\mathcal{H}_0^1(\omega)$ is a Hilbert space with the scalar product

$$(1.2) \quad (u, v)_{\mathcal{H}_0^1(\omega)} = (u, v)_1 = (\text{grad } u, \text{grad } v)_0 + (u, v)_0$$

and it is therefore clear that we have a surjective isomorphism $\mathcal{H}_0^1 \ni v \rightarrow w \in \mathcal{H}^{-1}$, given by

$$(1.3) \quad (u, v)_1 = (u, w)_0, \quad u \in \mathcal{H}_0^1(\omega).$$

Clearly this isomorphism is $1 - \Delta$ if Δ denotes the Laplacian on S^{n-1} . On $\mathcal{H}^{-1}(\omega)$ we choose the scalar product which makes $1 - \Delta$ unitary from \mathcal{H}_0^1 to \mathcal{H}^{-1} . If we consider $(1 - \Delta)^{-1}$ as a compact operator in \mathcal{H}^{-1} , it is clear that $(1 - \Delta)^{-1}$ as a compact operator in \mathcal{H}^{-1} , it is clear that $(1 - \Delta)^{-1}$ is self adjoint. In fact, if $u, v \in \mathcal{H}^{-1}$ we have $u = (1 - \Delta)u'$, $v = (1 - \Delta)v'$, $u', v' \in \mathcal{H}_0^1$ and

$$((1 - \Delta)^{-1}u, v)_{-1} = ((1 - \Delta)^{-1}u', v')_1 = (u', v')_0 = \dots = (u, (1 - \Delta)^{-1}v)_{-1}.$$

Let $\lambda_0 \geq \lambda_1 \geq \dots \geq 0$ be the eigenvalues of $(1 - \Delta)^{-1}$. We need the following more or less well known rough estimate of the λ_k .

LEMMA 1.1. *There exists a constant $C > 0$ such that*

$$(1.4) \quad C^{-1} \leq \lambda_k k^{2/(n-1)} \leq C.$$

PROOF. We recall the well known argument based on the mini-max principle. Let $\lambda_k(\omega)$ be the k -th element in the decreasing sequence of eigenvalues of $(1 - \Delta)^{-1}$ on $\mathcal{H}^{-1}(\omega)$. Let $\omega_1 \subset S^{n-1}$, be an open set containing ω . Then $\mathcal{H}_0^1(\omega)$ is a closed subspace of $\mathcal{H}_0^1(\omega_1)$. If H is a Hilbert space we denote by $T_k(H)$ the set of closed subspaces of codimension $\leq k$. We have the

mini-max formula:

$$\lambda_k(\omega_1) = \inf_{L \in T_k(\mathcal{H}^{-1}(\omega_1))} \sup_{u \in L} \frac{((1 - \Delta)^{-1}u, u)_{-1}}{\|u\|_{-1}^2}.$$

Representing $u = (1 - \Delta)u'$, $u' \in \mathcal{H}_0^1(\omega_1)$, we have

$$((1 - \Delta)^{-1}u, u)_{-1} = \|u'\|_0^2, \quad \|u\|_{-1}^2 = \|u'\|_1^2,$$

so the mini-max formula becomes

$$\lambda_k(\omega_1) = \inf_{L \in T_k(\mathcal{H}_0^1(\omega_1))} \sup_{u \in L} \|u\|_0^2 / \|u\|_1^2.$$

If $L \in T_k(\mathcal{H}_0^1(\omega_1))$ we have $L \cap \mathcal{H}_0^1(\omega) \in T_k(\mathcal{H}_0^1(\omega))$ so we get

$$\lambda_k(\omega_1) \geq \inf_{L \in T_k(\mathcal{H}_0^1(\omega_1))} \sup_{u \in L \cap \mathcal{H}_0^1(\omega)} \|u\|_0^2 / \|u\|_1^2 \geq \inf_{L \in T_k(\mathcal{H}_0^1(\omega))} \sup_{u \in L} \|u\|_0^2 / \|u\|_1^2 = \lambda_k(\omega).$$

Now choose $\omega_1 = S^{n-1}$ and $\omega_2 \subset \omega$ such that $\bar{\omega}_2$ is diffeomorphic to the unit cube in \mathbf{R}^{n-1} . Then it is well known that $\lambda_k(\omega_1)$ satisfy (1.4), and another application of the minimax principle, comparing the eigenvalues of Δ with the eigenvalues of the Laplacian on the unit cube (which can be calculated explicitly), shows that $\lambda_k(\omega_2)$ also satisfy (1.4). Then $\lambda_k(\omega)$ also satisfy (1.4) since $\lambda_k(\omega_2) \leq \lambda_k(\omega) \leq \lambda_k(\omega_1)$.

We now recall from Dunford-Schwartz [5], that a compact operator A in some Hilbert space is said to be of class C_p , $1 < p < \infty$ if the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots$ of $(A * A)^\dagger$ satisfy $\sum \lambda_j^p < \infty$. The operators of class C_p form a stable set under composition to the right or to the left by bounded operators.

LEMMA 1.2. *If $\mathcal{B}: \mathcal{H}^0(\omega) \times \mathcal{H}^{-1}(\omega) \rightarrow \mathcal{H}_0^1(\omega) \times \mathcal{H}^0(\omega)$ is a bounded operator, then \mathcal{B} is of class C_p for all $p > n - 1$, when considered as an operator in $\mathcal{H}^0(\omega) \times \mathcal{H}^{-1}(\omega)$.*

PROOF. The operator

$$\mathcal{D} = \begin{pmatrix} 0 & (1 - \Delta)^{-1} \\ 1 & 0 \end{pmatrix}$$

is an isomorphism from $\mathcal{H}^0 \times \mathcal{H}^{-1}$ onto $\mathcal{H}_0^1 \times \mathcal{H}^0$, so \mathcal{B} can be written as $\mathcal{D}\mathcal{C}$, where \mathcal{C} is bounded in $\mathcal{H}^0 \times \mathcal{H}^{-1}$. It is therefore enough to show that \mathcal{D} is of class C_p for $p > n - 1$. Now $(1 - \Delta)^{-1}: \mathcal{H}^{-1} \rightarrow \mathcal{H}^0$ and $1: \mathcal{H}^0 \rightarrow \mathcal{H}^{-1}$ are adjoints of each other, so \mathcal{D} is selfadjoint as an operator $\mathcal{H}^0 \times \mathcal{H}^{-1} \rightarrow \mathcal{H}^0 \times \mathcal{H}^{-1}$.

Let $(u_0, u_1) \in \mathcal{H}^0 \times \mathcal{H}^{-1}$ be an eigenvector with eigenvalue λ . Then we obtain: $\lambda u_0 = (1 - \Delta)^{-1} u_1$, $\lambda u_1 = u_0$ or equivalently: $(1 - \Delta)^{-1} u_1 = \lambda^2 u_1$, $u_0 = \lambda u_1$. The eigenvalues of \mathfrak{D} are therefore $\pm \sqrt{\lambda_k(\omega)}$, (where $\lambda_k = \lambda_k(\omega)$ are introduced above) and Lemma 1.2 follows from Lemma 1.1.

We will also need the following lemma:

LEMMA 1.3. *If $u \in C^\infty(\bar{\omega})$, $u|_{\partial\omega} = 0$, then $u \in \mathcal{H}_0^1(\omega)$.*

PROOF. For $\varepsilon > 0$, we put $K_\varepsilon = \{x \in \omega; d(x, \mathbf{C}\omega) \geq \varepsilon\}$, where d is some distance in S^{n-1} . It is well known that there exist functions $\chi_\varepsilon \in C_0^\infty(\omega)$, $0 < \varepsilon < 1$, such that $0 \leq \chi_\varepsilon \leq 1$, $\chi_\varepsilon = 1$ on K_ε , and $|\text{grad } \chi_\varepsilon| \leq C\varepsilon^{-1}$, where C is independent of ε . Put $u_\varepsilon = \chi_\varepsilon u \in C_0^\infty(\omega)$. Then $u_\varepsilon \rightarrow u$ in $L^2(\omega)$ when $\varepsilon \rightarrow 0$. Moreover

$$\text{grad } u_\varepsilon - \text{grad } u = u \text{ grad } \chi_\varepsilon + (\chi_\varepsilon - 1) \text{ grad } u.$$

Now the volume of $\omega \setminus K_\varepsilon$ tends to 0 with ε , and $u \text{ grad } \chi_\varepsilon$ and $(\chi_\varepsilon - 1) \text{ grad } u$ are uniformly bounded with support in $\omega \setminus K_\varepsilon$. Hence $\text{grad } u_\varepsilon \rightarrow \text{grad } u$ in $L^2(\omega)$ and it follows that $u \in \mathcal{H}_0^1(\omega)$.

2. - An elliptic operator, depending on a complex parameter.

Let $\omega \subset S^{n-1}$ be open and let $A_j(\theta, D_\theta)$, $j = 0, 1, 2$ be differential operators of order $\leq j$ with coefficients in $C^\infty(\bar{\omega})$. We shall study the operators

$$(2.1) \quad A_z = A(\theta, D_\theta, z) = \sum_{j=0}^2 A_j(\theta, D_\theta) z^{2-j}, \quad z \in \mathbf{C}.$$

Let $a_j(\theta, \eta) \in C^\infty(T^*S^{n-1}|_{\bar{\omega}})$ be the (j -th order) principal symbol of A_j and put

$$(2.2) \quad a(\theta, \eta, \tau) = a_2(\theta, \eta) + a_1(\theta, \eta)\tau + a_0(\theta)\tau^2,$$

so that $a(\theta, \eta, \tau)$ is the principal symbol of the operator $A(\theta, D_\theta, D_t)$. We assume that

$$(2.3) \quad A\left(\theta, D_\theta, \frac{\partial}{\partial t}\right) \text{ is elliptic on } \bar{\omega} \times \mathbf{R},$$

and even properly elliptic when $n = 2$.

Then $A_2(\theta, D_\theta)$ is elliptic and $A_0(\theta) = a_0(\theta)$ is non-vanishing. We put

$$(2.4) \quad \Gamma = \{z \in \mathbf{C}; a(\theta, \eta, z) = 0 \text{ for some } (\theta, \eta) \in T^*S^{n-1} \setminus 0, \theta \in \bar{\omega}\}.$$

Then Γ is a closed conic set in $\mathbf{C} \setminus \{0\}$ and (2.3) implies that

$$(2.5) \quad \Gamma \cap i\mathbf{R} = \emptyset.$$

We recall the following well known lemma. (For a proof, see [10, Lemma 3.1].)

LEMMA 2.1. *Let q be a complex valued quadratic form on \mathbf{R}^n such that $q(\xi) \neq 0$ for $\xi \neq 0$. For $n = 2$ we also assume that $\text{var.}_{\gamma} \arg. q = 0$ for every closed curve γ in $\mathbf{R}^2 \setminus \{0\}$. Then*

$$\{q(\xi) : \xi \in \mathbf{R}^n \setminus \{0\}\} = \{z \in \mathbf{C} \setminus \{0\} : \alpha_1 \leq \arg z \leq \alpha_2\},$$

where $0 \leq \alpha_2 - \alpha_1 < \pi$.

In order to apply Gårding's inequality we need

LEMMA 2.2. *Let $\Sigma \subset \mathbf{C} \setminus \{0\}$ be a closed cone such that $\Sigma \cap \Gamma = \emptyset$. When $n = 2$ we also assume that $\Sigma \cap i\mathbf{R} \neq \emptyset$ and that Σ is connected. Then there exists a function $\varrho \in C^\infty(\bar{\omega} \times (\Sigma \cap S^1))$ such that $\varrho(\theta, \zeta)A(\theta, D_\theta, \zeta D_\zeta)$ is strongly elliptic on $\bar{\omega} \times \mathbf{R}_t$, when $\zeta \in \Sigma \cap S^1$. In other words: $\text{Re } \varrho(\theta, \zeta) \cdot a(\theta, \eta, \zeta \tau) > 0$ when, $(\eta, \tau) \neq (0, 0)$ is real, $\theta \in \bar{\omega}$, $\zeta \in \Sigma \cap S^1$.*

PROOF. Clearly $A(\theta, D_\theta, \zeta D_\zeta)$ is elliptic at each point $(\theta, \zeta) \in \bar{\omega} \times (\Sigma \cap S^1)$ and also properly elliptic when $n = 2$, in view of the assumption that Σ is connected and $\Sigma \cap i\mathbf{R} \neq \emptyset$. (The assumption (2.3) implies that $A(\theta, D_\theta, \zeta D_\zeta)$ is properly elliptic for $\zeta = \pm i$.) Then Lemma 2.1 shows that for every $\alpha = (\theta_\alpha, \zeta_\alpha) \in \bar{\omega} \times (\Sigma \cap S^1)$, there exists $\varrho_\alpha \in \mathbf{C}$ such that

$$(2.6) \quad \text{Re } \varrho_\alpha a(\theta_\alpha, \eta, \zeta_\alpha \tau) > 0, \quad (\eta, \tau) \neq (0, 0).$$

Then by the continuity and the homogeneity, there exists a neighborhood $U_\alpha \subset \bar{\omega} \times (\Sigma \cap S^1)$ of α such that

$$(2.7) \quad \text{Re } \varrho_\alpha a(\theta, \eta, \zeta \tau) > 0, \quad (\eta, \tau) \neq (0, 0), (\theta, \zeta) \in U_\alpha.$$

We can now pick a finite partition of unity:

$$1 = \sum_1^N \chi_j, \quad \chi_j \in C^\infty(\bar{\omega} \times (\Sigma \cap S^1)), \quad \text{supp } \chi_j \subset U_{\alpha_j}, \quad 0 \leq \chi_j \leq 1.$$

Then the lemma follows if we put

$$\varrho(\theta, \zeta) = \sum_1^N \varrho_{\alpha_j} \chi_j(\theta, \zeta).$$

Now let Σ and ϱ be as in the lemma. We have then Gårding's inequality:

$$(2.8) \quad \|f\|_1^2 < C_1 \operatorname{Re}(\varrho(\theta, \zeta)A(\theta, D_\theta, \zeta D_t)f, f)_0 + C_2 \|f\|_0^2, \\ f \in \mathcal{H}_0^1(\omega \times]-1, 1[), \quad \zeta \in \Sigma \cap S^1$$

where the positive constants C_1 and C_2 are independent of f and of ζ . From (2.8) we see that there is a constant C , independent of f and of ζ such that

$$(2.9) \quad \|f\|_1 < C(\|A(\theta, D_\theta, \zeta D_t)f\|_{-1} + \|f\|_0), \\ f \in \mathcal{H}_0^1(\omega \times]-1, 1[), \quad \zeta \in \Sigma \cap S^1,$$

if Σ satisfies the assumptions of Lemma 2.2. Using this inequality and an idea of Agmon [1] we shall now prove

PROPOSITION 2.3. *Assume that (2.3) holds and let $\Sigma \subset \mathbf{C} \setminus \{0\}$ be a closed cone such that $\Sigma \cap \Gamma = \emptyset$. When $n = 2$ we also assume that Σ is connected and that $\Sigma \cap i\mathbf{R} \neq \emptyset$. Then there exists a constant C such that $A_z = A(\theta, D_\theta, z)$ is an isomorphism from $\mathcal{H}_0^1(\omega)$ onto $\mathcal{H}^{-1}(\omega)$ and*

$$(2.10) \quad \|u\|_1 + |z|\|u\|_0 + |z|^2\|u\|_{-1} < C\|A_z u\|_{-1}, \quad u \in \mathcal{H}_0^1(\omega),$$

when $z \in \Sigma$, $|z| > C$.

REMARK 2.4. In the next section it will be convenient to work with the norm depending on z :

$$\|u\|_1^z = \|u\|_1 + |z|\|u\|_0 + |z|^2\|u\|_{-1}$$

on \mathcal{H}_0^1 . Then the inequality (2.10) takes the form

$$(2.10') \quad \|u\|_1^z < C\|A_z u\|_{-1}.$$

PROOF OF PROPOSITION 2.3. Choose $\chi \in C_0(]-1, 1[)$ such that $\|\chi\|_0 = 1$. If $u \in \mathcal{H}_0^1(\omega)$, $z \in \mathbf{C}$, we put

$$f_z(\theta, t) = u(\theta)\chi(t) \exp [i|z|t] \in \mathcal{H}_0^1(\omega \times]-1, 1[).$$

Notice that

$$(2.11) \quad \|f_z\|_1^2 = \|u\|_1^2 \|\chi(t) \exp [i|z|t]\|_0^2 + \|u\|_0^2 \|D_t \chi(t) \exp [i|z|t]\|_0^2.$$

Since $\|\chi\|_0 = 1$, we have $\|D_t \chi(t) \exp [i|z|t]\|_0 = |z| + \mathcal{O}(1)$ and hence

$$(2.12) \quad \|f_z\|_1^2 = \|u\|_1^2 + (|z|^2 + \mathcal{O}(1)) \|u\|_0^2.$$

We also have $\|f_z\|_0 = \|u\|_0$. Now let $z \in \Sigma$ and put $\zeta = z/|z|$. An easy computation gives

$$(2.13) \quad A(\theta, D_\theta, \zeta D_t) f_z = \chi(t) \exp [i|z|t] A(\theta, D_\theta, z) u + \zeta A_1(\theta, D_\theta) u(\theta) \cdot \\ \cdot (D_t \chi) \exp [i|z|t] + \zeta^2 A_0(\theta) u(\theta) (D_t^2 \chi(t) + 2|z| D_t \chi(t)) \exp [i|z|t].$$

The inclusion $\mathcal{H}_0^1(\omega \times]-1, 1[) \subset L^2(]-1, 1[; \mathcal{H}_0^1(\omega))$ implies by duality that $L^2(]-1, 1[; \mathcal{H}^{-1}(\omega)) \subset \mathcal{H}^{-1}(\omega \times]-1, 1[)$, so if we use the corresponding inequality for the norms we get from (2.13):

$$(2.14) \quad \|A(\theta, D_\theta, \zeta D_t) f_z\|_{-1} \leq \|A(\theta, D_\theta, \zeta D_t) f_z\|_{L^2(]-1, 1[; \mathcal{H}^{-1}(\omega))} \\ \leq \|A_z u\|_{-1} + C(\|u\|_0 + |z| \cdot \|u\|_{-1}), \quad z \in \Sigma, u \in \mathcal{H}_0^1(\omega)$$

where, here and in the following, C denotes some positive constant, independent of u and z . Combining (2.9), (2.12) and (2.14) we get (with a new constant C):

$$(2.15) \quad \|u\|_1 + |z| \|u\|_0 \leq C(\|A_z u\|_{-1} + \|u\|_0 + |z| \|u\|_{-1}), \\ u \in \mathcal{H}_0^1(\omega), z \in \Sigma, |z| \geq C.$$

Using that $A_0(\theta)$ is non-vanishing and then applying (2.15) we get

$$(2.16) \quad |z|^2 \|u\|_{-1} \leq C'(\|A_z u\|_{-1} + |z| \cdot \|u\|_0 + \|u\|_1) \\ \leq C(\|A_z u\|_{-1} + \|u\|_0 + |z| \|u\|_{-1}), \\ u \in \mathcal{H}_0^1(\omega), z \in \Sigma, |z| \geq C.$$

Adding (2.15) and (2.16) gives:

$$(2.17) \quad \|u\|_1 + |z| \cdot \|u\|_0 + |z|^2 \|u\|_{-1} \\ \leq C(\|A_z u\|_{-1} + \|u\|_0 + |z| \|u\|_{-1}), \\ u \in \mathcal{H}_0^1(\omega), z \in \Sigma, |z| \geq C.$$

Choosing $|z|$ large enough we can absorb the last two terms and we obtain (2.10). In particular $A_z: \mathcal{H}_0^1 \rightarrow \mathcal{H}^{-1}$ is injective with closed image when $z \in \Sigma, |z| \geq C$. What we have proved so far, is also valid for the adjoint

$$A_z^* = A_2^*(\theta, D_\theta) + A_1^*(\theta, D_\theta) \bar{z} + A_0^*(\theta) \bar{z}^2,$$

when $z \in \Sigma$, $|z| \geq C'$, so A_z is also surjective and this completes the proof of Proposition 2.3.

We also need some control over A_z near Γ . Put

$$B_j(\theta, D_\theta) = - (A_0(\theta))^{-1} A_j(\theta, D_\theta), \quad j = 1, 2$$

and introduce the operator $\mathcal{A}: \mathcal{H}_0^1 \times \mathcal{H}^0 \rightarrow \mathcal{H}^0 \times \mathcal{H}^{-1}$ given by the matrix $\mathcal{A} = \begin{pmatrix} 0 & 1 \\ B_2 & B_1 \end{pmatrix}$. The equation $A_z u = v$ is then equivalent to $(z - \mathcal{A}) \cdot U = V$, where $U = (u, zu)$, $V = (0, A_0(\theta)^{-1}v)$ and A_z is an isomorphism from \mathcal{H}_0^1 onto \mathcal{H}^{-1} if and only if $(z - \mathcal{A})$ is an isomorphism from $\mathcal{H}_0^1 \times \mathcal{H}^0$ onto $\mathcal{H}^0 \times \mathcal{H}^{-1}$. Proposition 2.3 shows that there are values $z \in \mathbf{C}$ for which $(z - \mathcal{A})$ is an isomorphism and without any loss of generality we shall assume that 0 is one of these values. Then $T = \mathcal{A}^{-1}$ is compact as an operator in $\mathcal{H}^0 \times \mathcal{H}^{-1}$ and Lemma 1.2 shows that T is of class C_p for all $p > n - 1$. At this point we shall apply Proposition (II.1) of [4], which is an easy consequence of general results in the theory of C_p -operators (see Dunford-Schwartz [5]). There exists $x_0 \in]0, 1[$ such that if $D_j = \{z \in \mathbf{C}: \operatorname{Re} z = x_0 + j\}$, $j \in \mathbf{Z}$, then $(I - zT)^{-1}$ exists for $z \in D_j$ and satisfies

$$(2.18) \quad \|(I - zT)^{-1}\| \leq C_\varepsilon \exp [|z|^{n-1+\varepsilon}], \quad z \in D_j,$$

for all $\varepsilon > 0$. The norm is here the operator norm in $\mathcal{H}^0 \times \mathcal{H}^{-1}$. Since $(\mathcal{A} - z) = (I - zT)\mathcal{A}$, we conclude that $(\mathcal{A} - z)$ is invertible for $z \in D_j$ and that

$$(2.19) \quad \|(z - \mathcal{A})^{-1}\| \leq C_\varepsilon \exp [|z|^{n-1+\varepsilon}], \quad z \in D_j,$$

where C_ε is a new constant, and the norm is the operator norm: $\mathcal{H}^0 \times \mathcal{H}^{-1} \rightarrow \mathcal{H}_0^1 \times \mathcal{H}^0$. Passing back to scalar operators, we see that $A_z: \mathcal{H}_0^1 \rightarrow \mathcal{H}^{-1}$ is invertible for $z \in D_j$ and that

$$(2.20) \quad \|u\|_1^z \leq C_\varepsilon \exp [|z|^{n-1+\varepsilon}] \|A_z u\|_{-1}, \quad u \in \mathcal{H}_0^1, z \in D_j, \varepsilon > 0.$$

3. - End of the proof.

From now on the proof is very similar to the proof in [4] so we shall not repeat all the details. Let Ω and P be as in the introduction, satisfying all the assumptions there. After an analytic diffeomorphism we may assume that $\Omega = \mathfrak{C}_0(\Omega) = \{(r, \theta); r > 0, \theta \in \omega\}$ is conic and that the angle of $\hat{\Gamma}_+$

is strictly smaller than π/n . In polar coordinates P will take the form

$$(3.1) \quad P(x, D) = r^\kappa \left[\sum_{j=0}^2 A_j(\theta, D_\theta) \left(r \frac{\partial}{\partial r} \right)^{2-j} + \sum_{k=1}^\infty \sum_{j=0}^2 r^k A_{j,k}(\theta, D_\theta) \left(r \frac{\partial}{\partial r} \right)^{2-j} \right].$$

C.f. formula (4.8) in [4]. The operators A_j are here the same as in the introduction, the operators $A_{j,k}$ are of order $\leq j$ and the infinite series converges uniformly with all its derivatives in a neighborhood of the origin. In fact, there exists a constant M , such that for every choice of local coordinates $\theta_1, \dots, \theta_{n-1}$ on S^{n-1} and every $N > 0$, there exists a constant C_N such that every derivative of order $\leq N$ of any coefficient of $A_{j,k}$ can be estimated by $C_N \cdot M^k$. This implies that there is a constant C such that

$$(3.2) \quad \|A_{2,k}\|_{1,-1} + \|A_{1,k}\|_{0,-1} + \|A_{0,k}\|_{-1,-1} \leq CM^k,$$

if $\| \cdot \|_{1,-1} (\| \cdot \|_{0,-1}, \| \cdot \|_{-1,-1})$ is the operator norm from $\mathcal{H}_0^1 (\mathcal{H}^0, \mathcal{H}^{-1})$ into \mathcal{H}^{-1} . After a change of variables $(r, \theta) \mapsto (\lambda r, \theta)$, $\lambda \geq 1$, the operators $A_{j,k}$ will be replaced by $\lambda^{-k} A_{j,k}$ so we may assume that M in (3.2) is as small as we like, although C will remain unchanged. Notice that the operators A_j remain unchanged and that the unit ball with respect to the new coordinates will have radius $1/\lambda$ in the old coordinates.

Let B be the closed unit ball in \mathbf{R}^n and let $u \in C^\infty(\bar{\Omega} \cap B)$. We introduce the Mellin transform

$$\tilde{u}(z, \theta) = \int_0^1 r^{-z-1} u(r\theta) dr, \quad \text{Re } z < 0.$$

Then (c.f. [4]) $\tilde{u}(z, \theta)$ extends to a meromorphic function in \mathbf{C} with values in $C^\infty(\bar{\omega})$ and simple poles at the points $z = 0, 1, 2, \dots$. These poles are the only ones and the residue at the point $z = k$ is $-u_k(\theta)$ if $u(r, \theta) \sim \sum_{k=0}^\infty r^k u_k(\theta)$ is the Taylor expansion of u , rewritten in polar coordinates.

Taking Mellin transforms of the equation $Pu = v$, we get

$$(3.3) \quad A_z \tilde{u}(z, \theta) + \sum_{k=1}^\infty \sum_{j=0}^2 (z-k)^{2-j} A_{j,k} \tilde{u}(z-k, \theta) = \tilde{v}(z+K, \theta) + C_0(\theta) + C_1(\theta)z,$$

where C_0 and C_1 are certain linear combinations of $u(1, \theta)$ and $(\partial u / \partial r)(1, \theta)$, and $A_z = \sum_0^2 A_j(\theta, D_\theta) z^{2-j}$ satisfies all the assumptions of section 2.

LEMMA 3.1. *If $u \in C^\infty(\bar{\Omega} \cap B)$ vanishes to infinite order at 0 and satisfies $Pu = 0$, $u|_{\partial\Omega} = 0$, then $u = 0$.*

PROOF. We get from (3.3) that

$$(3.4) \quad A_z \tilde{u}(z, \theta) = C_0(\theta) + C_1(\theta)z - \sum_{k=1}^{\infty} \sum_{j=0}^2 (z-k)^{2-j} A_{j,k} \tilde{u}(z-k, \theta).$$

Moreover \tilde{u} is now an entire function with values in $\mathcal{H}_0^1(\omega)$ in view of Lemma 1.3. From (3.2) it follows that

$$(3.5) \quad \|(z-k)^{2-j} A_{j,k} \tilde{u}(z-k, \theta)\|_{-1} \leq C \cdot M^k \|\tilde{u}(z-k, \theta)\|_1^{z-k}.$$

If $\Sigma \subset \mathbf{C} \setminus \{0\}$ is a closed connected cone satisfying $\Sigma \cap \Gamma = \emptyset$, $\Sigma \cap \hat{\Gamma}_+ = \emptyset$, $\Sigma \cap i\mathbf{R} \neq \emptyset$, we deduce from (3.4), (3.5) and Proposition 2.3, that

$$(3.6) \quad \|\tilde{u}(z, \theta)\|_1^z \leq C(1 + |z| + \sum_{k=1}^{\infty} M^k \|\tilde{u}(z-k, \theta)\|_1^{z-k}), \quad z \in \Sigma.$$

When $\operatorname{Re} z < -\frac{1}{2}$ it is clear that $\|\tilde{u}(z, \theta)\|_1^z \leq C_1 |z|^2$ and after a change of variables $(r, \theta) \mapsto (\lambda r, \theta)$, $\lambda \geq 1$, we may assume that M is as small as we like, without increasing C and C_1 . Working in the domains $\{z \in \Sigma; \operatorname{Re} z < -\frac{1}{2} + k\}$, $k = 0, 1, 2, 3, \dots$ and using induction over k , we see that there exists a constant C such that

$$(3.7) \quad \|\tilde{u}(z, \theta)\|_1^z \leq C(1 + |z|^2), \quad z \in \Sigma.$$

For more details we refer to [4].

Combining (3.4), (3.5) and (2.20) we obtain

$$(3.8) \quad \|\tilde{u}(z, \theta)\|_1^z \leq C_\varepsilon \exp[|z|^{n-1+\varepsilon}] (1 + |z| + \sum_{k=1}^{\infty} M^k \|\tilde{u}(z-k, \theta)\|_1^{z-k}) \quad z \in D_j,$$

where D_j , $j \in \mathbf{Z}$ are the vertical lines introduced in the end of section 2. Again, by recurrence over j (starting with $j = -1$) we obtain

$$(3.9) \quad \|\tilde{u}(z, \theta)\|_1^z \leq C_\varepsilon \exp[|z|^{n+\varepsilon}], \quad z \in D_j, \quad j \in \mathbf{Z}, \quad \varepsilon > 0,$$

where C_ε is a new constant. For more details we refer to [4].

Now $\hat{\Gamma}_+$ is defined by $\alpha_1 < \arg z < \alpha_2$, where $-\pi/2 < \alpha_1 < \alpha_2 < \pi/2$, $\alpha_2 - \alpha_1 < \pi/n$. Choosing Σ to be defined by $-\pi/2 < \arg z < \alpha_1 - \varepsilon$ or by

$\alpha_2 + \varepsilon < \arg z < \pi/2$, we see that (3.7) is valid when $\arg z = \alpha_2 + \varepsilon$ and when $\arg z = \alpha_1 - \varepsilon$. Choosing $\varepsilon > 0$ small enough, we deduce from (3.9) and the Phragmén-Lindelöf principle that (3.7) is valid in the whole complex plane. Hence $\tilde{u}(z, \theta)$ is a polynomial in z and this implies that $u = 0$.

LEMMA 3.2. *Let $u \in C^\infty(\bar{\Omega} \cap B)$, $u|_{\partial\Omega} = 0$ and assume that Pu has an analytic extension to a full neighborhood of the origin. Then the Taylor series of u converges in a complex neighborhood of the origin.*

PROOF. (C.f. Lemma (V.5) in [4]). After a change of variables of the form $(r, \theta) \mapsto (\lambda r, \theta)$, $\lambda > 1$, we may assume that the Taylor series $\sum_0^\infty r^k v_k(\theta)$ of $v = Pu$ converges to v in a neighborhood of the unit ball and that $\|v_k\|_{-1} \leq CM^k$, where C is fixed but M may be assumed arbitrarily small. Then

$$\tilde{v}(z, \theta) = \sum_0^\infty \frac{1}{(k-z)} v_k(\theta)$$

$\tilde{u}(z, \theta)$ will also have simple poles but the functions $\hat{u}(z, \theta) = \sin(2\pi z) \tilde{u}(z, \theta)$, $\hat{\theta}(z, \theta) = \sin(2\pi z) \tilde{\theta}(z, \theta)$ are entire. Moreover

$$\|\hat{\theta}(z, \theta)\|_{-1} \leq C \exp [2\pi |\operatorname{Im} z|].$$

From (3.3) we get

$$(3.10) \quad A_z \hat{u}(z, \theta) = \int \hat{v}(z + K, \theta) + \sin(2\pi z)(C_0(\theta) + C_1(\theta)z) - \sum_{k=1}^\infty \sum_{j=0}^2 (z-k)^{2-j} A_{j,k} \hat{u}(z-k, \theta) \Big],$$

and as in the proof of Lemma 3.1, we get

$$(3.11) \quad \|\hat{u}(z, \theta)\|_1^z \leq C \exp [C|z|], \quad z \in \mathbf{C},$$

for some constant C . Now $\hat{u}(k, \theta) = -2\pi u_k(\theta)$, $k = 0, 1, 2, \dots$, if $u \sim \sum_0^\infty r^k u_k(\theta)$ is the Taylor series expansion of u . From (3.11) we see that there exists a constant C such that

$$(3.12) \quad \|u_k\|_{\mathcal{H}^0(\omega)} \leq C^{k+1}, \quad k = 0, 1, 2, \dots$$

Now we write $P_k(x) = r^k u_k(\theta)$ so that $P_k(x)$ is a homogeneous polynomial of degree k . Then we get from (3.12)

$$(3.13) \quad \|P_k\|_{H^0(\Omega \cap B)} \leq C^{k+1}.$$

At this point we recall the classical Markov and Bernstein inequalities (see [8] and also [2], [3]). If $p(x)$ is a polynomial in one variable of degree $\leq k$, then

$$(M) \quad \sup_{-1 \leq x \leq 1} |p'(x)| \leq k^2 \sup_{-1 \leq x \leq 1} |p(x)|,$$

$$(B) \quad \sup_{z \in E_\rho} |p(z)| \leq \rho^k \sup_{-1 \leq x \leq 1} |p(x)|,$$

where E_ρ is the interior of the ellipse $x^2/a^2 + y^2/b^2 = 1$ with focal points at ± 1 and with $a + b = 2\rho$. Now let $\omega' \subset \omega$ have smooth boundary and let $\Omega' \subset \Omega$ be the corresponding cone. Then using (M) and (3.13) it is easy to show that

$$(3.14) \quad \sup_{x \in \Omega' \cap B} |p_k(x)| \leq (C')^{k+1},$$

for some constant C' . (This also follows from general inequalities in [2]). Then using (B) it is easy to show that

$$(3.15) \quad \sup_{x \in V} |p_k(x)| \leq (C'')^{k+1},$$

where $V \subset C^n$ is a small neighbourhood of the origin. Using also the fact that p_k is homogeneous of degree k we see that the Taylor series $\sum_0^\infty p_k$ converges uniformly in some complex neighbourhood of 0 and this completes the proof of the lemma. ((3.15) also follows more directly from general Bernstein type inequalities of [3].)

Now Theorem 0.1 follows easily from Lemma 3.1 and Lemma 3.2. Let $u \in C^\infty(\bar{\Omega})$ and assume that Pu and $u|_{\partial\Omega}$ extend to analytic functions near the origin. After subtracting an analytic function we may assume that $u|_{\partial\Omega} = 0$ near the origin. Then Lemma 3.2 shows that the Taylor series of u converges to an analytic function u' and Lemma 3.1 shows that $u - u' = 0$.

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